(1)

M. ANCONA – Algebraization in complex analytic geometry II. Géométrie Algébrique et Géométrie Analytique

GAGA (Serre 1956)

X projective variety $/\mathbb{C}$

$\mathcal{O}_X$ sheaf of regular functions

X^h, $\mathcal{H}_X$ = holomorphic functions

$$\left\{ \begin{array}{c} \text{algebraic coherent} \\ \text{sheaves on } X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{analytic coherent} \\ \text{sheaves on } X^h \end{array} \right\}$$

$$\mathcal{F} \longmapsto \mathcal{F}^h$$

Čech cohomology groups are preserved.

$\mathcal{F}$ alg. coherent sheaf

$id: X^h \to X$

$\mathcal{F}' = id^{-1}\mathcal{F}$

$$\mathcal{F}^h = \mathcal{F}' \otimes_{\mathcal{O}^h} \mathcal{H} =$$

<u>Fact</u>: $\mathcal{F}$ alg. coh. sheaf $\Longrightarrow$ $\mathcal{F}^h$ is an analytic coherent sheaf

$\varphi: \mathcal{F} \to \mathcal{G}$ alg. morphisms

$\varphi^h: \mathcal{F}^h \to \mathcal{G}^h$ analytic morphism

s section of $\mathcal{F}'$ on union of Zariski opens

$s' \otimes 1$ section of $\mathcal{F}^h$ over $\mathcal{U}^h$

This commutes with restriction maps and coboundary maps.

$$\rightsquigarrow \varepsilon : H^q(X, \mathcal{F}) \longrightarrow H^q(X^h, \mathcal{F}^h)$$

Theorem 1: ε is an isomorphism

Theorem 2: $\mathcal{F}, \mathcal{G}$ alg. coherent sheaves

$\Psi : \mathcal{F}^h \to \mathcal{G}^h$ analytic morphism

$\exists ! \ \varphi : \mathcal{F} \to \mathcal{G}$ such that $\varphi^h = \Psi$

Theorem 3: $\mathcal{M}$ an analytic coherent sheaf on X^h. Then $\exists \ \mathcal{F}$ alg. coh. sheaf on X such that $\mathcal{F}^h \simeq \mathcal{M}$. ($\mathcal{F}$ is unique up to isomorphism)

Proof of Theorem 1

(i) It suffices to prove Theorem 1 for $X = \mathbb{P}^n(\mathbb{C})$.

(ii) Theorem 1 is true for $\mathcal{O}_{\mathbb{P}^2}$

(iii) Thm 1 is true for $\mathcal{O}(m)$

(iv) Thm 1 for all $\mathcal{F}$

(i) $X \overset{i}{\hookrightarrow} \mathbb{P}^2$

$\mathcal{F}$ on X $\quad$ $i_*\mathcal{F}$ is an alg. coherent sheaf on $\mathbb{P}^2$

$$H^q(X, \mathcal{F}) \simeq H^q(\mathbb{P}^2, i_*\mathcal{F})$$

$$H^q(X^h, \mathcal{F}^h) \simeq H^q(\mathbb{P}^2, i_*\mathcal{F}^h)$$

(ii) We'll restrict to our study on $\mathbb{P}^2$

$$H^0(\mathbb{P}^2, \mathcal{O}) = \mathbb{C} \qquad H^0(\mathbb{P}^2, \mathcal{H}) = \mathbb{C}$$

$$H^i(\mathbb{P}^2, \mathcal{O}) = 0 \quad \forall i > 0, \qquad H^i(\mathbb{P}^2, \mathcal{H}) = H^{0,i}(\mathbb{P}^2) = 0 \quad \forall i > 0$$

(iii) By induction on r

$r = 0$ ok

$E = \{t_0 = 0\} \subset \mathbb{P}^2$ $\quad t_0, \dots t_2$ homog. coordinates

$$0 \to \mathcal{O}(m-1) \to \mathcal{O}(m) \to \mathcal{O}_E(m) \to 0$$

long exact sequence:

$$H^i(\mathbb{P}^2, \mathcal{O}(m-1)) \to H^i(\mathbb{P}^2, \mathcal{O}(m)) \to H^i(E, \mathcal{O}_E(m))$$
$$\checkmark$$

$$\to H^{i+1}(\mathbb{P}^2, \mathcal{O}(m-1)) \to H^{i+1}(\mathbb{P}^2, \mathcal{O}(m))$$

$$\to H^{i+1}(E, \mathcal{O}_E(m))$$
$$\checkmark$$

By the five lemma:

true for $\mathcal{O}(m-1)$ $\Rightarrow$ true for $\mathcal{O}(m)$

It is true for $\mathcal{O}$ by (ii), hence true for all $\mathcal{O}(m)$.

(iv) $\mathcal{F}$ alg. coh. sheaf on $\mathbb{P}^2$

By a Theorem of Serre, $\exists\, n_0$ s.t. $\forall n \geq n_0$

$\mathcal{F}(n)_x$ is generated by $H^0(\mathbb{P}^2, \mathcal{F}(n))$.

$$0 \to K \to \mathcal{O}(-n)^{\ell} \to \mathcal{F} \to 0 \quad \text{on } \mathbb{P}^2$$

$$H^i(\mathbb{P}^2, K) \to H^i(\mathbb{P}^2, \mathcal{O}(-n)^{\ell}) \to H^i(\mathbb{P}^2, \mathcal{F}) \to H^{i+1}(\mathbb{P}^2, K) \to H^{i+1}(\mathcal{O}(-n)^{\ell})$$

① ② ✓ ③ ④ ⑤ ✓

For $i > 2$, $H^i(\mathbb{P}^2, \mathcal{F}) = 0$ for all $\mathcal{F}$.

in analytic & algebraic case.

②, ④, ⑤ isomorphism $\Rightarrow$ ③ is a surjection

we have a surjection in ① too.

By the five lemma, we have Theorem 1 □

<u>Proof of Theorem</u> 2

$\varphi : \mathcal{F}^h \to \mathcal{G}^h$ $\qquad H^0(X, \underline{\mathcal{H}om}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}))$

analytic morphism

$$H^0(X^h, \underline{\mathcal{H}om}_{\mathcal{H}}(\mathcal{F}^h, \mathcal{G}^h))$$

But $\mathcal{H}om_{\mathcal{H}}(\mathcal{F}^h, \mathcal{G}^h) \simeq \underline{\mathcal{H}om}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})^h$. □

<u>Proof of Theorem 3</u>

It suffices to prove Thm 3 for $X = \mathbb{P}^2$.

$X^h \hookrightarrow \mathbb{P}^2$ $\quad \mathcal{M}$ analytic coherent sheaf on X^h

$\rightsquigarrow i_* \mathcal{M}$ on $\mathbb{P}^2$

$i_* \mathcal{M} \simeq \mathcal{G}^h$ for some $\mathcal{G}$ algebraic coherent sheaf on $\mathbb{P}^2$.

$\mathcal{G} = i_* \mathcal{F}$ for some $\mathcal{F}$ alg. coherent sheaf on X

Prop. Let $\mathcal{M}$ be an analytic coherent sheaf on $\mathbb{P}^2$. Then $\exists\, n_0$ such that $\forall n \geq n_0$ $\mathcal{M}(n)_x$ is generated by $H^0(\mathbb{P}^2, \mathcal{M}(n))$ $\forall x \in \mathbb{P}^2$.

Proposition + Theorem 2 $\Longrightarrow$ Theorem 3

$$\mathcal{H}(n)^{\ell} \longrightarrow \mathcal{M} \longrightarrow 0$$

globally on $\mathbb{P}^2$

Let $0 \to K \to \mathcal{H}(n)^{\ell} \to \mathcal{M} \to 0$. We apply the proposition for k

$$\mathcal{H}(n_2)^{l_2} \xrightarrow{\Psi} \mathcal{H}(n_1)^{l_1} \to \mathcal{M} \to 0$$

for some $n_i \in \mathbb{Z}$

$$\mathcal{O}(n_2)^{l_2} \xrightarrow{\varphi} \mathcal{O}(n_1)^{l_1} \quad \text{by Theorem 2}$$

with $\varphi^h = \Psi$.

The sheaf we want is $\operatorname{coker} \varphi = \mathcal{F}$, $\mathcal{F}^h = \mathcal{M}$ $\square$

We are reduced to prove:

$A_2)$ $\mathcal{M}(n)_x$ generated by $H^0(\mathbb{P}^2, \mathcal{M}(n))$

$B_2)$ $H^i(\mathbb{P}^2, \mathcal{M}(n)) = 0 \quad \forall i > 0$

By induction:

$A_{2-1}) + B_{2-1}) \Rightarrow A_2)$

$A_2) \Rightarrow B_2)$

<u>Remark</u>: If $H^0(\mathbb{P}^2, \mathcal{M}(n))$ generates $\mathcal{M}(n)_x$, then it generates $\mathcal{M}(n)_y$ for all y in a

neighborhood of x.

Remark + $\mathbb{P}^2$ compact $\Rightarrow$ we can prove Prop A_2 for one $x \in \mathbb{P}^2$

$E = \{t_0 = 0\} \subset \mathbb{P}^2$

E passing at $x \in \mathbb{P}^2$

$$0 \to \mathcal{B}(n) \to \mathcal{M}(n-1) \xrightarrow{\text{mult by } t_0} \mathcal{M}(n) \longrightarrow \mathcal{M}_E(n) \to 0$$

gives

$$0 \to \mathcal{B}(n) \to \mathcal{M}(n-1) \to \mathcal{L} \to 0$$

$$0 \to \mathcal{L} \to \mathcal{M}(n) \to \mathcal{M}_E(n) \to 0$$

induced long exact sequences:

$$H^1(\mathbb{P}^2, \mathcal{M}(n-1)) \to H^1(\mathbb{P}^2, \mathcal{L})$$

$$\to \cancel{H^2(\mathbb{P}^2, \mathcal{B}(n))}^{\,0} \to$$

$$H^1(\mathbb{P}^2, \mathcal{L}) \to H^1(\mathbb{P}^2, \mathcal{M}(n)) \to \cancel{H^1(\mathbb{P}^2, \mathcal{M}_E(n))}^{\,0}$$

By induction hypothesis, we can take large n

such that $H^2(\mathbb{P}^2, \mathcal{B}(n)) = 0$ $H^1(\mathbb{P}^2, \mathcal{M}_E(n)) = 0$

$$H^1(\mathbb{P}^2, \mathcal{M}(n-1)) \twoheadrightarrow H^1(\mathbb{P}^2, \mathcal{L})$$

$$H^1(\mathbb{P}^2, \mathcal{L}) \twoheadrightarrow H^1(\mathbb{P}^2, \mathcal{M}(n)).$$

Key point: for a compact complex variety X and $\mathcal{M}$ analytic coherent sheaf

$H^q(X, \mathcal{M})$ is a $\mathbb{C}$-vector space of finite dimension

$$\dim H^1(\mathbb{P}^1, \mathcal{M}(n-1)) \geq \dim H^1(\mathbb{P}^2, \mathcal{L})$$
$$\geq \dim H^1(\mathbb{P}^2, \mathcal{M}(n))$$

For large n, $H^1(\mathbb{P}^1, \mathcal{M}(n))$ stabilizes.

$$H^0(\mathbb{P}^2, \mathcal{M}(n)) \to H^0(\mathbb{P}^2, \mathcal{M}_E(n))$$

$$\to H^1(\mathbb{P}^2, \mathcal{L}) \underset{\text{isomorphism}}{\twoheadrightarrow} H^1(\mathbb{P}^2, \mathcal{M}(n))$$

$$\Rightarrow H^0(\mathbb{P}^2, \mathcal{M}(n)) \twoheadrightarrow H^0(\mathbb{P}^2, \mathcal{M}_E(n)) = H^0(E, \mathcal{M}_E(n))$$

is a surjection.

By induction, for $n \geq n_0$

$\mathcal{M}_E(n)_x$ is generated by $H^0(E, \mathcal{M}_E(n))$.

$A = \mathcal{O}_x$, $I = \mathcal{I}_{E,x}$ $\quad A/I = \mathcal{O}_{x,E}$

$M = \mathcal{M}(n)_x \qquad N = H^0(\mathbb{P}^2, \mathcal{M}(n)) \cdot \mathcal{M}(n)_x$

We want $M = N$.

We have $M/IM = N/IM$.

$$\mathcal{M}_E(n)_x = M \otimes A/I = M/IM$$

$$= H^0(E, \mathcal{M}_E(n)) \cdot \mathcal{M}_E(n)_x$$

$$= H^0(\mathbb{P}^2, \mathcal{M}(n)) \cdot \mathcal{M}_E(n)_x$$

$\Rightarrow M = N + IM$

By Nakayama's lemma, $M = N$.

$A_2) \Rightarrow B_2)$ dévissage + induction

Kodaira-Spencer:

$$\begin{array}{c} L \\ \downarrow \\ X \end{array} \quad \text{line bundle over } X$$

$H^1(X, L)$ is $\mathbb{C}$-vector space of finite dimension

$$H^q(X, \Omega^p(L)) \simeq \mathcal{H}^{p,q}(X, L)$$

(p,q) harmonic forms

$$= \{ \Delta \alpha = 0 \}$$

Δ is an elliptic operator

Theorem: over a compact manifold, an elliptic operator is Fredholm.

$$0 \to K \to \mathcal{H}^1 \to M \to 0$$

Comments

$X = \mathbb{P}^N$
$\ni x$

Noetherian

$$\mathcal{O}_{X,x} \hookrightarrow \mathcal{H}_{X,x} \hookrightarrow \mathbb{C}[[z_1, \dots z_N]]$$

$\boxed{\text{flat}}$

$\mathcal{H}_{X,x}$ is $\mathbb{C}\{z_1, \dots z_N\}$

$\mathcal{F}$ coh. analytic

$\mathcal{F} \otimes \mathcal{O}(n)$ is generated by its global sections

and has no higher cohomology.