A. SMEETS. –

Algebraic foliations

1. Foliations in diff. geometry
   - 1.1. Foliated manifolds and their leaves
   - 1.2. Frobenius integrability theorem
2. Foliations in algebraic geometry
   - 2.1. Algebraic foliations & formal germs of leaves
   - 2.2. Algebraicity and integrability

M $\mathcal{C}^\infty$ manifold, dim n

topological manifold + $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n$

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta) \ \underline{\text{smooth}}$$

Foliated diffeomorphism

$U, U' \subseteq \mathbb{R}^n$ $\qquad$ $p : \mathbb{R}^n \to \mathbb{R}^d$

$d \leq n \quad \mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$ $\qquad$ $q : \mathbb{R}^n \to \mathbb{R}^{n-d}$

$\underline{\Psi} : U \to U'$

The following are equivalent:

- $D\underline{\Psi}$ takes values in the subspace of $M_n(\mathbb{R})$

consisting of $\left(\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right)$.

- $\forall p \in \mathcal{U}$ $\exists$ opens $I \subseteq \mathbb{R}^d$ containing $p(p)$
$J \subseteq \mathbb{R}^{n-d}$ $q(p)$

such that $I \times J \subseteq \mathcal{U}$, and $\exists$ $\mathcal{C}^\infty$ mappings

$\Psi_1 : I \times J \longrightarrow \mathbb{R}^d$

$\Psi_2 : J \longrightarrow \mathbb{R}^{n-d}$

such that $\Psi(x, y) = (\Psi_1(x, y), \Psi_2(y))$

- given $y' \in \mathbb{R}^{n-d}$ the connected component of $(q \circ \Psi)^{-1}(y')$ of dim d in $\mathcal{U}$ are of the form $\underset{\mathbb{R}^d}{\underset{\cap}{\Omega}} \times \{y\}$.

We say that Ψ is <u>foliated</u> of dimension (d, n).

$\mathcal{C}^\infty$ manifold M: a foliated atlas on M is an atlas for which the transition maps
$\varphi_\beta \circ \varphi_\alpha^{-1}$ are foliated of dimension (d, n).

A ~~foliated~~ foliation $\mathcal{F}$ on M is an equivalence class of foliated atlases of dimension (d, n).

A foliated chart for $(M, \mathcal{F})$ is one occurring in some foliated atlas defining $(M, \mathcal{F})$.

<u>Leaves</u>

$(M, \mathcal{F})$ as before, foliated of dim (d, n).

Cook up $M^{\mathcal{F}}$ as follows:

equip M with the unique topology finer than the one on M such that given any foliated chart $\varphi : (U \subseteq M) \longrightarrow (V \subseteq \mathbb{R}^n)$

$$\mathbb{R}^n = \underset{\text{standard topology}}{\mathbb{R}^d} \times \underset{\text{discrete topology}}{\mathbb{R}^{n-d}}$$

φ becomes a homeomorphism

$M^{\mathcal{F}}$ a d-dimensional $\mathcal{C}^\infty$-manifold

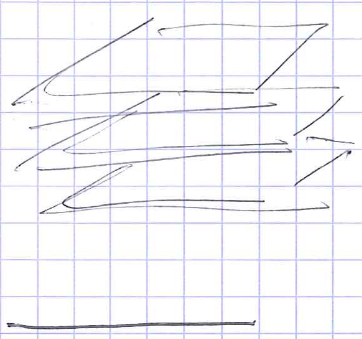
The connected components of $M^{\mathcal{F}}$ are the <u>leaves</u> of $(M, \mathcal{F})$.

Given a leaf L, there is $L \hookrightarrow M$ "injective immersion" but the image may <u>not</u> be closed, can be dense

<u>Example</u>: submersions

dim n M $M \to N$ submersion

dim $n-d$ N The connected components of the fibres give a foliation with d-dimensional leaves



$\mathbb{R}^{n-d}$

$$\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$$

G a Lie group

H closed Lie subgroup $H \to G \to G/H$

<u>Frobenius integrability theorem</u>

<u>Def</u>. M C^∞ manifold, tangent bundle TM

An <u>involutive distribution</u> of rank d on M is a subbundle $F \subseteq TM$ of rank d closed under Lie bracket, i.e. $[F, F] \subseteq F$

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

A chart for M adapted to F is a chart $\mathcal{U} \xrightarrow{\varphi} V$ such that, for all $x \in \mathcal{U}$,

$$\begin{aligned} D\varphi(x) : T_x M &\longrightarrow T_x \mathbb{R}^n = \mathbb{R}^n \\ F_x &\longmapsto \mathbb{R}^d \times \{0\} \end{aligned}$$

Theorem: With F, M as above, $\forall p \in M$ there is a chart $\varphi : \underset{p}{\overset{\forall}{\mathcal{U}}} \subset M \longrightarrow V \subseteq \mathbb{R}^n$ adapted to F.

Given $(M, \mathcal{F})$, get $F \subset TM$ as follows:

$\forall p \in M$, F_p = tangent space to $\mathcal{F}_p$

Conversely, given an involutive distribution F of rank d, the atlas consisting of charts adapted to F defines a foliation of dimension (d, n) on M.

Tangent bundle $T_{\mathcal{F}}$ = subbundle of TM defined via this correspondence.

Sketch of proof: $p \in M$

$$\varphi : (\mathcal{U}_0 \ni p) \longrightarrow \mathbb{R}^n$$
$$p \longmapsto 0$$
$$(x_1, \dots x_n)$$

Permutting the x_i, shrinking $\mathcal{U}_0$ if necessary, may assume that $\forall i \leq d$, there is a unique regular section v^i of $F_{|\mathcal{U}_0}$ of the form

$$v^i = \frac{\partial}{\partial x_i} + \sum_{k=d+1}^{n} a_k^i \frac{\partial}{\partial x_k} \qquad a_k^i \in \mathcal{C}^\infty(\mathcal{U}_0, \mathbb{R})$$

$[v^i, v^j] = 0 \quad (\Leftarrow \text{involutivity})$

So d commuting vector fields $v^1, v^2, \dots v^d$

flow: $\Psi : V \longrightarrow \mathcal{U}_0$

V is an open neighborhood of $\{0\} \times \mathcal{U}_0$ in $\mathbb{R}^d \times \mathcal{U}_0$

$\exists W_0 \subseteq \mathbb{R}^n$ open neighborhood of 0 such that

$$\Psi\left((t_1, \dots t_d), \varphi^{-1}(0 \dots 0, t_{d+1} \dots t_n)\right)$$

defines a map $\Psi : W_0 \to U_0$ maps $0 \mapsto p$, étale at 0. Restricting & inverting gives a chart adapted to F $\square$

Algebraic setting

k field

X/k smooth quasi-projective k-variety

F sub vector bundle of TX.

Def. If F is stable under Lie bracket, (X, F) is an algebraic foliation.

If F is a saturated coherent subsheaf of TX stable under Lie bracket, (X, F) is an algebraic foliation with singularities

$\exists\, U \subset X$, $X \setminus U$ codim 2 such that (U, F_U) is an algebraic foliation.

Conversely, given (X, F) alg. foliation

$X \hookrightarrow \bar{X}$ smooth $(\bar{X}, \bar{F})$ .... with singularities

$k = \mathbb{C}$ $\qquad$ (X, F) algebraic foliation

char $k = 0$

$\Rightarrow (X(\mathbb{C}), F^{a})$ complex analytic foliation

(X, F) $\qquad$ $P \in X(k)$

Formal germ of leaf through P

$x_1, \dots x_n$ on $\mathcal{U} \ni P$ induces an étale morphism to $\mathbb{A}^n_k$ $(P \mapsto 0)$

T_X free on $\mathcal{U}$, with basis $\left(\frac{\partial}{\partial x_i}\right)_{i=1,\dots n}$ $\qquad$ $\forall i \leq 1$

$\exists!$ v^i regular section of T_X over $\mathcal{U}$ of the form

$[v^i, v^j] = 0$ (involutivity)

"Formal flow" $\qquad \hat{\mathbb{A}}^d_k \times \hat{X}_P \to \hat{X}_P$

Formal leaf of F through P = image of

$\hat{V}$

$\Psi(-, P) : \hat{\mathbb{A}}^d_k \longrightarrow \hat{X}_P$

formal subscheme of $\hat{X}_\varphi$

Algebraicity of leaves & integrability

$(X, \mathcal{F})/_k$ foliation of dim d

$\Sigma \hookrightarrow X$ is invariant under $\mathcal{F}$ if the ideal sheaf I_Σ is closed under the action of $\mathcal{F}$.

If $\Sigma \hookrightarrow X$, then $\overline{\Sigma}$ is invariant under $\mathcal{F}$.

Definition. An algebraic leaf of $\mathcal{F}$ through P is an integral closed subscheme of X containing P, of dimension d, invariant under $\mathcal{F}$.

A smooth family of algebraic leaves of $(X, \mathcal{F})$ parametrized by B (a k-variety)

$$\begin{tikzcd}
V \arrow[r, hook] \arrow[dr, "\text{smooth}"'] & B \times_k X \arrow[d, "\pi"'] \arrow[dr] & \\
& B & X
\end{tikzcd}$$

$T_{V/B} = \pi^* \mathcal{F}$ as subbundles of $\pi^* T_{X/k}$.

Spec K = generic point of B

$V_K \hookrightarrow X_K$, $V \hookrightarrow B \times_K X$

if V_K algebraic leaf of (X_K, F_K) $\exists\, \mathcal{U} \subset B$ open

such that $V_{\mathcal{U}} \to \mathcal{U}$ smooth family of algebraic leaves over $\mathcal{U}$.

Proposition / Definition: $\forall\, P \in X(k)$ TFAE

① the formal leaf $\hat{\mathcal{F}}_P$ of (X, F) through P is algebraic

② $\exists\, V \hookrightarrow X$ smooth integral subscheme of dim d, $P \in V$ invariant under F

$\hat{\mathcal{F}}_P^{\,\mathrm{Zar}}$ is called the algebraic leaf of F through P.

Definition. X connected, smooth quasi-proj.

$(X, \mathcal{F})$ is algebraically integrable if the leaf of F through η_X is algebraic.

Remark: Then $\forall P \in X(K)$, K/k the leaf of (X, F) through P is algebraic.

Theorem. (Gómez-Mont) (X, F) is algebraically integrable $\iff$ on $U \hookrightarrow X$ dense open $(U, F_{|U})$ is defined by a submersion, i.e. $\exists\, B$ smooth quasi-proj. of dimension $n-d$, and $U \xrightarrow[\substack{\text{smooth}\\\text{surjective}}]{\pi} B$ such that

$$F_{|U} = \ker\left(T_{U/k} \to \pi^* T_{B/k}\right).$$

Example: G/k algebraic group

$\mathfrak{g} = \mathrm{Lie}\, G$

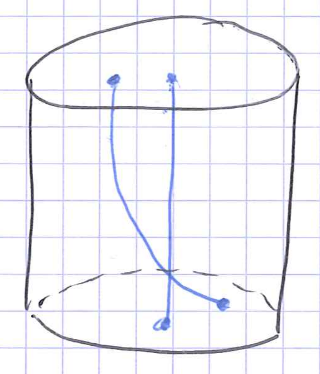
$\mathfrak{h} = k$-Lie subalgebra of $\mathfrak{g}$

$F =$ left-G-invariant subbundle of $T_{G/k}$ such that $F_e = \mathfrak{h}$.

(G, F) algebraic foliation

This is algebraically integrable $\Leftrightarrow$ $\mathfrak{h} = \mathrm{Lie}\, H$

for some sub-algebraic group $H \subset G$.



+ glue

Open question: 3 dim smooth proj. variety $/\overline{\mathbb{Q}}$

one dim sub vector bundle of TX