Y. BRUNEBARBE — Algebraization in complex analytic geometry I. From Puiseux to Chow

Theorem (Chow 1949)

Any closed $\mathbb{C}$-analytic subvariety in $\mathbb{P}^N(\mathbb{C})$ is algebraic.

① Rank of evaluation maps and dimension of Zariski closures

$F \subset \mathbb{C}^N$ subset

$$\forall K \geq 1 \qquad \eta_K : \mathbb{C}[X_1, \dots X_N]_{\leq K} \longrightarrow \overset{\text{continuous maps}}{\mathcal{C}(F, \mathbb{C})}$$

$$P \longmapsto P_{|F}$$

Set $\overline{F}^{zar} \subset \mathbb{P}^N(\mathbb{C})$

Proposition 1 $\quad \mathrm{rk}(\eta_K) \underset{K \to +\infty}{\sim} \dfrac{\deg(\overline{F})}{(\dim \overline{F})!} K^{\dim \overline{F}}$

Proof:

$$P(1, \dots) \\ \uparrow \\ P$$

$$\begin{array}{ccc}
\mathbb{C}[X_1, \dots X_N]_{\leq K} & \xrightarrow{\eta_K} & \mathcal{C}(F, \mathbb{C}) \\
\uparrow s & & \uparrow \\
\mathbb{C}[X_0, \dots X_N]_{\leq K} & \longrightarrow & \mathbb{C}[X_0, \dots X_N] / I(\overline{F})_K
\end{array}$$

$$\mathrm{rk}(\eta_k) = \dim \left(\mathbb{C}[X_0, \dots X_N] \big/ I(F)_k \right) \qquad \square$$

② An algebraic criterion for analytic submanifolds

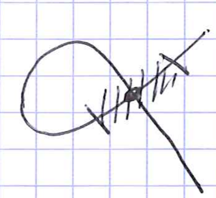
Let $\mathbb{B}^d \xrightarrow{i} \mathbb{P}^N(\mathbb{C})$ be an analytic embedding, $\mathbb{B}^d \subset \mathbb{C}^d$

and $F = i(\mathbb{B}^d) \subset \mathbb{P}^N(\mathbb{C})$.

<u>Lemma</u>: $\overline{F}^{\mathrm{Zar}}$ is an <u>irreducible</u> algebraic variety such that $\dim \overline{F}^{\mathrm{Zar}} \geq d$.

<u>Proof</u>: $F \not\subset \overline{F}_{\mathrm{sing}} \subsetneq \overline{F}$ by definition of Zariski closure

$\Rightarrow F \cap \overline{F}_{\mathrm{reg}} \neq \emptyset$ $\square$



<u>Proposition 2</u>. Assume there exists

$$(*) \begin{cases} C > 0 \text{ such that} \\ \forall k \in \mathbb{N}, \ \forall s \in \Gamma(\mathbb{P}^N(\mathbb{C}), \mathcal{O}(k)) \\ i^* s \neq 0 \Rightarrow \mathrm{mult}_0 (i^* s) \leq C \cdot k \end{cases}$$

Indeed, an equivalence

Then $\dim(\overline{F}) = d$.

Recall:

let $\mathcal{O}_0^{an}$ be the ring of germs of analytic functions at 0 on $\mathbb{B}^d$. It is a local ring with maximal $\mathfrak{m}$.

$$\forall f \in \mathcal{O}_0^{an}, \quad \mathrm{mult}_0(f) = \sup\{k \mid f \in \mathfrak{m}^k\}$$
$$\in \mathbb{N} \cup \{\infty\}$$

Proof: We want to estimate $\mathrm{rt}(\eta_k)$

$$E_k = \Gamma(\mathbb{P}^N(\mathbb{C}), \mathcal{O}(k)) = \mathbb{C}[X_0, \dots X_N]_k$$

Define a non-increasing filtration as follows:

$$s \in E_k^{\ell} \iff \mathrm{mult}_0(i^* s) \geq \ell$$

$(z_1, \dots z_d)$ coordinates of $\mathbb{B}^d$. Fix a trivialization of $i^*\mathcal{O}(k)$.

$$s \longmapsto \text{degree } \ell \text{ term of Taylor's expansion}$$

$$E_k^{\ell} / E_k^{\ell+1} \hookrightarrow \mathbb{C}[z_1, \dots z_d]_\ell$$

$$rk(\eta_k) = \dim_{\mathbb{C}} \left(E_k \Big/ \bigcap_{l \geq 0} E_k^l \right)$$

$$= \sum_{l \geq 0} \dim_{\mathbb{C}} \left(E_k^l / E_k^{l+1} \right)$$

$$\underset{\substack{\uparrow \\ \text{assumption}}}{=} \sum_{l=0}^{[C \cdot K]} \cdots\cdots$$

$$\leq \sum_{l=0}^{[C \cdot K]} \dim_{\mathbb{C}} \left(\mathbb{C}[z_1, \ldots z_d]_l \right)$$

$$= \dim \left(\mathbb{C}[z_1, \ldots z_d]_{\leq [C \cdot K]} \right) = \binom{[C \cdot K] + d}{d}$$

$$\underset{\text{Prop. 1}}{\Longrightarrow} \quad \dim(\overline{F}) \leq d.$$

③ Proof of Chow's theorem in the smooth case

<u>Proposition 3</u> : X connected closed complex

submanifold of $\mathbb{P}^N(\mathbb{C})$. Let $P \in X$.

$$\forall k \geq 0, \ \forall s \in \Gamma(\mathbb{P}^N(\mathbb{C}), \mathcal{O}(k))$$

$$s|_X \neq 0 \implies \mathrm{mult}_P(s|_X) \leq \deg(X) \cdot k$$

<u>Recall</u>: topological degree, $X \subset \mathbb{P}^N(\mathbb{C})$

$$H^{2i}(\mathbb{P}^N(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} \cdot [H]^i$$

$$[X] \in H^{2\,\mathrm{codim}_{\mathbb{C}}(X)}(\mathbb{P}^N(\mathbb{C}), \mathbb{Z})$$

$$\| \quad \deg(X) \cdot [H]^{\mathrm{codim}\, X}$$

$X \subset \mathbb{P}^N(\mathbb{C})$ connected submanifold

$P \in X \qquad i : \mathbb{B}^d \longrightarrow X \subseteq \mathbb{P}^N(\mathbb{C})$ an embedding

$$\overline{i(\mathbb{B}^d)}^{\mathrm{Zar}} = \overline{X}^{\mathrm{Zar}}$$

is an irreducible algebraic variety.

Prop. 2 + 3 $\implies$ $\dim(\overline{X}) = d = \dim(X)$

<u>Claim</u>: $\overline{X}_{\mathrm{reg}} \cap X$ is a non-empty open

and closed subset of $\overline{X}_{reg}$ (for the analytic topology).

$\Rightarrow \overline{X}_{reg} \cap X = \overline{X}_{reg}$

$\Rightarrow \overline{X}_{reg} \subseteq X$

$\Rightarrow \overline{X} \subseteq X$

But $X \subseteq \overline{X}$, hence $X = \overline{X}$.

<u>We are using</u>:

- let X be an irreducible complex algebraic variety. Then $X_{reg}(\mathbb{C})$ is dense in $X(\mathbb{C})$ for the analytic topology.
- $X(\mathbb{C})$ is connected for the analytic topology

<u>Proof of the claim</u>

non-empty: empty $\Leftrightarrow X \subseteq \overline{X}_{sing} \subsetneq \overline{X}$ ↯

closed because $X \subseteq \mathbb{P}^N(\mathbb{C})$ closed

open: two complex submanifolds of $\mathbb{P}^N(\mathbb{C})$,

$\overline{X}_{reg} \cap X \subseteq \overline{X}_{reg}$

of the same dimension

$\Rightarrow \overline{X}_{reg} \cap X$ open complex submanifold of $\overline{X}_{reg}$ $\square$

Proof of Proposition 3

X connected complex submanifold of $\mathbb{P}^N(\mathbb{C})$

$p \in X$, $\forall k \geq 0$ $s \in \Gamma(\mathbb{P}^N(\mathbb{C}), \mathcal{O}(k))$

$s_{|X} \Rightarrow \operatorname{mult}_p s_{|X} \leq \deg(X) \cdot k$

- If $\dim(X) = 1$, $\operatorname{mult}_p(s_{|X}) \leq \deg(s_{|X})$

$$= \deg_X(\mathcal{O}(k)_{|X})$$
$$= k \deg(\mathcal{O}(1)_{|X})$$

- If $\dim(X) \geq 2$

$G_2 = \{ L \subset \mathbb{P}^N(\mathbb{C}) \mid (N-d+1)\text{-plane containing } p \}$

$L \cap X$

$Z = \{ L \mid \forall x \in X \cap L \quad T_x L + T_x X = T_x \mathbb{P}^N(\mathbb{C}) \}$
$\subset G_2$

Lemma: ① $Z \subset Gr$ open dense subset

② $\{ T_p(L \cap X) \mid L \in Z \}$ is an open dense subset of $\mathbb{P}(T_p X)$.

③ $\forall k \geq 0$, $\forall s \in \Gamma(\mathbb{P}^N(\mathbb{C}), \mathcal{O}(k))$

$$\mathrm{mult}_p(s|_X) = \min_{L \in Z} \mathrm{mult}_p(s|_{X \cap L})$$

Application:

Chow theorem for the graph shows two alg. varieties are isomorphic $\Leftrightarrow$ their analytifications are isomorphic

quasi-projective counterexamples!