Grothendieck's Lefschetz-type theorems.

SAVA

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I Intro

k field $\quad X \subseteq \mathbb{P}^N_k$ smooth projective of pure dim. d

$t \in H^0(\mathbb{P}^N_k, \mathcal{O}(1))$ intersecting X properly:

$\{t=0\} = Y \subseteq X$ has dimension $d-1$.

We wish to compare Y and X.

Introduce $Y_n = \{t^{n+1} = 0\} \subseteq X$, $\quad \hat{X}_Y := \varinjlim_n Y_n$.

Three steps:

$$Y \underset{\text{deformation}}{\rightsquigarrow} \hat{X}_Y \underset{\text{algebraization}}{\rightsquigarrow} \begin{matrix}\text{some open neighbourhood} \\ Y \subseteq U \subseteq X\end{matrix} \underset{\text{extension}}{\rightsquigarrow} X.$$

II Main theorems.

Th 1 Let $E \to X$ be a vector bundle.

If $d \geq 2$, then $H^0(X, E) \xrightarrow{\sim} H^0(\hat{X}_Y, E|_{\hat{X}_Y})$

Th 2 Let $\mathcal{E} = (E_n) \longrightarrow \hat{X}_Y$ be a vector bundle $\left[\begin{array}{l} E_n \to Y_n \text{ vector bundle} \\ + \text{ isos } E_n|_{Y_{n-1}} \xrightarrow{\sim} E_{n-1} \end{array}\right.$

If $d \geq 3$, there exists $E \to X$ coherent such that $\mathcal{E} \simeq E|_{\hat{X}_Y}$.

Rk SGA2 contains : weaker hypotheses on singularities of X

variants over a base

local variants.

Rk Th 1 fails if $d = 1$.

If $X = \mathbb{P}^1_k$, $Y = \operatorname{Spec} k$, $E = \mathcal{O}_X$, then:

$H^0(X, E) = k$

$H^0(\hat{X}_Y, E) = H^0(\operatorname{Spf} k[[t]], \mathcal{O}) = k[[t]]$.

Rk Th 2 fails if $d = 2$

If $X = \mathbb{P}^2_k$, $Y = \mathbb{P}^1_k$, there are line bundles on $\hat{X}_Y$ that do not algebraize (to be explained later).

III. Comparison theorem ($d \geq 2$)

Proof of Th 1 : $0 \longrightarrow E(-n-1) \xrightarrow{t^{n+1}} E \longrightarrow E|_{Y_n} \longrightarrow 0$

$$\Downarrow$$

$$\begin{array}{ccccccc}
H^0(X, E(-n-1)) & \longrightarrow & H^0(X, E) & \longrightarrow & H^0(Y_n, E) & \longrightarrow & H^1(X, E(-n-1)) \\
\| & & & & & & \| \\
H^d(X, E^\vee \otimes K_X(n+1))^\vee & & & & & & H^{d-1}(X, E^\vee \otimes K_X(n+1))^\vee \\
\| \; n \gg 0 & & & & & & \| \; n \gg 0 \text{ because } d \geq 2 \\
0 & & & & & & 0
\end{array}$$

thus $H^0(X, E) \xrightarrow{\sim} \varprojlim_n H^0(Y_n, E) = H^0(\hat{X}_Y, E)$.

IV. Existence theorem ($d \geq 3$)

We rely on . Lemma : Let $\mathcal{E}$ be a vector bundle on $\hat{X}_Y$. Then :

(i) $H^i(\hat{X}_Y, \mathcal{E})$ is of finite dimension for $i = 0, 1$.

(ii) If $\ell \gg 0$, $H^0(\hat{X}_Y, \mathcal{E}(\ell)) \otimes \mathcal{O}_{\hat{X}_Y} \twoheadrightarrow \mathcal{E}(\ell)$.

Proof of Th 2. By the lemma, there is ℓ, r, and a surjection

$$\mathcal{O}_{\hat{X}_Y}(-\ell)^{\oplus r} \twoheadrightarrow \mathcal{E} \longrightarrow 0.$$

Applying this argument again to the kernel of $\twoheadrightarrow$, we get an exact sequence :

$$\mathcal{O}_{\hat{X}_Y}(-\ell')^{\oplus r'} \xrightarrow[\hat{\phi}]{} \mathcal{O}_{\hat{X}_Y}(-\ell)^{\oplus r} \longrightarrow \mathcal{E} \longrightarrow 0.$$

By Th 1, $\hat{\phi} \in H^0(\hat{X}_Y, \mathcal{H}om(\mathcal{O}(-\ell')^{\oplus r'}, \mathcal{O}(\ell)^{\oplus r}))$

lifts to $\phi \in H^0(X, \text{———————}) = \mathrm{Hom}(\mathcal{O}_X(-\ell')^{\oplus r'}, \mathcal{O}_X(-\ell)^{\oplus r})$

Define $E := \mathrm{Coker}(\phi)$.

<u>Proof of the Lemma:</u>

(i) Write:

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
 & 0 \arrow[d] & & \mathcal{O}_Y(-n) \arrow[d] \\
0 \arrow[r] & \mathcal{O}_X(-n-1) \arrow[r] \arrow[d, "t"] & \mathcal{O}_X \arrow[r] \arrow[d, equal] & \mathcal{O}_{Y_n} \arrow[d] \\
0 \arrow[r] & \mathcal{O}_X(-n) \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] & \mathcal{O}_{Y_{n-1}} \arrow[d] \\
 & \mathcal{O}_Y(-n) \arrow[d] & & 0 \\
 & 0 & &
\end{tikzcd}
\quad \otimes_{\mathcal{O}_{\hat{X}_Y}} \mathcal{E} + \text{coho}$$

$$0 \to H^0(Y, \mathcal{E}_0(-n)) \to H^0(Y_n, \mathcal{E}_n) \to H^0(Y_{n-1}, \mathcal{E}_{n-1})$$

$$H^0(Y, \mathcal{E}_0(-n)) = H^{d-1}(Y, \mathcal{E}_0^\vee \otimes \omega_Y(n))^\vee \overset{n \gg 0 \text{ since } d \geq 2}{=} 0$$

Thus, if $n_0 \gg 0$,

$$H^0(\hat{X}_Y, \mathcal{E}) = \varprojlim_n H^0(Y_n, \mathcal{E}_n) \hookrightarrow H^0(Y_{n_0}, \mathcal{E}_{n_0})$$

is finite-dimensional.

The same argument shows, since $d \geq 3$ that for $n_0 \gg 0$

$$H^1(\hat{X}_Y, \mathcal{E}) \underset{\uparrow}{=} \varprojlim_m H^1(Y_m, \mathcal{E}_m) \hookrightarrow H^1(Y_{n_0}, \mathcal{E}_{n_0})$$

[Mittag-Leffler] satisfied because $H^1(Y_m, \mathcal{E}_m)$ is finite-dimensional $\forall m$.

is finite-dimensional

(ii) $0 \to \mathcal{O}_X(-1) \xrightarrow{t} \mathcal{O}_X \longrightarrow \mathcal{O}_Y \to 0$

$\quad \wr \otimes \mathcal{O}_{\hat{X}_Y} \otimes \mathcal{E}$

$0 \to \mathcal{E}(-1) \xrightarrow{t} \mathcal{E} \to \mathcal{E}_0 \to 0$

$\quad \wr \otimes \mathcal{O}(\ell) \quad$ + coho

$$H^0(\hat{X}_Y, \mathcal{E}(\ell)) \to H^0(Y, \mathcal{E}_0(\ell)) \to H^1(\hat{X}_Y, \mathcal{E}(\ell-1)) \xrightarrow{t} H^1(\hat{X}_Y, \mathcal{E}(\ell)) \to H^1(Y, \mathcal{E}_0(\ell))$$

If $\ell \gg 0$, $H^1(Y, \mathcal{E}_0(\ell)) = 0$, hence $H^1(\hat{X}_Y, \mathcal{E}(\ell-1)) \xrightarrow{t} \!\!\!\!\to H^1(\hat{X}_Y, \mathcal{E}(\ell))$.

Since $H^1(\hat{X}_Y, \mathcal{E}(\ell))$ has finite dim by (i), this shows that its dimension decreases with $\ell\ (\gg 0)$, hence stabilizes, hence that for $\ell \gg 0$,

$$H^1(\hat{X}_Y, \mathcal{E}(\ell-1)) \xrightarrow[\sim]{t} H^1(\hat{X}_Y, \mathcal{E}(\ell)).$$

Consequently $H^0(\hat{X}_Y, \mathcal{E}(\ell)) \longrightarrow\!\!\!\!\to H^0(Y, \mathcal{E}_0(\ell))$ for $\ell \gg 0$.

If moreover ℓ has been chosen big enough so that $\mathcal{E}_0(\ell)$ is globally generated,

$$H^0(\hat{X}_Y, \mathcal{E}(\ell)) \otimes \mathcal{O}_{\hat{X}_Y} \longrightarrow\!\!\!\!\to \mathcal{E}_0(\ell).$$

By Nakayama, it follows that:

$$H^0(\hat{X}_Y, \mathcal{E}(\ell)) \otimes \mathcal{O}_{\hat{X}_Y} \longrightarrow\!\!\!\!\to \mathcal{E}(\ell),$$

as wanted.

V) Application to π_1

Th: If $d \geq 3$, then $\pi_1(Y) \xrightarrow{\sim} \pi_1(X)$.

Proof: We need to show that $\mathrm{Et}(X) \xrightarrow[\sim]{\text{équivalence}} \mathrm{Et}(Y)$

Look at $\mathrm{Et}(X) \xrightarrow{(a)} \varinjlim_{\substack{Y \subseteq U \subseteq X \\ \text{open}}} \mathrm{Et}(U) \xrightarrow{(b)} \mathrm{Et}(\hat{X}_Y) \xrightarrow{(c)} \mathrm{Et}(Y)$

(a) is an equivalence because a finite étale cover of U extends uniquely to a normal finite cover of X.

By Zariski–Nagata, the locus where it is not étale is a divisor, that avoids Y. But Y is ample, so that the extension is étale.

(b) Th 1 + Th 2 in fact show that

$$\varinjlim_{\substack{Y \subseteq U \subseteq X \\ \text{open}}} \text{VectorBundles}(U) \xrightarrow[\sim]{\text{equivalence}} \text{VectorBundles}(\hat{X}_Y)$$

But a finite étale cover is the data of a vector bundle E + $\mathcal{O} \to E$
+ $E \otimes E \to E$
+ axioms

This implies formally that (b) is an equivalence.

(c) is an equivalence because finite étale covers do not see nilpotent thickenings:

$$\mathrm{Et}(Y_{n+1}) \xrightarrow{\sim} \mathrm{Et}(Y_n).$$

Rk: Same proof shows that if $d \geq 2$, $\pi_1(Y) \twoheadrightarrow \pi_1(X)$.

VI Application to Pic

Th: If $d \geq 3$ and $H^i(Y, \mathcal{O}(-n)) = 0$ for $n \geq 1$, $i = 1, 2$,

then $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(Y)$.

ex: $X = \mathbb{P}^d$ $d \geq 4$, $Y \subseteq X$ hypersurface. then $\mathrm{Pic}(Y) \simeq \mathbb{Z}.\mathcal{O}(1)$.

Proof: look at $\mathrm{Pic}(X) \xrightarrow{(a)} \varinjlim_{\substack{Y \subseteq U \subseteq X \\ \text{open}}} \mathrm{Pic}(U) \xrightarrow{(b)} \mathrm{Pic}(\hat{X}_Y) \xrightarrow{(c)} \mathrm{Pic}(Y)$

(a) is an ~~equivalence~~ isomorphism because a line bundle on U extends to a line bundle on X by regularity, uniquely because $\mathrm{codim}_X(X \setminus U) > 1$.

(b) is an isomorphism by Th 1 + Th 2

(c)
$$
\begin{array}{ccccccc}
0 \to & \mathcal{O}(-Y_{n-1})|_{Y_n} & \longrightarrow & \mathcal{O}^*_{Y_n} & \longrightarrow & \mathcal{O}^*_{Y_{n-1}} & \to 0 \\
& \wr\wr & & & & & \\
& \mathcal{O}_Y(-n) \ni f & \longmapsto & 1+f & & &
\end{array}
$$

$$\Downarrow$$

$$
\begin{array}{ccccccc}
H^1(Y, \mathcal{O}(-n)) \to & H^1(Y_n, \mathcal{O}^*_{Y_n}) & \to & H^1(Y_{n-1}, \mathcal{O}^*_{Y_{n-1}}) & \to & H^2(Y, \mathcal{O}(-n)) \\
\| & \| & & \| & & \| \\
0 & \mathrm{Pic}(Y_n) & \xrightarrow{\sim} & \mathrm{Pic}(Y_{n-1}) & & 0
\end{array}
$$

thus $\mathrm{Pic}(Y) \xleftarrow{\sim} \varprojlim_n \mathrm{Pic}(Y_n) = \mathrm{Pic}(\hat{X}_Y)$.

Rk: This last computation also shows that if $Y = \mathbb{P}^1 \underset{\text{line}}{\hookrightarrow} \mathbb{P}^2 = X$,

$\mathrm{Pic}(\hat{X}_Y)$ is huge, although $\underset{\mathbb{Z}.\mathcal{O}(1)}{\mathrm{Pic}(X)} \simeq \varinjlim_{Y \subseteq U \subseteq X} \mathrm{Pic}(U)$.