

M. ANCONA -

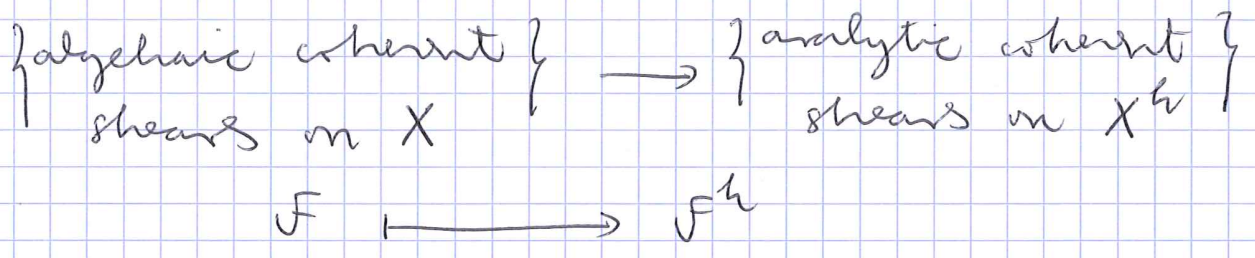
Algebraization in complex
analytic geometry II. Géométrie
Algébrique et Géométrie
Analytique

GAGA (Seme 1956)

X projective variety / \mathbb{C}

\mathcal{O}_X sheaf of regular functions

X^h , $\mathcal{H}_X =$ holomorphic functions



Each cohomology groups are preserved.

F alg. coherent sheaf

$\text{id}: X^h \rightarrow X$

$F^h = F' \otimes_{\mathcal{O}^h} \mathcal{H} =$

$F' = \text{id}^{-1} F$

Fact: F alg. coh. sheaf $\Rightarrow F^h$ is an analytic coherent sheaf

$\varphi: F \rightarrow G$ alg. morphism

$\varphi^h: F^h \rightarrow G^h$ analytic morphism

s section of F^1 on union of Zariski opens

$s^1 \otimes 1$ section of F^h over U^h

This commutes with restriction maps and coboundary maps.

$$\leadsto \varepsilon: H^q(X, F) \rightarrow H^q(X^h, F^h)$$

Theorem 1: ε is an isomorphism

Theorem 2: F, g alg. coherent sheaves

$\psi: F^h \rightarrow g^h$ analytic morphism

$\exists! \varphi: F \rightarrow g$ such that $\varphi^h = \psi$

Theorem 3: \mathcal{M} an analytic coherent sheaf

on X^h . Then $\exists F$ alg. coh. sheaf on X such

that $F^h \simeq \mathcal{M}$. (F is unique up to isomorphism)

Proof of Theorem 1

(i) It suffices to prove Theorem 1 for $X = \mathbb{P}^n(\mathbb{C})$.

(2)

(ii) Theorem 1 is true for $\mathcal{O}_{\mathbb{P}^2}$ (iii) Thm 1 is true for $\mathcal{O}(m)$ (iv) Thm 1 for all \mathcal{F}

(i) $X \xrightarrow{i} \mathbb{P}^2$

 \mathcal{F} on X $i_* \mathcal{F}$ is an alg. coherent sheaf on \mathbb{P}^2

$$H^q(X, \mathcal{F}) \cong H^q(\mathbb{P}^2, i_* \mathcal{F})$$

$$H^q(X^h, \mathcal{F}^h) \cong H^q(\mathbb{P}^2, i_* \mathcal{F}^h)$$

(ii) We'll restrict to our study on \mathbb{P}^2

$$H^0(\mathbb{P}^2, \mathcal{O}) = \mathbb{C} \quad H^0(\mathbb{P}^2, \mathcal{H}) = \mathbb{C}$$

$$H^i(\mathbb{P}^2, \mathcal{O}) = 0 \quad \forall i > 0, \quad H^i(\mathbb{P}^2, \mathcal{H}) = H^i(\mathbb{P}^2) = 0 \quad \forall i > 0$$

(iii) By induction on r $r=0$ ok $E = \{t_0 = 0\} \subset \mathbb{P}^2$ t_0, \dots, t_2 homog. coordinates

$$0 \rightarrow \mathcal{O}(m-1) \rightarrow \mathcal{O}(m) \rightarrow \mathcal{O}_E(m) \rightarrow 0$$

long exact sequence:

$$H^i(\mathbb{P}^2, \mathcal{O}(m-1)) \rightarrow H^i(\mathbb{P}^2, \mathcal{O}(m)) \rightarrow H^i(E, \mathcal{O}_E(m))$$

$$\rightarrow H^{i+1}(\mathbb{P}^2, \mathcal{O}(m-1)) \rightarrow H^{i+1}(\mathbb{P}^2, \mathcal{O}(m))$$

$$\rightarrow H^{i+1}(E, \mathcal{O}_E(m))$$

By the five lemma:

true for $\mathcal{O}(m-1) \Rightarrow$ true for $\mathcal{O}(m)$

It is true for \mathcal{O} by (ii), hence true for all $\mathcal{O}(m)$.

(iv) \mathcal{F} alg. coh. sheaf on \mathbb{P}^2

By a theorem of Serre, $\exists n_0$ s.t. $\forall n \geq n_0$

$\mathcal{F}(n)_x$ is generated by $H^0(\mathbb{P}^2, \mathcal{F}(n))$.

$$0 \rightarrow K \rightarrow \mathcal{O}(-n) \rightarrow \mathcal{F} \rightarrow 0 \quad \text{on } \mathbb{P}^2$$

$$\begin{array}{ccccccc}
 H^i(\mathbb{P}^2, \mathcal{K}) & \rightarrow & H^i(\mathbb{P}^2, \mathcal{O}(-n)) & \rightarrow & H^i(\mathbb{P}^2, \mathcal{F}) & \rightarrow & H^{i+1}(\mathbb{P}^2, \mathcal{K}) & \textcircled{3} \\
 \textcircled{1} & & \textcircled{2} & \checkmark & \textcircled{3} & & \textcircled{4} & \\
 & & & & & & \rightarrow H^{i+1}(\mathcal{O}(-n)^{\oplus l}) & \\
 & & & & & & \checkmark \textcircled{5} &
 \end{array}$$

For $i > 2$, $H^i(\mathbb{P}^2, \mathcal{F}) = 0$ for all \mathcal{F} .

in analytic & algebraic case.

②, ④, ⑤ isomorphism \rightarrow ③ is a surjection

We have a surjection in ① too.

By the five lemma, we have Theorem 1 \square

Proof of Theorem 2

$\psi: \mathcal{F}^h \rightarrow \mathcal{G}^h$
analytic morphism

$$H^0(X, \underline{\text{Hom}}_{\mathcal{O}}(\mathcal{F}, \mathcal{G}))$$

$$H^0(X^h, \underline{\text{Hom}}_{\mathcal{O}^h}(\mathcal{F}^h, \mathcal{G}^h))$$

But $\underline{\text{Hom}}_{\mathcal{O}^h}(\mathcal{F}^h, \mathcal{G}^h) \cong \underline{\text{Hom}}_{\mathcal{O}}(\mathcal{F}, \mathcal{G})^h$. \square

Proof of Theorem 3

It suffices to prove Thm 3 for $X = \mathbb{P}^2$.

$X^h \hookrightarrow \mathbb{P}^2$ \mathcal{M} analytic coherent sheaf on X^h

$\implies i_* \mathcal{M}$ on \mathbb{P}^2

$i_* \mathcal{M} \simeq \mathcal{G}^h$ for some \mathcal{G} algebraic coherent sheaf on \mathbb{P}^2 .

$\mathcal{G} = i_* \mathcal{F}$ for some \mathcal{F} alg. coherent sheaf on X

Prop. Let \mathcal{M} be an analytic coherent sheaf

on \mathbb{P}^2 . Then $\exists n_0$ such that $\forall n \geq n_0$
 $\mathcal{M}(n)_x$ is generated by $H^0(\mathbb{P}^2, \mathcal{M}(n))$
 $\forall x \in \mathbb{P}^2$.

Proposition + Theorem 2 \implies Theorem 3

$$\mathcal{H}(n)^{\ell} \rightarrow \mathcal{M} \rightarrow 0$$

globally on \mathbb{P}^2

Let $0 \rightarrow \mathcal{K} \rightarrow \mathcal{H}(n)^{\ell} \rightarrow \mathcal{M} \rightarrow 0$. We apply

④

the proposition for \mathbb{R}

$$\mathcal{H}(n_2)^{l_2} \xrightarrow{\Psi} \mathcal{H}(n_1)^{l_1} \rightarrow \mathcal{M} \rightarrow 0$$

for some $n_i \in \mathbb{Z}$

$$\mathcal{O}(n_2)^{l_2} \xrightarrow{\varphi} \mathcal{O}(n_1)^{l_1} \quad \text{by Theorem 2}$$

with $\varphi^h = \Psi$.

The sheaf we want is either $\mathcal{F} = \mathcal{F}$, $\mathcal{F}^h = \mathcal{M}$ \square

We are reduced to prove:

$$A_2) \mathcal{M}(n)_x \text{ generated by } H^0(\mathbb{P}^2, \mathcal{M}(n))$$

$$B_2) H^i(\mathbb{P}^2, \mathcal{M}(n)) = 0 \quad \forall i > 0$$

By induction:

$$A_{2-1}) + B_{2-1}) \Rightarrow A_2)$$

$$A_2) \Rightarrow B_2)$$

Remark: If $H^0(\mathbb{P}^2, \mathcal{M}(n))$ generates $\mathcal{M}(n)_x$, then it generates $\mathcal{M}(n)_y$ for all y in \mathcal{V}

neighborhood of x .

Remark + \mathbb{P}^2 compact \Rightarrow we can find P and A_2
for $\forall x \in \mathbb{P}^2$

$$E = \{x_0 = 0\} \subset \mathbb{P}^2$$

E passing at $x \in \mathbb{P}^2$

$$0 \rightarrow \mathcal{B}(n) \rightarrow \mathcal{M}(n-1) \xrightarrow{\text{mult by } x_0} \mathcal{M}(n) \rightarrow \mathcal{M}_E(n) \rightarrow 0$$

gives

$$0 \rightarrow \mathcal{B}(n) \rightarrow \mathcal{M}(n-1) \rightarrow \mathcal{L} \rightarrow 0$$

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{M}(n) \rightarrow \mathcal{M}_E(n) \rightarrow 0$$

induced by exact sequences:

$$H^1(\mathbb{P}^2, \mathcal{M}(n-1)) \rightarrow H^1(\mathbb{P}^2, \mathcal{L})$$

$$\rightarrow H^2(\mathbb{P}^2, \mathcal{B}(n)) \rightarrow 0$$

$$H^1(\mathbb{P}^2, \mathcal{L}) \rightarrow H^1(\mathbb{P}^2, \mathcal{M}(n)) \rightarrow H^1(\mathbb{P}^2, \mathcal{M}_E(n)) \rightarrow 0$$

By induction hypothesis, we can take large n

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such that $H^2(\mathbb{P}^2, \mathcal{O}(n)) = 0$ $H^1(\mathbb{P}^2, \mathcal{M}_E(n)) = 0$

$$H^1(\mathbb{P}^2, \mathcal{M}(n-1)) \rightarrow H^1(\mathbb{P}^2, \mathcal{O})$$

$$H^1(\mathbb{P}^2, \mathcal{O}) \rightarrow H^1(\mathbb{P}^2, \mathcal{M}(n)).$$

Key point: for a compact complex variety X

and \mathcal{M} analytic coherent sheaf

$H^1(X, \mathcal{M})$ is a \mathbb{C} -vector space of finite dimension

$$\begin{aligned} \dim H^1(\mathbb{P}^1, \mathcal{M}(n-1)) &\geq \dim H^1(\mathbb{P}^2, \mathcal{O}) \\ &\geq \dim H^1(\mathbb{P}^2, \mathcal{M}(n)) \end{aligned}$$

For large n , $H^1(\mathbb{P}^1, \mathcal{M}(n))$ stabilizes.

$$H^0(\mathbb{P}^2, \mathcal{M}(n)) \rightarrow H^0(\mathbb{P}^2, \mathcal{M}_E(n))$$

$$\rightarrow H^1(\mathbb{P}^2, \mathcal{O}) \rightarrow H^1(\mathbb{P}^2, \mathcal{M}(n))$$

↑
isomorphism

$$\Rightarrow H^0(\mathbb{P}^2, \mathcal{M}(n)) \rightarrow H^0(\mathbb{P}^2, \mathcal{M}_E(n)) = H^0(E, \mathcal{M}_E(n))$$

is a surjection.

By induction, for $n \geq n_0$

$M_E(n)_x$ is generated by $H^0(E, \mathcal{M}_E(n))$.

$$A = \mathcal{H}_x, \quad I = \mathcal{J}_{E,x}, \quad A/I = \mathcal{H}_{x,E}$$

$$M = M(n)_x, \quad N = H^0(\mathbb{P}^2, \mathcal{M}(n)) \cdot M(n)_x$$

We want $M = N$.

$$\text{We have } M/IM = N/IM.$$

$$M_E(n)_x = M \otimes A/I = M/IM$$

$$= H^0(E, \mathcal{M}_E(n)) \cdot M_E(n)_x$$

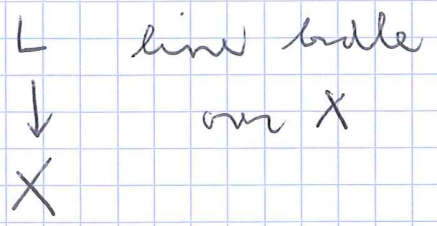
$$= H^0(\mathbb{P}^2, \mathcal{M}(n)) \cdot M_E(n)_x$$

$$\Rightarrow M = N + IM$$

By Nakayama's lemma, $M = N$.

$A_2) \Rightarrow B_2)$ dérivage + induction

Kodaira-Spencer:



$H^1(X, L)$ is \mathbb{C} -vector space of finite dimension

$$\begin{aligned}
 H^q(X, \Omega^p(L)) &\cong \mathcal{H}^{p,q}(X, L) \\
 &\text{(p,q) harmonic forms} \\
 &= \{ \Delta \alpha = 0 \}
 \end{aligned}$$

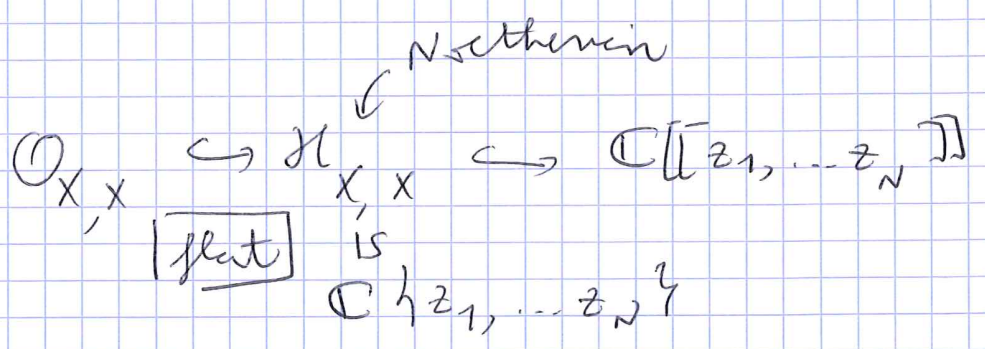
Δ is an elliptic operator

Theorem: over a compact manifold, an elliptic operator is Fredholm.

$$0 \rightarrow K \rightarrow \mathcal{H}^p \rightarrow M \rightarrow 0$$

Comments

$$\begin{array}{l}
 X = \mathbb{P}^n \\
 \mathbb{C} \\
 X
 \end{array}$$



\mathbb{R} wh. analytic

$F \otimes \mathcal{O}(n)$ is generated by its global sections
and has no higher cohomology.