L. KÜHNE – Arithmetic algebraization à la Schneider-Lang I. Statement and proofs

1 Overview

Theorem 1 (classical Schneider-Lang)

K number field (always embedded $\sigma_0 : K \hookrightarrow \mathbb{C}$)

1 $f_1, \dots f_n$ meromorphic functions of finite growth $\leq \rho$

(i.e. $f_i = g_i / g_{i+i}$ g_i entire

$\log |g_i(z)| \leq |z|^{\rho}$ for $|z| \gg 1$)

2 $\mathrm{tr\,deg}_K K(f_1, \dots f_N) \geq 2$

3 diff. eq. $\frac{d}{dz} \circlearrowright K[f_1, \dots f_N]$

Then

$\# \{ z \in \mathbb{C} \setminus \text{poles} \mid f_i(z) \in K \} \leq 2\rho \, [K : \mathbb{Q}]$

Observation: one can relax 3 to a local one

Theorem 2 (simple case of Bost/Bost 2010)

X/K quasi-projective smooth, dim ≥ 2

$\gamma : \mathbb{C} \to X(\mathbb{C})$ holomorphic map

① of finite growth $\le \rho$

i.e. $X \overset{i}{\hookrightarrow} \mathbb{P}^k$ $\quad (i\circ\gamma)(z) = [\varphi_0(z), \dots \varphi_k(z)]$

φ_i growth $\le \rho$

② $\gamma(\mathbb{C})$ has Zariski dense image

Then

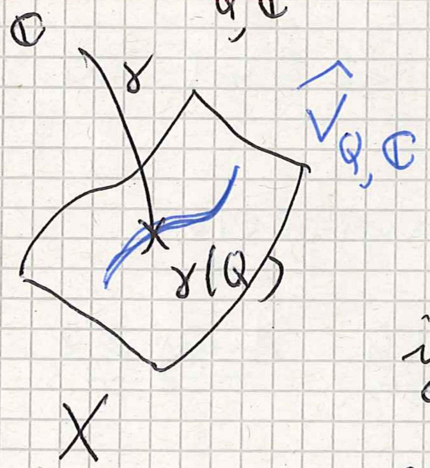
$\{ Q \in \mathbb{C} \mid \gamma(Q) \in X(k)$ "α-good" with respect to γ $\}$

$$\le \frac{N+1}{N-1}\,\rho\,\alpha\,[k:\mathbb{Q}]$$

$N = \dim X$

For any $Q \in \mathbb{C}$ with $\gamma(Q) \in X(k)$, γ induces

a $\hat{\gamma}_{Q,\mathbb{C}} : \hat{\mathbb{A}}^1_{\mathbb{C}} = \mathrm{Spf}(\mathbb{C}[[t]]) \longrightarrow \hat{V}_{Q,\mathbb{C}} \subseteq \hat{X}_{P,\mathbb{C}} \simeq \hat{\mathbb{A}}^N_{\mathbb{C}}$



Roughly, Q is "α-good"

if $\hat{\gamma}_{Q,\mathbb{C}}$ comes (after change of coordinates) from something nice over k

There should exist S of places of K

($\supseteq$ inf. places)

(1) $\exists$ models $\mathfrak{X} \to (\mathcal{O}_K \setminus S)$

$\gamma(Q) = p$ has extension

$\tilde{p} : \mathcal{O}_K \setminus S \to \mathfrak{X}$ $\leftarrow$ smooth nearby $\tilde{p}$

(2) $\hat{\gamma}_Q$, $\hat{V}_Q$ $/K$

$$\mathbb{A}^1_K = \mathrm{Spf}(K[[T]]) \to \hat{V}_Q \subseteq \hat{X}_p \simeq \hat{\mathbb{A}}^N_K$$

such that

$$\hat{\gamma}_Q \otimes_{\sigma_0} \mathbb{C} \simeq \hat{\gamma}_{Q,\mathbb{C}}$$

$$\hat{V}_Q \otimes_{\sigma_0} \mathbb{C} = \hat{V}_{Q,\mathbb{C}}$$

(3) $$\hat{\gamma}_Q(T) = (z_1(T), \ldots z_N(T)) \in K[[T]]^N$$

$$z_j(T) = \sum_{i=0}^{\infty} a_i^{(j)} T^i$$

we have constants C_p $(p \notin S)$ with $\prod_{p \notin S} C_p < \infty$

$$\forall i \quad |a_i^{(j)}|_p \leq \frac{C_p}{|i!|_p^{\alpha}}.$$

(4) furthermore, $\underline{a} = (a_1^{(1)}, \ldots a_1^{(N)}) \in \mathcal{O}_S^N \setminus \{0\}$

(5) $\forall p$ place of K, z_i has p-adic positive convergence radius

<u>Most important corollary</u>:

F = line bundle $\subseteq$ tangent bundle (X)

$\mathcal{F}$ = (analytic) integral curve of $F_{\mathbb{C}}$

$\gamma: \mathbb{C} \to \mathcal{F}$ étale holomorphic of finite growth

Then $\# \{ Q \in \mathbb{C} \mid \gamma(Q) \in X(K) \} \leq 3\rho\, [K:\mathbb{Q}]$

or $\mathcal{F}$ is algebraic curve.

(every point is 1-good)

② <u>Jets</u>

<u>Setting</u>: L = ample line bundle on X

F = set of α-good points (finite)

$Q \in \mathbb{C}$

<u>Recall</u>: $Q \in F$, $P = \gamma(Q) \in X(K)$

and $\hat{V}_Q \subseteq \hat{X}_P$ formal subscheme of dim 1

Important: $\hat{V}_Q$ Zariski dense in X

③

<u>choose</u>:

⊛ model $\mathcal{X}/\mathcal{O}_K$

hermitian integral line bundle $\mathcal{L}/\mathcal{X}$, $\mathcal{L}_K = L$

(ml-ample)

⊛ hermitian integral structure on $T_p \hat{V}_Q$

$$E_D = \Gamma(\mathcal{X}, \mathcal{L}^{\otimes D}) \qquad D \geq 1$$

and recall

$$\mathrm{Ker}\big(L^{\otimes D}_{|\hat{V}_{Q,i}} \longrightarrow L^{\otimes D}_{|\hat{V}_{Q,i-1}}\big) = S^i(T_p^* \hat{V}_Q) \otimes L_p^{\otimes D}$$

i^{th} infinitesimal neighborhood

Taylor coeff.

$$s \in E_D \xmapsto[i^{th}\text{ jet}]{} j^i_Q(s)$$

<u>Study</u>: $E_D \xhookrightarrow{\Psi_D} \bigoplus_{Q \in F} H^0(\hat{V}_{Q,n}, L^{\otimes D})$

$n >> 0$

Zariski density

$$E_D^i = \bigcap_{Q \in F} \{ s \in E_D \mid s_{|\hat{V}_{Q,i-1}} = 0 \}$$

Ψ_D induces maps

$$\Psi_D^i : E_D^i / E_D^{i+1} \longrightarrow \bigoplus_{\substack{Q \in F \\ P = \gamma(Q)}} H^0(\overset{\gamma(Q)}{\overset{\parallel}{P}}, S^i(T_P^* \hat{V}_Q) \otimes L_{1,P}^{\otimes D})$$

<u>Lemma</u> (Algebraicity)

Assume

(1) $\forall i$ $$\frac{1}{[K:\mathbb{Q}]} \sum_{P \text{ non-arch.}} \log \|\Psi_D^i\|_P \leq \alpha\, i \log(i) + c_1(i+D)$$

(α-good $\Rightarrow$ this)

(2) $$\frac{1}{[K:\mathbb{Q}]} \sum_P \log \|\Psi_D^i\|_P \leq c_2(i+D)$$

($\Leftarrow$ positive convergence radius)

(3) if $\frac{i}{D} = \lambda >> 0$, $$\frac{1}{[K:\mathbb{Q}]} \log \|\Psi_D^i\|_{\sigma_0} \leq -\beta \log\left(\frac{i}{D}\right) + c_3(i+D)$$

(many good points)

Then either $\beta \leq \alpha \frac{N+1}{N-1}$ or $\hat{V}_Q$ is algebraic.

$\not\exists$

Tools: ① slope inequalities for ψ_D $(\psi_D^i, \dots)$

$$(*) \quad \frac{\widehat{\deg}(E_D)}{[K:\mathbb{Q}]} \leq \sum_{i=0}^{\infty} \mathrm{rk}\left(E_D^i / E_D^{i+1}\right) \cdot \hat{\mu}_{\max}\left(S^i(T_p^* \hat{V}_Q) \otimes L_p^{\otimes p}\right)$$

$$+ \sum_{p \text{ all primes}} \log \|\psi_D^i\|_p .$$

② Arithmetic Hilbert–Samuel

$$(**) \quad \frac{\widehat{\deg}(E_D)}{[K:\mathbb{Q}]} \geq -c_4 \cdot D^{N+1}$$

Proof: Assume $\beta > \alpha \frac{(N+1)}{N-1}$

Choose $r \in \left[\frac{\beta}{\beta-1}, \frac{N+1}{2}\right)$ $\qquad \left(\frac{i}{D} > \lambda >> 0\right)$

Assume $D >> 0$ such that $D^2 > \lambda D$

(**) in (*):

$$-c_4 D^{N+1} \leq \underbrace{\overset{\mathrm{I}}{\sum_{i \leq D^2}} (\dots)}_{(1)+(2)} + \underbrace{\overset{\mathrm{II}}{\sum_{i > D^2}} (\dots)}_{(3)}$$

$$I \leq \sum_{i=0}^{D^2} \mathcal{H}\left(E_D^{i} / E_D^{i+1}\right)\left(c_4 D^2 + c_5 D^2 (\log D)\right)$$

$$\lim_{D \to \infty} \frac{I}{D^{N+1}} = 0$$