D. IZQUIERDO — The formalism of slopes. Vector bundles on projective curves

$X/\mathbb{C}$ smooth connected projective curve

1 Slopes and semi-stability

Recall:

$$\left\{ \begin{array}{c} \text{vector bundles} \\ \text{over } X \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{locally free finite type} \\ \mathcal{O}_X\text{-modules} \end{array} \right\}$$

$$(p : E \to X) \longmapsto \left(\mathcal{E} : \mathcal{U} \mapsto \left\{ \begin{array}{c} \text{sections } s : \mathcal{U} \to E \\ \text{of } p \end{array} \right\} \right)$$

Two important invariants associated to a coherent $\mathcal{O}_X$-module $\mathcal{F}$.

* rank : $\exists\, \mathcal{U} \hookrightarrow X$ $\mathcal{F}_{|\mathcal{U}}$ is free.

$$\mathrm{rk}(\mathcal{F}) = \mathrm{rk}(\mathcal{F}_{|\mathcal{U}})$$
$$\mathrm{rk}(E) = \mathrm{rk}(\mathcal{E})$$

* degree :

$$\deg(\mathcal{F}) = \chi(\mathcal{F}) - \mathrm{rk}(\mathcal{F}) \cdot \chi(\mathcal{O}_X)$$

Remark : Riemann-Roch : when $\mathcal{F}$ is an invertible $\mathcal{O}_X$-module, $\deg(\mathcal{F})$ is the degree of the associated Weil divisor

- $\det \mathcal{F} = \Lambda^{\mathrm{rk}(\mathcal{F})} \mathcal{F} \in \mathrm{Pic}\, X$

degree of the associated Weil divisor is $\deg(\mathcal{F})$

E

If $\mathrm{rk}\, \mathcal{F} > 0$, define the slope of $\mathcal{F}$ as:

$$\mu(\mathcal{F}) = \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}$$

Remark: $\mu(E \otimes E') = \mu(E) + \mu(E')$

$\mu(E^*) = -\mu(E)$

$\mu(\mathrm{Hom}(E, F)) = \mu(F) - \mu(E)$

Lemma: $E' \subseteq E$ non-zero subbundle of E

Equivalent:

(i) $\mu(E') < \mu(E)$

(ii) $\mu(E') < \mu(E/E')$

(iii) $\mu(E) < \mu(E/E')$

(just use

$0 \to E' \to E \to E/E' \to 0$)

Same for $\leq$, $>$

<u>Definition</u>: E vector bundle of $rk > 0$ and slope μ. We say that E is <u>semi-stable</u> (resp. <u>stable</u>) if for any subbundle $0 \neq E' \subsetneq E$

$$\mu'(E') \leq \mu(E)$$

(resp. $\mu(E') < \mu(E)$)

(equivalent for quotients)

<u>Remark</u>: If E is semistable, then for any $\mathcal{E}'' \subseteq \mathcal{E}$ coherent $\mathcal{O}_X$-submodule, $\mu(\mathcal{E}'') \leq \mu(\mathcal{E})$. Indeed, there is a subbundle E' of E such that

$$\mathcal{E}'' \subseteq \mathcal{E}'$$

$$rk(\mathcal{E}'/\mathcal{E}'') = 0$$

So $\deg(\mathcal{E}') = \deg(\mathcal{E}'/\mathcal{E}'') + \deg(\mathcal{E}'')$

$$= \dim H^0(X, \mathcal{E}'/\mathcal{E}'') + \deg(\mathcal{E}'')$$

$$\geq \deg(\mathcal{E}'').$$

$rk(\mathcal{E}') = rk(\mathcal{E}'')$. $\Rightarrow \mu(\mathcal{E}') \geq \mu(\mathcal{E}'')$.

equality $\Leftrightarrow$ $\mathcal{E}''$ corresponded to a subbundle

Examples: • line bundles are stable

• $X = \mathbb{P}^1$, $\mathcal{E} = \mathcal{O}_X(a_1) \oplus \cdots \oplus \mathcal{O}_X(a_r)$

the only semistable bundles are $\mathcal{O}_X(k)^r$.

② Categories of semistable bundles

Prop. (i) E, F s.s. bundles $/X$, $f: E \to F$, $f \neq 0$

Then $\mu(E) \leq \mu(F)$.

(ii) If E, F are stable of the same slope μ, then f is an isomorphism

(iii) E stable: $\mathrm{End}(E) \simeq \mathbb{C}$

Proof. (i) $I = \mathrm{im}(\mathcal{E} \to \mathcal{F})$

semi-stability $\Rightarrow \mu(\mathcal{E}) \leq \mu(I) \leq \mu(\mathcal{F})$

(ii) all equal to μ

hence I corresponds to a subbundle of F.

<u>stability</u>: $I = F$

$f : E \xrightarrow{\sim} I$

(iii) $f \in \mathrm{End}(E)$, $f \neq 0$

$\underbrace{\mathbb{C}[f]}_{\text{finite extension of } \mathbb{C}} = \mathbb{C}$

□

$\mathcal{C}$ = category of vector bundles $/X$

$\mathcal{C}$ is additive, not abelian

$\mu \in \mathbb{Q}$

$\mathcal{C}(\mu)$ = full subcategory of semistable bundles of slope μ

<u>Theorem</u>: $\mathcal{C}(\mu)$ is abelian, stable by extension.

[$f : E \to F$, $f \neq 0$, $I = \mathrm{Im}(f)$

$\mu = \mu(E) \leq \mu(I) \leq \mu(F) = \mu$

here I is a subbundle of F.]

③ <u>Filtrations</u>

a) <u>Jordan-Hölder</u>

E vector bundle semi-stable of slope μ / X

There exists a stable subbundle E_1 of slope μ.

Start again with $E/_{E_1}$.

<u>Get</u>: $0 \subseteq E_1 \subseteq \dots \subseteq E_k = E$ such that,

for all i, $gr_i = E_i/_{E_{i-1}}$ stable of slope μ

$\Rightarrow$ Jordan-Hölder filtration

b) <u>Harder-Narasimhan filtration</u>

<u>Proposition</u>. E vector bundle / X

There exist vector subbundles

$$0 \subseteq E_1 \subseteq \dots \subseteq E_k = E$$

such that:

(i) $\forall i$ $\quad gr_i = E_i/_{E_{i-1}}$ are semi-stable

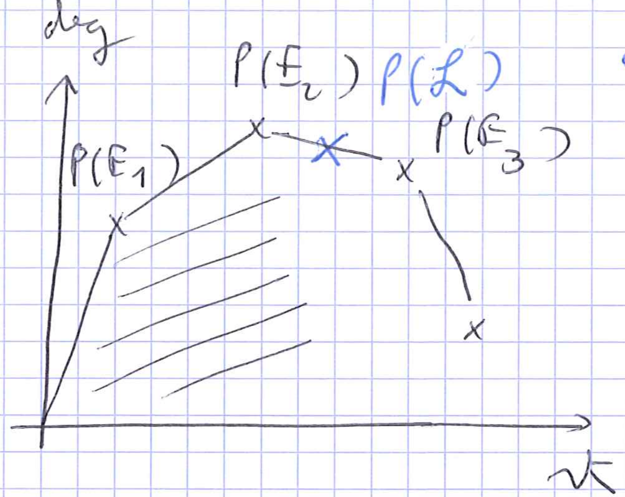
(ii) $\forall i$ $\quad \mu(gr_i) > \mu(gr_{i+1})$

This filtration is unique.

<u>Proof</u>. Consider coherent sub $\mathcal{O}_X$-modules of $\mathcal{E}_1$ of maximal slope, and among them, choose one of maximal rank.

$\mathcal{E}_1$ is a subbundle E_1, E/E_1 start again.

Newton polygon



$\mathcal{L}$ is a subbundle $E_2 \subseteq \mathcal{L} \subseteq E_3$

$\mathcal{L}$ coherent sub $\mathcal{O}_X$-module of $\mathcal{E}$

4 Moduli spaces

<u>Recall</u>: • If F is a coherent $\mathcal{O}_X$-module of rank r, we have the Hilbert polynomial

$$P_F(n) = \chi(F(n)) = n\, r \deg(X) + \chi(F)$$

$$F(n) = F \otimes \mathcal{O}_X(n)$$

• P a degree 1 polynomial $\qquad S/\mathbb{C}$ variety

E vector bundle $/X$

$\underline{\mathrm{Hilb}}^P(E, S) = \{ \mathcal{G}$ coherent $\mathcal{O}_{X\times S}$ - module

$\mathcal{G}$ is a quotient of $E|_{X\times S}$

$\mathcal{G}$ is flat over S

$\forall s \in S \quad P_{\mathcal{G}(s)} = P \}$

Grothendieck: It is representable by a projective variety $\mathrm{Hilb}^P(E)$.

Say that a family $\mathcal{V}$ of vector bundles over X is bounded if there exists a variety S and a vector bundle E on $S\times X$ such that $\mathcal{V}$ is contained in the set of vector bundles classes $E(s)$, $s \in S$.

Theorem: The set $S(r, d)$ of s.s. vector bundles over X of rank r and degree d is bounded, whereas the set of vector bundles $T(r, d)$

of rank $r \geq 2$ and degree d is <u>not</u>.

<u>Proof</u>: $T(n,d)$

<u>Fact</u>: If $\mathcal{V}$ is bounded, then

$$\{h^1(E) \mid E \in \mathcal{V}\} \text{ is finite.}$$

Fix $a \in X$.

$$\mathcal{E} = \underbrace{\mathcal{O}(-ka)}_{\deg k} \oplus \underbrace{\mathcal{O}((d-k)a)}_{\deg d-k} \oplus \underbrace{\mathcal{O}^{r-2}}_{\deg 0} \in T(n,d)$$

$$h^1(\mathcal{E}) \geq h^1(\mathcal{O}(-ka)) = k+g-1 \xrightarrow[k\to\infty]{} \infty$$

$S(r,d)$ $\quad \mu = \frac{d}{r}$

r integer $> 2g-1-\mu$

$a \in X$

For $\eta \geq r-1$, $\mu(E(\eta a)) = \mu(E) + \mu(\mathcal{O}(\eta a))$

$$= \mu + \eta > 2g-2 = \mu(\omega_X)$$

$\Rightarrow \mathrm{Hom}(E(\eta a), \omega_X) = 0$

Serre duality: $H^1(E(\eta a)) = 0$

$H^1(E(ra)) = 0$

$0 \to E((r-1)a) \to E(ra) \to E_a \to 0$

$0 \to H^0(E((r-1)a)) \to H^0(E(ra)) \to H^0(E_a) \to 0$

$\Rightarrow E(ra)$ is generated by its global sections.

H vector space of dim

$$N = \chi(E(ra)) = \dim H^0(E(ra))$$

By choosing an isomorphism, $H \simeq H^0(E(ra))$

then surjective map

$$H \otimes \mathcal{O}(-ra) \to E \quad (*)$$

P = Hilbert polynomial of any element of $S(r, d)$

$(*)$ corresponds to a point of $\mathrm{Hilb}^P(H \otimes \mathcal{O}(-ra))$.

$\Rightarrow$ bounded $\square$

$\Omega \subseteq \mathrm{Hilb}^P(H \otimes \mathcal{O}(-ra))$

the set of E such that E is semi-stable

and $H \simeq H^0(E(ra))$.

Corollary: Natural bijection

$$\Omega / SL(H) \xrightarrow{\sim} S(r, d)$$

Point: prove that Ω is the open subset given by Mumford's GIT. ✓

$$M(r, d) = \Omega /\!/ SL(H)$$

Theorem: $\underline{M}(r, d) : S \mapsto \left\{ \begin{array}{l} \text{isomorphism class} \\ \text{of vector bundles of} \\ \text{rank } r \text{ and deg } d \\ \text{over } X \times S \text{ such that} \\ F(s) \text{ s.s. deg } d, \\ \text{rk } r \quad \forall s \in S \end{array} \right\}$

There is a functorial morphism

$\varphi : \underline{M}(r, d)(S)$

$\to \mathrm{Mor}(S, M(r, d))$

such that, for all N alg. variety, $\forall$ funct. morphism $\psi : \underline{M}(r, d)(S) \to N(S)$.

$\exists!\, f : M(r,d) \to N$

$$\begin{tikzcd} \underline{M}(r,d)(S) \arrow[r] \arrow[dr] & M(r,d)(S) \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\ & N(S) \end{tikzcd}$$

$S(r,d) \to M(r,d)$

is surjective and the fibers consist of vector bundles which have the same associated bundle $\bigoplus gr_i$ for the Jordan-Hölder filtration.

There is an open subset of $M(r,d)$ which parametrizes isomorphism classes of stable vector bundles.