

SUGGESTED PROGRAM FOR JAVA 2026

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The outlines below of the content of the lectures are uneven in their level of detail. Some are brief, and would just ask to cover a certain section. Others are more detailed, when the content is to diverge from the reference suggested in order to simplify it, or in case the translation from the material in the text suggested to the content of the lecture is less obvious. So a longer description does not mean more material to be covered. When I suggest to state a result and omit the proof it means that the time plan for the lecture does not include time for the proof. If you feel that you do have time, by all means, include any proof you wish.

Notation: Given a free abelian group V and a field K , set $V_K := V \otimes_{\mathbb{Z}} K$. Similarly, if V is a vector space over \mathbb{Q} and K is a field of characteristic 0 we set $V_K := V \otimes_{\mathbb{Q}} K$. Given an abelian variety A we set $\text{End}_{\mathbb{Q}}(A) := \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ and refer to it as the ring of rational endomorphisms of A .

1. THE HODGE CONJECTURE FOR ABELIAN VARIETIES

The main reference is Sections 1 to 4 of [vG1] (for complex multiplication by quadratic imaginary number fields), which is easier to read than the original paper of Weil [W]. A generalization for CM-fields is discussed in [DM].

- Survey the first 4 sections of [vG1] ending with the overview of results (Weil's Theorem, The reduction of the Hodge conjecture for abelian 4-folds to abelian varieties of Weil type due to Moonen-Zarhin, Scheon's Theorem). This should be the content of most of the lecture.
- State the main result of [M2] that the Weil classes on abelian sixfolds of split Weil type are algebraic.
- State Tankeev theorem that the Hodge ring on simple abelian varieties of prime dimension is generated by divisor classes.

- State the theorem of Moonen-Zarhin [MZ3, Th. 0.2] about the generation of the Hodge ring of non-simple abelian 5-folds by divisor classes and pull back of Weil classes from quotient abelian 4-folds. Conclude that the Hodge conjecture holds for abelian varieties of dimension ≤ 5 .

2. ABELIAN VARIETIES OF WEIL TYPE

This lecture consists of a detailed exposition of Section 5 of [vG1].

- State and prove Lemma 5.2 on the cohomology of abelian varieties of Weil type. The polarization, the hermitian form, its discriminant, and the definition of the two dimensional space of Hodge Weil classes $HW(A, \eta)$ on an abelian variety of Weil type $(A, \eta : K \rightarrow \text{End}_{\mathbb{Q}}(A))$.
- Describe the construction of the n^2 -dimensional family of polarized abelian varieties of Weil type (H_n in the notation of [vG1]), the transitive $SU(n, n)$ action, and the complex structure.
- Go over [vG1, Example 5.12].
- Emphasize that up to isogeny the three discrete invariants $\dim(A)$, K , and

$$\det(A, \eta, h) \in \mathbb{Q}^\times / \text{Nm}(K^\times)$$

determine the connected component of moduli up to isogenies, by the main result of reference [L] in [vG1]. In other words, given a polarized abelian variety of Weil type (A, η, h) and any connected component of moduli with the same values of these three discrete invariants, the component contains the isomorphism class of a polarized abelian variety of Weil type (A', η', h') , such that there exists an isogeny between A and A' inducing a K -linear isomorphism of $H_1(A, \mathbb{Q})$ and $H_1(A', \mathbb{Q})$ compatible with the hermitian forms associated to (η, h) and (η', h') .

- Conclude that it suffices to prove the algebraicity of the Weil classes for all polarized abelian varieties of Weil type in one component, for each value of these three discrete invariants.

3. MUMFORD-TATE GROUPS

Present Section 6 of [vG1]. See also the survey paper of Griffiths [Gr]. The main results are [vG1, Th. 6.11] that the Special Mumford Tate group of the general polarized abelian variety of Weil type is SU_H , and [vG1, Th. 6.12] about the Hodge ring of the general polarized abelian variety of Weil type.

4. SCHOEN'S PROOF OF THE ALGEBRAICITY OF THE WEIL CLASSES ON ABEIAN FOURFOLDS OF WEIL TYPE, CASE OF $\mathbb{Q}(\sqrt{-3})$

C. Schoen proved the algebraicity of the Weil classes on abelian fourfolds of split Weil type with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ in [S1]. He claimed in this paper to have proven it for all abelian fourfolds of Weil type, but corrected this assertion and proved it in [S2] for abelian sixfolds of split Weil type with complex multiplication by $\mathbb{Q}(\sqrt{-3})$, thereby proving it for all abelian fourfolds. Similar results were obtained for $\mathbb{Q}(\sqrt{-1})$ in [K]. Present the sketch of the proof of the algebraicity of the Weil classes on abelian fourfolds of split Weil type with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-1})$ from [vG1, Sec. 7]. I suggest that you concentrate on the case $\mathbb{Q}(\sqrt{-3})$ and expand the presentation by referring to [S1].

5. SEMIREGULARITY AND DEFORMATIONS

The most important reference is [BF1]. We need however just the statement of their semiregularity theorem. Most of the effort will be to generalize it to twisted sheaves [M2, Sec. 7] (a generalization was discussed also in [Pr] and [HP]).

- A discussion of Chern classes, Chern characters, and some basic properties.
- Define the *Atiyah class* at_E of a coherent sheaf E over an N -dimensional smooth projective variety (or Kähler manifold) M using the universal Atiyah class (see [HL, Sec. 10.1] and [M2, (7.3.2)]).
- Relate the Chern character to the trace of the exponential Atiyah class [HL, Sec. 10.1].

- Define the *semiregularity map*

$$\sigma_E : \text{Ext}^2(E, E) \rightarrow \bigoplus_{q=0}^{N-2} H^{q+2}(M, \Omega_M^q)$$

and the commutative diagram of [BF1, Cor. 4.3] (displayed in [M2, Diagram (7.1.2)]).

- Explain that $at_E : H^1(M, TM) \rightarrow \text{Ext}^2(E, E)$ is the obstruction homomorphism to lifting classes in $H^1(TM)$ to infinitesimal deformations of the pair (M, E) (see [T]). For simplicity, you may explain the latter only for a locally free sheaf E , in which case the first order deformations of (M, E) are parametrized by $H^1(\text{Diff}^1(E)_s)$, where $\text{Diff}^1(E)_s$ is the sheaf of first order differential operators with scalar symbol. The latter fits in the short exact sequence

$$(1) \quad 0 \rightarrow \mathcal{E}nd(E) \rightarrow \text{Diff}^1(E)_s \rightarrow TM \rightarrow 0$$

and contraction with at_E is the connecting homomorphism from $H^1(TM)$ to $H^2(\mathcal{E}nd(E)) \cong \text{Ext}^2(E, E)$. Note that the coherent sheaf of first order differential operators $\text{Diff}^1(E)$ is isomorphic to $\mathcal{H}om(j^1(E), E)$, where $j^1(E)$ is the first jet bundle¹ of E . Applying $R\mathcal{H}om(\bullet, E)$ to the short exact sequence

$$0 \rightarrow E \otimes \Omega_M^1 \rightarrow j^1(E) \rightarrow E \rightarrow 0$$

we get the short exact

$$0 \rightarrow \mathcal{E}nd(E) \rightarrow \text{Diff}^1(E) \rightarrow \mathcal{E}nd(E) \otimes TM \rightarrow 0.$$

The short exact sequence (1) is the pullback of the one above via the homomorphism $TM \rightarrow \mathcal{E}nd(E) \otimes TM$ of tensoring with the identity endomorphism of E . A local splitting of the short exact sequence (1) is given by a local holomorphic connection. The above can be generalized to any coherent sheaf E by replacing E by a complex of locally free sheaves. Local holomorphic connections of such complexes are discussed in [HL, Sec. 10.1].

- Interpret the injectivity of the semiregularity map σ_E as equality of the kernels of two arrows in Diagram [M2, (7.1.2)]. Then state it as the equivalence of the vanishing of two obstructions, where the vanishing of $\xi \lrcorner \text{ch}(E)$, for $\xi \in H^1(TM)$, is the condition for $\text{ch}(E)$ to remain of Hodge type along ξ , by Griffiths Transversality [Vo3, Theorem 10.21].

6. THE SEMIREGULARITY THEOREM AND ITS GENERALIZATION TO TWISTED SHEAVES

- State the Buchweitz-Flenner Semiregularity Theorem [BF1, Th. 5.1].
- The Semiregularity Theorem enables us to deform a coherent sheaf E flatly to every fiber over an open analytic neighborhood U of a point in the base B of the family. Hence, $\text{ch}(E)$ remains algebraic in the fibers over points of U (assuming now that the fibers are projective). Point out that if the base B of the family is smooth and connected, then $\text{ch}(E)$ remains in fact algebraic in every fiber (over the whole base), since the locus in the base, where a class remains algebraic, is a countable union² of closed complex analytic subvarieties of B (see [Vo2, Sec. 4.2]).
- (1) Prove that every simple sheaf on a K3 or abelian surface is semiregular.
- (2) Use the main result of [EGS], that the minimal class $\Theta^3/3!$ on the generic principally polarized abelian fourfold (A, Θ) is not algebraic, to conclude that the ideal sheaf $\mathcal{I}_{AJ(C)}$ of the Abel-Jacobi image of a curve C of genus 4 in its Jacobian $\text{Pic}^1(C)$ is not semiregular.
- (3) Point out that the codomain of the semiregularity map σ_E has a fixed dimension $D := \sum_{q=0}^{N-2} h^{q,q+2}(M)$, so as soon as $\dim(\text{Ext}^2(E, E)) > D$, the sheaf E can not be semiregular in the strict sense of [BF1].
- Let E be a rank $r > 0$ coherent sheaf. Explain that the condition that $\text{ch}(E)$ remains a Hodge class over the base (which appears in the Semiregularity Theorem) is equivalent to the conjunction of the following two independent conditions.
 - (1) $c_1(E)$ remains of Hodge type, and

¹Let \mathcal{I}_Δ be the ideal sheaf of the diagonal $\Delta \subset M \times M$. The first order neighborhood Δ_1 of the diagonal is the scheme with structure sheaf $\mathcal{O}_{\Delta_1} := \mathcal{O}_M \times M/\mathcal{I}_\Delta^2$. The first jet bundle is $j^1(E) := \pi_{2,*}(\pi_1^*(E))$, where $\pi_i : \Delta_1 \rightarrow M$, $i = 1, 2$, is the restriction to Δ_1 of the projection $M \times M \rightarrow M$ onto the i -th factor.

²Under the additional assumption that B is quasiprojective, that locus is in fact a closed analytic subset [Vo1, Sec. 17.3.1] and even algebraic, by a theorem of Cattani-Deline-Kaplan.

(2) $\kappa(E) := \text{ch}(E) \exp(-c_1(E)/r)$ remains of Hodge type over the base.

We would like to strengthen the theorem, by dropping the condition on $c_1(E)$ and deforming instead a related, possibly twisted, coherent sheaf with vanishing first Chern class. The simplest case is the following.

Example 6.1. Let E be a rank r coherent sheaf with the property that $c_1(E)/r \in H^{1,1}(M, \mathbb{Q})$ is in fact integral, so that $\det(E) \cong L^r$, for some line bundle r and $E \otimes L^{-1}$ has trivial determinant. In this case $\kappa(E) = \text{ch}(E \otimes L^{-1})$, E is semiregular if and only if $E \otimes L^{-1}$ is, and if in addition $\kappa(E)$ remains of Hodge type, then the semiregularity theorem applies to $E \otimes L^{-1}$.

- Define twisted coherent sheaves. The elementary language of [M2] follows [Ca1] avoiding the language of stacks.
- Present [M2, Construction 7.3.3] of the twisted sheaf with trivial determinant line-bundle associated to a coherent sheaf of positive rank.
- Define the Atiyah class of a twisted sheaf and its Chern character ([M2, Def. 7.3.5, 7.3.6], defined in other texts earlier, e.g., in [Li]). Relate the Chern character of the twisted sheaf in [M2, Construction 7.3.3] to that of the original untwisted coherent sheaf [M2, Lemma 7.3.7].
- State [M2, Conjecture 7.3.9]. If time permits, sketch the proof of the Conjecture in the case of abelian varieties found in [M2, Sec. 7.4]. (I can provide notes for a simplified proof of a slightly less general statement, suggested by [M2, Footnote 19], which suffices for our application).

7. THE CLIFFORD ALGEBRA AND THE SPIN GROUP ASSOCIATED TO AN ABELIAN VARIETY

The best reference on Clifford algebras and spin groups for our purpose is [Ch]. The paper [GLO] includes an excellent summary of the basic facts we need, but it does not contain the material on pure spinors we need later. Let X be an n -dimensional abelian variety and set $\hat{X} := \text{Pic}^0(X)$. For motivation, mention that the semiregular sheaf we need will be constructed over $X \times \hat{X}$, where X is an abelian surface [M1] or the Jacobian of a curve of genus 3 [M2], and that we hope to find examples for all higher dimensional abelian varieties. Point out that $V := H^1(X, \mathbb{Z}) \oplus H^1(\hat{X}, \mathbb{Z})$ has a natural symmetric even unimodular bilinear pairing $(\bullet, \bullet)_V$ given in [M2, (1.2.2)].

- Define the Clifford algebra $C(V)$ over \mathbb{Z} as in³ [GLO]. Define the Clifford group $G(V)$ and its subgroup $\text{Spin}(V)$. Define the norm homomorphism and its kernel $\text{SO}_+(V_\bullet)$, for $\bullet = \mathbb{Z}$ or a field, where $V_\bullet := V \otimes_{\mathbb{Z}} \bullet$. Point out that $\text{SO}_+(V_\bullet)$ is the image of $\text{Spin}(V_\bullet)$ in $\text{SO}(V_\bullet)$ (see⁴ [Ch, II.2.3] in the field case, and over \mathbb{Z} , by [M1, Cor. 5.1], which exhibits generators for $\text{SO}_+(V)$ which clearly belong to the image of the homomorphism ρ in Eq. (2) below). Set $S := H^*(X, \mathbb{Z})$. Define the isomorphism

$$m : C(V) \rightarrow \text{End}(S)$$

and the two half spin representations $S^+ := H^{\text{ev}}(X, \mathbb{Z})$ and $S^- := H^{\text{odd}}(X, \mathbb{Z})$ of $\text{Spin}(V)$. Use the notation

$$(2) \quad \begin{aligned} m : \text{Spin}(V) &\rightarrow \mathcal{B}(S), \\ \rho : \text{Spin}(V) &\rightarrow \text{SO}^+(V) \end{aligned}$$

for the spin and vector representations.

- For motivation (to be expanded in lecture 10) point out the following. Let $\text{Aut}(D^b(X))$ be the group of isomorphism classes of autoequivalences of the derived category of coherent sheaves on a projective variety X . The group $\text{Aut}(D^b(X))$ acts on $H^*(X, \mathbb{Q})$, not preserving the grading but preserving the Mukai pairing ([M2, (1.2.3)] in the case of abelian varieties, [Ca2] in general), and it acts by conjugation on its identity component. When X is an abelian variety, $\text{Aut}(D^b(X))$ preserves the integral structure $H^*(X, \mathbb{Z})$ and the identity component of $\text{Aut}(D^b(X))$ is $X \times \hat{X}$, where $\hat{X} := \text{Pic}^0(X)$. Note that $H^*(X \times \hat{X}, \mathbb{Z})$ is the exterior algebra $\wedge^* V$ of the vector representation V . Orlov proved that the action of $\text{Aut}(D^b(X))$ on $H^*(X, \mathbb{Z})$ and on $H^*(X \times \hat{X}, \mathbb{Z})$ is given by the composition of the isomorphism m and the action of $\text{Aut}(D^b(X))$ on $H^*(X \times \hat{X}, \mathbb{Z})$.

³One needs to use the quadratic form $Q(\bullet) = \frac{1}{2}(\bullet, \bullet)_V$ and not the bilinear pairing, as was unfortunately done in early versions of [M2, Sec. 2.1], in order to avoid two-torsion in $C(V)$.

⁴In [Ch] $\text{SO}(V)$ is denoted by G^+ and $\text{SO}_+(V)$ by G_0^+ .

\hat{X}, \mathbb{Z}) both factor through a homomorphism $\text{Aut}(D^b(X)) \rightarrow \text{Spin}(V)$ via the spin and vector representations of the latter. Orlov proved that one gets the short exact sequence

$$(3) \quad 0 \rightarrow X \times \hat{X} \times \mathbb{Z} \rightarrow \text{Aut}(D^b(X)) \rightarrow \text{Spin}_{\text{Hdg}}(V) \rightarrow 0,$$

where $\text{Spin}_{\text{Hdg}}(V)$ is the subgroup of $\text{Spin}(V)$ preserving the Hodge structure of V and the factor \mathbb{Z} corresponds to even shifts (the shift [1] maps to -1 in $\text{Spin}(V)$). This is the way Orlov's results are presented in [GLO, Prop. 4.3.7 and Cor. 4.3.9]. This explains the central role spin groups play in the geometry of abelian varieties.

- The way we relate the vector and spin representations of $\text{Spin}(V_{\mathbb{C}})$ is via Chevalley's theory of pure spinors [Ch]. Let $\text{IGr}(2n, V_{\mathbb{C}})$ be the grassmannian of maximal isotropic subspaces of $V_{\mathbb{C}}$. It has two connected components (when X is an elliptic curve, $\text{IGr}(2, V_{\mathbb{C}}) = \mathbb{P}^1 \coprod \mathbb{P}^1$ corresponding to the two rulings of the quadric surface in $\mathbb{P}(V_{\mathbb{C}})$ cut out by the quadratic form of $V_{\mathbb{C}}$ associated to the symmetric bilinear pairing). So $\text{IGr}(2n, V_{\mathbb{C}}) = \text{IGr}_+(2n, V_{\mathbb{C}}) \coprod \text{IGr}_-(2n, V_{\mathbb{C}})$ in general. Describe the $\text{Spin}(V)$ -equivariant embedding

$$\ell : \text{IGr}(2, V_{\mathbb{C}}) \rightarrow \mathbb{P}(S_{\mathbb{C}})$$

mapping $\text{IGr}_+(2n, V_{\mathbb{C}})$ into $\mathbb{P}(S_{\mathbb{C}}^+)$ and $\text{IGr}_-(2n, V_{\mathbb{C}})$ into $\mathbb{P}(S_{\mathbb{C}}^-)$ (see [Ch, III.1.5]. The variety $\ell(\text{IGr}_+(2n, V_{\mathbb{C}}))$ is called the *even spinorial variety*. Include the formula for the line ℓ_W of *pure spinors* in $S_{\mathbb{C}}$ corresponding to a maximal isotropic subspace $W \subset V_{\mathbb{C}}$

$$\ell_W = \{\lambda \in S_{\mathbb{C}} : w \wedge \lambda + \theta \rfloor \lambda = 0, \forall (w, \theta) \in W\}.$$

Note that this formula also determine the maximal isotropic subspace W associated to a pure spinor $\lambda \in S_{\mathbb{C}}$. Check that $1 \in H^0(X)$ and $[pt] \in H^{2n}(X)$ are pure spinors and $\ell_{H^1(\hat{X})} = H^0(X)$ and $\ell_{H^1(X)} = H^{4n}(X)$.

- Mention that the spinorial variety is $(2n^2 - n)$ -dimensional, where $n = \dim(X)$. When X is an abelian surface, the even spinorial variety is the 6-dimensional quadric in $\mathbb{P}H^{ev}(X, \mathbb{C})$ cut out by the Mukai pairing. When $n = 3$, the 31-dimensional $\mathbb{P}H^{ev}(X, \mathbb{C})$ is equal to the secant variety to the 15-dimensional even spinorial variety. When $n \geq 4$, the $(4n^2 - 2n + 1)$ -dimensional secant variety is a proper subvariety of the $(2^{2n-1} - 1)$ -dimensional $\mathbb{P}(S_{\mathbb{C}}^+)$.
- Mention that when $\dim(X) = 3$, the subring $[\text{Sym}^*(S_{\mathbb{Q}}^+)]^{\text{Spin}(V_{\mathbb{Q}})}$, of $\text{Spin}(V_{\mathbb{Q}})$ -invariant polynomials in the homogeneous coordinate ring of $\mathbb{P}(S_{\mathbb{Q}}^+)$, is a polynomial ring $\mathbb{Q}[J]$ generated by a single polynomial J of degree 4 (the Igusa quartic⁵). The zero locus of J is the tangential variety $\mathbb{P}(T\ell(\text{Gr}_+(6, V_{\mathbb{Q}})))$ (see [I] and [M2, Sec. 10]). Let $w \in S_{\mathbb{Q}}^+$. If $J(w) \neq 0$, then there exists through w a unique secant and the value $J(w)$ determines the field of definition of the points of intersection of the secant with the spinorial variety (and so the field of definition of the corresponding maximal isotropic subspaces of $V_{\mathbb{C}}$). If $J(w) > 0$, then the field of definition is the imaginary quadratic number field $K := \mathbb{Q}(\sqrt{-J(w)})$ [M2, Lemma 10.2.1]. The complex multiplication $\eta_P : K \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$ to be constructed below from the secant P through w will involve this field K .
- Present the example of the rational K -secant in formula [M2, (1.2.4)]. It is a 2-dimensional subspace $P \subset S_{\mathbb{Q}}^+$, such that $\mathbb{P}(P_K)$ is secant to the even spinorial variety in $\mathbb{P}(S_K^+)$, for each imaginary quadratic number fields K .

8. POLARIZED ABELIAN VARIETIES OF WEIL TYPE FROM K-SECANTS

A general Strategy for the proof of the algebraicity of the Weil classes: Let X be an abelian n -fold. Let d be a positive integer and set $K := \mathbb{Q}(\sqrt{-d})$. Choose a 2-dimensional subspace $P \subset S_{\mathbb{Q}}^+$, non-degenerate with respect to the Mukai pairing and spanned by Hodge classes, such that $\mathbb{P}(P_K)$ intersects the spinorial variety $\text{IGr}_+(2n, V_{\mathbb{C}})$ at two complex conjugate points defined over K , but not over \mathbb{Q} . Our goal in the rest of the lectures is to carry out for⁶ $n = 3$ the following program.

⁵When X is an abelian surface, the analogous statement holds, but replacing the Igusa quartic polynomial by the Mukai quadratic form. When n is odd, the Mukai pairing is anti-symmetric, and so its quadratic form vanishes. For $n \geq 4$, the $\text{Spin}(V_{\mathbb{C}})$ -orbits in $\mathbb{P}(S_{\mathbb{C}}^+)$ are no longer hypersurfaces, so the ring of invariants can not be generated by a single polynomial.

⁶With the exception of the semiregularity Condition (2a), we will carry the program for all n

- (1) Construct a complex multiplication $\eta_P : K \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$, such that the two maximal isotropic subspaces associated to the two pure spinors $\mathbb{P}(P_K) \cap \text{IGr}(2n, V_K)$ are associated to the two eigenspaces W_1 and W_2 of η_P . Construct also an η_P -compatible polarization h_P , so that $(X \times \hat{X}, \eta_P, h_P)$ is a polarized abelian variety of Weil type.
- (2) Construct over $X \times \hat{X}$ a coherent sheaf E of positive rank r , such that
 - (a) E is semiregular (possibly only in an equivariant sense with respect to a subgroup $G \subset X \times \hat{X}$ of translations, so that it descends to the isogenous abelian variety⁷ $Y := (X \times \hat{X})/G$ as a semiregular sheaf).
 - (b) The class $\kappa(E) := \text{ch}(E) \exp(-c_1(E)/r)$ remains of Hodge type over every deformation of $(X \times \hat{X}, \eta_P, h_P)$ as a polarized abelian variety of Weil type.
 - (c) The graded summand $\kappa_n(E)$ and h^n are linearly independent.

The semiregularity theorem implies that E deforms to a twisted coherent sheaf E' over any deformation (A, η, h) of $(X \times \hat{X}, \eta_P, h_P)$. Hence, $\kappa_n(E')$ is algebraic. Weil's result from Lecture 2, that $\dim H^{n,n}(A') = 3$ for the generic abelian variety of Weil type, combines with Condition (2b) to imply that $\kappa_n(E')$ belongs to $\text{span}\{h^n\} \oplus \text{HW}(A, \eta)$. Condition (2c) implies that there exists $t \in \mathbb{Q}$, such that $\kappa_n(E) - th^n$ is a non-zero algebraic class in $\text{HW}(A, \eta)$. Now K acts on $H^{2n}(A, \mathbb{Q})$ via algebraic correspondences and $\text{HW}(A, \eta)$ is a 1-dimensional K -subspace. Hence, every class in $\text{HW}(A, \eta)$ is algebraic.

- Define the group $\text{Spin}(V_K)_{\ell_1, \ell_2}$, the characters \det_i , and the group $\text{Spin}(V_{\mathbb{Q}})_P$ associated to a K -secant P . State Lemmas 2.2.1 and 2.2.2⁸ in [M2]. Note that $\text{Spin}(V_{\mathbb{Q}})_P$ will be identified with the Mumford Tate group of the generic polarized abelian variety of Weil type from Lecture 3.
- State and prove [M2, Lemma 2.2.6] stating that if the two dimensional subspace $P \subset S_K^+$ is spanned by Hodge classes, where $\mathbb{P}(P)$ a K -secant line, then the two maximal isotropic subspaces W_1 and W_2 satisfy the Hodge theoretic condition on the eigenspaces of the complex multiplication from the definition of an abelian variety of Weil type.
- State and prove [M2, Lemma 2.2.7] about the $\text{Spin}(V)_P$ -invariant subring $H^*(X \times \hat{X}, \mathbb{Q})^{\text{Spin}(V)_P}$. Conclude that a class γ in $H^{n,n}(X \times \hat{X}, \mathbb{Q})$ remains algebraic under every deformation of $(X \times \hat{X}, \eta_P, h_P)$ as a polarized abelian variety of Weil type, if and only if γ is $\text{Spin}(V)_P$ -invariant. Restate Condition (2b) of the general strategy in this language.

Next we relate the construction of the period domains of polarized abelian varieties of split Weil type (Weil's construction of these n^2 -dimensional moduli spaces from Lecture 3) to K -secants. This will achieve Step (1) of the above strategy.

- Cover the construction of the complex multiplication $\eta : K \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$ and the η -compatible polarization h of [M2, Prop. 2.4.4] carried out in [M2, Sec. 2.4].
- Show that $\text{Spin}(V_{\mathbb{Q}})_P$ preserves a K -valued hermitian form H and is isomorphic to $\text{SU}(V_{\mathbb{Q}}, H)$ [M2, Lemma 3.1.2].
- Show that $(X \times \hat{X}, \eta, h)$ is of *split Weil-type*, i.e., that $\det(X \times \hat{X}, \eta, h)$ is the coset of $(-1)^n$ in $\mathbb{Q}^\times / \text{Nm}(K^\times)$, see [M2, Lemma 3.1.3]. Note that this is equivalent to $H^1(X \times \hat{X}, \mathbb{Q})$ admitting a K -subspace of half the dimension isotropic with respect to the Hermitian form. A verification of this property (and so a different proof of the split Weil-type property) is given in [M3, 11.1.2].
- Show that the adjoint orbit of the complex structure $I_{X \times \hat{X}}$ in $\text{Spin}(V_{\mathbb{R}})_P$ is a period domain for abelian varieties of Weil type (sections 3.2 and 4 of [M2]).

9. EQUIVALENCES OF DERIVED CATEGORIES OF COHERENT SHEAVES

- Present in detail the basic facts about equivalences of derived categories of coherent sheaves on smooth projective algebraic varieties following Section 4.1 of [GLO] (see also [M2, Sec. 5]).
- Define Mukai's derived equivalence $\Phi_{\mathcal{P}} : D^b(X) \rightarrow D^b(\hat{X})$ associated to the Poincaré line bundle \mathcal{P} over $X \times \hat{X}$.

⁷Note that $H^1(Y, \mathbb{Q}) = H^1(X, \mathbb{Q})$, h_P determines a polarization \bar{h}_P on Y , and (Y, η_P, \bar{h}_P) is a polarized abelian variety of Weil type.

⁸There is a misprint in [M2, Lemma 2.2.2], the group should be $\text{SL}_{2n}(K)$.

- State and prove [GLO, Prop. 4.2.2.1] relating the cohomological isomorphism associated to Mukai's equivalence to Poncaré duality (see also [Hu, Lemma 9.23]).
- State [Hu, Exercise 9.21] and prove it if time permits.
- If time permits, which is very unlikely, briefly mention some of the results in [Hu, Sec. 9.3]. The convolution product $* : D^b(\hat{X}) \times D^b(\hat{X}) \rightarrow D^b(\hat{X})$ corresponds to the tensor product on $D^b(X)$ via Mukai's equivalence. We will not use the convolution product.

10. CHEVALLEY'S ISOMORPHISM AND ORLOV'S DERIVED EQUIVALENCE

The general strategy presented in Lecture 7 calls for a coherent sheaf E on $X \times \hat{X}$, such that $\kappa(E)$ remains a Hodge class on every deformation of $(X \times \hat{X}, \eta_P, h_P)$, where $P \subset S_{\mathbb{Q}}^+$ is a K-secant, $\eta_P : K \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$ the complex multiplication associated to P , and h_P the polarization associated to P . We have seen that $\text{Spin}(V_{\mathbb{Q}})_P$ is the special Mumford Tate group of the generic deformation of $(X \times \hat{X}, \eta, h)$ (by deforming the complex structure in the adjoint orbit of $I_{X \times \hat{X}}$ in $\text{Spin}(V_{\mathbb{R}})_P$), so we need $\kappa(E)$ to be $\text{Spin}(V)_P$ -invariant. If F_1 and F_2 are two coherent sheaves on X with $\text{ch}(F_i)$ in the secant P , then the Chern character of the sheaf $F_1 \boxtimes F_2 := \pi_1^* F_1 \otimes \pi_2^* F_2$ over $X \times X$ is $\text{Spin}(V)_P$ -invariant with respect to the diagonal action of $\text{Spin}(V)$ on $H^*(X) \otimes H^*(X)$. If we can find an equivalence of derived categories $D^b(X \times X) \cong D^b(X \times \hat{X})$, which induces a $\text{Spin}(V)$ -equivariant isomorphism $H^*(X \times X, \mathbb{Z}) \cong H^*(X \times \hat{X}, \mathbb{Z})$, it would transform $F_1 \boxtimes F_2$ to an object with a $\text{Spin}(V)_P$ -invariant Chern character. Such an equivalence does not exist. In this lecture we recall that Orlov found an equivalence, which induced cohomology isomorphism is sufficiently close to being $\text{Spin}(V)$ -equivariant, so that the transpose of $F_1 \boxtimes F_2$ has a $\text{Spin}(V)_P$ -invariant κ class.

- Define Chevalley's isomorphism

$$\tilde{\varphi} : H^*(X \times X, \mathbb{Z}) = S \otimes_{\mathbb{Z}} S \rightarrow \wedge^* V = H^*(X \times \hat{X}, \mathbb{Z})$$

(see [M2, Eq. (2.3.2)]). Point out that it is not $\text{Spin}(V)$ -equivariant, but it preserves a filtration on $\wedge^* V$ and induces $\text{Spin}(V)$ -equivariant isomorphisms of the graded pieces.

- The goal of the rest of the lecture is to lift Chevalley's isomorphism $\tilde{\varphi}$ to the level of derived categories and study its properties.
- Introduce the isomorphism ϕ in [M2, Eq. (1.2.5)] (a composition of $\tilde{\varphi}$ with two standard automorphisms).
- Define Orlov's equivalence

$$(4) \quad \Phi : D^b(X \times X) \rightarrow D^b(X \times \hat{X})$$

from [M2, Sec. 1.3]. Good references include [GLO] and [Hu]. Point out that Φ^{-1} maps the sky-scraper sheaf of a point $(a, L) \in X \times \hat{X}$ to the Fourier-Mukai kernel in $D^b(X \times X)$ of the derived equivalence $\tau_x^*(\bullet) \otimes L : D^b(X) \rightarrow D^b(X)$ (see section 4.2.4 of [GLO]).

- The following comment is to be skipped in the lecture. It answers the question: *How the hell did Orlov's equivalence⁹ got entangled with the algebraicity of the Weil classes?*

O'Grady was the first to realize the relationship between abelian fourfolds of Weil type and moduli spaces of sheaves on abelian surfaces [O'G]. Given a moduli space $M := M(v)$ of stable sheaves with Chern character v on an abelian variety X we have a morphism $a : X \times \hat{X} \rightarrow \text{Aut}(M)$ sending (x, L) to the automorphism $\tau_x^*(\bullet) \otimes L$. Assume that there exists a universal sheaf \mathcal{U} over $X \times M$. Over $M \times M$ there is a canonical object \mathcal{E} , whose "fiber" over (F, F') is $\text{Ext}^*(F, F')$. Canonically, it is $R\pi_{23,*}(L\pi_{12}^* \mathcal{U}^\vee \otimes L\pi_{23}^* \mathcal{U})$. When X is an abelian surface, $\text{Ext}^i(F, F')$ vanishes, whenever $i \neq 1$ and $F \not\cong F'$, so that away from the diagonal \mathcal{E} is a locally free sheaf (sitting in degree 1). In the paper [M1] the Weil classes on $X \times \hat{X}$ were constructed via the κ -class of the pullback of \mathcal{E} to $X \times \hat{X}$ via the morphism

$$\begin{aligned} \iota_{F_1, F_2} : X \times \hat{X} &\rightarrow \{F_1\} \times M \subset M \times M \\ (x, L) &\mapsto (F_1, \tau_x^*(F_2) \otimes L). \end{aligned}$$

⁹One may ask even why do we define the complex multiplication on the abelian variety $X \times \hat{X}$ and not on $X \times X$. Note that the simplest way to introduce complex multiplication is not on $X \times \hat{X}$, but rather on $X \times X$ by sending $\sqrt{-d}$ to the rational endomorphism $\begin{pmatrix} 0 & 1_X \\ -d_X & 0 \end{pmatrix}$, where d_X is d times the identity endomorphism of X , as in [vG1, Example 5.12].

The pullback $\iota_{F_1, F_2}^* \mathcal{E}$ is shown to be a shift of $\Phi(F_2 \boxtimes F_1^\vee)$ in [M2, Lemma 6.2.1] answering the above question.

- State without proof Lemma 6.1.1 relating the cohomological isomorphism induced by Orlov's equivalence Φ to Chevalley's isomorphism $\tilde{\varphi}$.
- State [GLO, Prop. 4.2.5.3 and 4.2.5.4] about the homomorphism

$$\gamma_X : \text{Aut}(D^b(X)) \rightarrow \text{Aut}(X \times \hat{X})$$

and its kernel.

- State [GLO, Prop. 4.3.7 and Lemma 4.3.9] and relate it to the short exact sequence (3) from Lecture 7.
- Mention Equation [M2, (6.1.9)] describing the difference between the homomorphism γ_X and the homomorphism

$$\text{Aut}(D^b(X)) \rightarrow \text{Aut}(D^b(X \times X)) \rightarrow \text{Aut}(D^b(X \times \hat{X})).$$

(composition of the “diagonal” embedding with conjugation by Φ).

- Present the $\text{Spin}(V)$ -equivariant homomorphism

$$(5) \quad \tilde{\phi} : H^*(X \times X, \mathbb{Z}) = S \otimes_{\mathbb{Z}} S \rightarrow \wedge^* V_{\mathbb{Q}} = H^*(X \times \hat{X}, \mathbb{Q})$$

in Equation (1.3.2) in Section 1.3 in the introduction of [M2] (without the proof of [M2, Prop. 1.3.1] but with the proof of [M2, Cor. 1.3.2]).

- Point out that the $\text{Spin}_{\text{Hdg}}(V)$ -action on $H^*(X \times X, \mathbb{Z})$ is induced by the action of derived equivalences and it does not preserve the grading. It is conjugated by $\tilde{\phi}$ to a $\text{Spin}_{\text{Hdg}}(V)$ -action on ¹⁰ $H^*(X \times \hat{X}, \mathbb{Z})$ induced by automorphisms of $X \times \hat{X}$, which preserve the grading. Automorphisms act on cohomology via monodromy operators, and in [GLO] this conjugation by the cohomological isomorphism induced by (essentially) Φ is interpreted as a version of homological mirror symmetry, relating equivalences of derived categories on the complex “side” to (symplectic) monodromy operators on its mirror.

11. HODGE WEIL CLASSES ON $X \times \hat{X}$ FROM TENSOR SQUARES OF PURE SPINORS

- State [Ch, III.3.2] relating the top exterior power $\bigwedge^{2n} W \subset \bigwedge^{2n} V_{\mathbb{C}}$ of a maximal isotropic subspace $W \subset V_{\mathbb{C}}$ to the image of the tensor square $\ell_W \otimes \ell_W \subset S_{\mathbb{C}} \otimes S_{\mathbb{C}}$ of the line ℓ_W spanned by pure spinors corresponding to W .
- Prove part (1) of [M2, Prop. 6.4.1] using the following simplified argument. Set $\phi' := \phi \circ (\text{id} \otimes \tau)$, where $\phi : H^*(X \times X, \mathbb{Z}) \rightarrow H^*(X \times \hat{X}, \mathbb{Z})$ is the isomorphism induced by Orlov's derived equivalence and τ is the dualization involution given in [M2, Eq. (1.2.3)] acting on $H^i(X)$ by multiplication by $(-1)^{i(i-1)/2}$. Let $\ell_i \subset H^{ev}(X, K)$ be the line of pure spinors corresponding to W_i , $i = 1, 2$. Note that $\ell_1 \oplus \ell_2 = P_K$ is defined over \mathbb{Q} . The lines $\ell_i^2 := \ell_i \otimes \ell_i$ are in $H^*(X \times X, K)$ and their direct sum is defined over \mathbb{Q} . [M2, Prop. 6.4.1(1)] states that $\phi'(\ell_1^2 \oplus \ell_2^2)$ is contained in $\bigoplus_{i \geq 2n} H^i(X \times \hat{X}, \mathbb{Q})$ and its projection to $H^{2n}(X \times \hat{X}, \mathbb{Q})$ maps it isomorphically onto $HW(X \times \hat{X}, \mathbb{Q})$.

Let \mathcal{P} be the Poincaré line bundle over $X \times \hat{X}$. Set

$$\tilde{\phi}(\bullet) := \exp(-c_1(\mathcal{P})/2) \cup \phi'(\bullet).$$

The isomorphism $\tilde{\phi} : H^*(X \times X, \mathbb{Q}) \rightarrow H^*(X \times \hat{X}, \mathbb{Q})$ is the $\text{Spin}(V)$ -equivariant isomorphism of [M2, Diagram (1.3.1)] (see Equation (5) in Lecture 10).

Proof of [M2, Prop. 6.4.1(1)]. The $\text{Spin}(V)_P$ -invariant subring $H^*(X \times \hat{X}, \mathbb{Q})^{\text{Spin}(V)_P}$ decomposes as the direct sum

$$H^*(X \times \hat{X}, \mathbb{Q})^{\text{Spin}(V)_P} = \text{span}\{h_P^i\}_{i=0}^{2n} \oplus HW(X \times \hat{X}, \eta_P),$$

by [M2, Lemma 2.2.7] presented in Lecture 7. The first summand is $\text{Spin}(V)_{\ell_1, \ell_2}$ -invariant, and $HW(X \times \hat{X}, \eta_P)$ is the direct sum of the two one-dimensional $\text{Spin}(V)_{\ell_1, \ell_2}$ -representations $\wedge^{2n} W_i$, with characters \det_i , $i = 1, 2$, by [M2, Lemma 2.2.7].

¹⁰Although $\tilde{\phi}$ is not integral, it conjugates elements of $\text{Spin}(V)$ to integral graded automorphisms of the ring $H^*(X \times \hat{X}, \mathbb{Z})$.

The subspace $P \otimes P$ is contained in $H^*(X \times X, \mathbb{Q})^{\text{Spin}(V)_P}$ and decomposes as

$$P \otimes P = [(\ell_2 \otimes \ell_1) \oplus (\ell_1 \otimes \ell_2)] \oplus [\ell_1^2 \oplus \ell_2^2].$$

Each of the summands $(\ell_1 \otimes \ell_2)$ and $(\ell_2 \otimes \ell_1)$ is the trivial $\text{Spin}(V)_{\ell_1, \ell_2}$ -representation, by [Ch, III.3.3]. The summand ℓ_i^2 in $P \otimes P$ corresponds to the character \det_i of $\text{Spin}(V)_{\ell_1, \ell_2}$, by [Ch, III.3.2]. Hence, the isomorphism $\tilde{\phi}$ maps ℓ_i^2 onto $\wedge^{2n} W_i$. Thus,

$$\phi'(\ell_i^2) = \exp(c_1(\mathcal{P})/2) \cup \wedge^{2n} W_i$$

is contained in $\bigoplus_{i \geq 2n} H^i(X \times \hat{X}, \mathbb{Q})$ and projects isomorphically onto $\wedge^{2n} W_i$. \square

- Present the short [M2, Example 8.2.2] that every stable sheaf on an abelian surface is a semiregular secant sheaf.
- Let C be a nonsingular curve of genus 3 and let $X = \text{Pic}^0(C)$. Let $\text{AJ} : C \rightarrow \text{Pic}^1(C)$ be the Abel-Jacobi morphism. Let C_i , $1 \leq i \leq d+1$, be generic $d+1$ disjoint translates of the image $\text{AJ}(C)$ into X . Let Θ be a translate of the theta divisor from $\text{Pic}^2(C)$ into X . Denote by Θ also the class Poincaré dual to Θ in $H^{1,1}(X, \mathbb{Z})$. Prove Lemma [M2, 8.2.1] stating that the sheaf

$$(6) \quad F := \mathcal{I}_{\bigcup_{i=1}^{d+1} C_i}(\Theta)$$

(the ideal sheaf tensored with $\mathcal{O}X(\Theta)$) is a $\mathbb{Q}(\sqrt{-d})$ -secant sheaf in the secant

$$P := \text{span}_{\mathbb{Q}} \left\{ 1 - \frac{d}{2}\Theta^2, \Theta - \frac{d}{6}\Theta^3 \right\}.$$

- State and prove [M2, Lemma 8.3.1] that the unique, up to scalar factor, $\text{Spin}(V)_P$ -invariant polarization $h \in H^{1,1}(X \times \hat{X}, \mathbb{Z})$ satisfies: h^n and $\kappa_n(\Phi(F \boxtimes F))$ are linearly independent. You will use [M2, Prop. 6.4.1] presented in Lecture 11. Relate to Condition¹¹ (2c) of the general strategy in Lecture 7.

12. SEMIREGULAR SECANT SHEAVES ON ABELIAN 3-FOLDS

Keep the notation of lecture 11. In this lecture we prove that the sheaf F given in (6) in the previous lecture is close to being semiregular, and when the collection $\{C_i\}_{i=1}^{d+1}$ is chosen to be an orbit of a cyclic group $G \subset X$, then the ideal $\mathcal{I}_{\bigcup_{i=1}^{d+1} C_i}$ is the pullback of a semiregular sheaf over X/G .

- State and prove [M2, Lemma 8.3.2] that the natural homomorphism

$$H^0(TX)^{\oplus(d+1)} \oplus H^1(X, \mathcal{O}X) \rightarrow \text{Ext}^1(F, F)$$

is an isomorphism.

- Review the definition of Hochschild cohomology $\text{HH}^i(X)$ and the algebra homomorphism

$$\text{ev}_F : \text{HH}^*(X) \rightarrow \text{Ext}^*(F, F)$$

of [M2, Eq. (8.3.3)]. Denote its restriction to $\text{HH}^2(X)$ by ob_F . Briefly review the HKR isomorphism $\text{HH}^*(X) \cong \text{HT}^*(X)$. Note that $H^*(X, \mathbb{C})$ is an $\text{HT}^*(X)$ -module.

- Let F be a coherent sheaf over an N -dimensional compact Kähler manifold M . Introduce the commutative diagram [M2, Eq. (8.3.4)].

$$(7) \quad \begin{array}{ccc} \text{HT}^2(M) & \xrightarrow{\text{ob}_F} & \text{Ext}^2(F, F) \\ & \searrow \text{ch}(F) & \swarrow \sigma_F \\ & \prod_{q=0}^{N-2} H^{q+2}(\Omega_M^q). & \end{array}$$

¹¹The Condition is stated for E arising from two sheaves F_1 and F_2 on X and we will use two sheaves, but they will have the same Chern character (see (9)).

- State and prove [M2, Lemma 8.3.4]. It states that if ob_F is surjective and the kernels of ob_F is equal to the kernel of $\lrcorner \text{ch}(F)$, then F is semiregular. Point out that the statement of the Lemma is invariant under equivalences of derived categories [M2, Rem. 8.3.6] and [M2, Lemma 8.3.10].
- State and prove [M2, Lemma 8.3.7] that the ideal sheaf \mathcal{J}_{C_j} , of a single translate of the Abel-Jacobi curve in X , is semiregular.
- State (without proof) a shorter version of [M2, Lemma 8.3.8] that the Yoneda algebra $\text{Ext}^*(F, F)$ of the sheaf F , displayed above in (6), is generated by $\text{Ext}^1(F, F)$. Conclude that ob_F is surjective.
- State (without proof) [M2, Prop. 8.3.9] that the rank of ob_F is 6. Point out that when the set $\{C_j\}_{j=1}^{d+1}$ is the orbit of C_1 under translations by a subgroup $G \subset X$ of order $d+1$, then the quotient morphism $q : X \rightarrow \bar{X} := X/G$ maps C_1 isomorphically onto a curve $\bar{C} \subset \bar{X}$, the ideal sheaf $\mathcal{J}_{\bigcup_{i=1}^{d+1} C_i}$ is the pullback of $\mathcal{J}_{\bar{C}}$, and

$$Lq^* : \text{Ext}^*(\mathcal{J}_{\bar{C}}, \mathcal{J}_{\bar{C}}) \rightarrow \text{Ext}^*(\mathcal{J}_{\bigcup_{i=1}^{d+1} C_i}, \mathcal{J}_{\bigcup_{i=1}^{d+1} C_i})^G$$

is an isomorphism. Furthermore, $\mathcal{J}_{\bar{C}}$ is semiregular and $\text{ob}_{\mathcal{J}_{\bar{C}}}$ has rank 6, essentially by the same proof as that of [M2, Lemma 8.3.7]. In that case [M2, Prop. 8.3.9] follows from the fact that $\text{ob}_{\mathcal{J}_{\bigcup_{i=1}^{d+1} C_i}}$ is G -equivariant, the G -action on $\text{HT}^*(X)$ is trivial, $q^* : \text{HT}^*(\bar{X}) \rightarrow \text{HT}^*(X)$ is an isomorphism, and the diagram

$$(8) \quad \begin{array}{ccc} \text{HT}^*(X) & \xrightarrow{\text{ev}_{\mathcal{J}_{\bigcup_{i=1}^{d+1} C_i}}} & \text{Ext}^*(\mathcal{J}_{\bigcup_{i=1}^{d+1} C_i}, \mathcal{J}_{\bigcup_{i=1}^{d+1} C_i})^G \\ q^* \uparrow \cong & & \uparrow \cong Lq^* \\ \text{HT}^*(\bar{X}) & \xrightarrow{\text{ev}_{\mathcal{J}_{\bar{C}}}} & \text{Ext}^*(\mathcal{J}_{\bar{C}}, \mathcal{J}_{\bar{C}}) \end{array}$$

is commutative.

- State and prove [M2, Cor. 8.3.12] that the kernels in $\text{HT}^2(X)$ of ob_F and $\lrcorner \text{ch}(F)$ are equal.

13. AN OBJECT E OF $D^b(X \times \hat{X})$ WITH A $\text{Spin}(V)_P$ -INVARIANT $\kappa(E)$ AND A 9 DIMENSIONAL $\ker(\text{ob}_E) = \ker(\lrcorner \text{ch}(E))$

We have seen in Lecture 10 how to construct an object E in $D^b(X \times \hat{X})$ with a $\text{Spin}(V)_P$ -invariant $\kappa(E)$ from two secant sheaves F_1 and F_2 on X . In lecture 12 we found an equivariantly semiregular secant sheaf F . In this lecture we construct an equivariantly semiregular object E on $X \times \hat{X}$ with a $\text{Spin}(V)_P$ -invariant $\kappa(E)$ from two equivariantly semiregular secant sheaves F_1 and F_2 on X . Denote by $F_1 \boxtimes F_2$ be the object $\pi_1^* F_1 \overset{L}{\otimes} \pi_2^* F_2$ in $D^b(X \times X)$.

Let F_1 and F_2 be objects in $D^b(X)$ with $\text{ch}(F_i) \in P$ (so secant objects), such that $\ker(\lrcorner \text{ch}(F_i)) = \ker(\text{ob}_{F_i})$, for $i = 1, 2$. Assume, furthermore, that $\text{ev}_{F_i} : \text{HT}^1(X) \rightarrow \text{Ext}^1(F_i, F_i)$ is injective. For example, F_1 could be the sheaf (6) and F_2 could be its analogue $\mathcal{J}_{\bigcup_{i=1}^{d+1} \Sigma_i}(\Theta)$, where $\Sigma = -\text{AJ}(C) \subset \text{Pic}(C)^{-1}$ and Σ_j , $1 \leq j \leq d+1$, is a generic collection of disjoint translates of Σ into X . Note that the classes $[C_i]$ and $[\Sigma_j]$ in $H^6(X, \mathbb{Z})$ are equal, since multiplication by -1 acts trivially on $H^{\text{ev}}(X, \mathbb{Z})$. Hence,

$$(9) \quad \text{ch}(F_1) = \text{ch}(F_2)$$

in this case.

- State and prove the equation [M2, (8.4.3)]

$$\ker(\text{ob}_{F_1 \boxtimes F_2}) = [\pi_1^* \ker(\text{ob}_{F_1}) \otimes \pi_2^* H^0(\mathcal{O}X)] \oplus [\pi_1^* H^0(\mathcal{O}X) \otimes \pi_2^* \ker(\text{ob}_{F_2})].$$

- Set $E := \Phi(F_1 \boxtimes F_2)$, where $\Phi : D^b(X \times X) \rightarrow D^b(X \times \hat{X})$ is Orlov's equivalence given in (4). State and prove [M2, Lemma 8.4.1] about the equality

$$\ker(\text{ob}_E) = \ker(\lrcorner \text{ch}(E))$$

of subspaces of $\text{HT}^2(X \times \hat{X})$.

- Define the isomorphism $\Phi^{\text{HT}} : \text{HT}(X \times X) \rightarrow \text{HT}(X \times \hat{X})$ induced by Orlov's equivalence Φ and the operator $(\bullet)^* : \text{HT}^*(X) \rightarrow \text{HT}^*(X)$ from [M2, Sec. 8.5]. State, without proof, [M2, Cor. 8.5.2] relating
 - the 9-dimensional space $\ker(\text{ob}_{F_1})$, of generalized first order deformations of the pair (X, F_1) , to
 - the 9-dimensional subspace $\ker(\text{ob}_E)$ of $H^1(T[X \times \hat{X}]) \oplus H^2(\mathcal{O}X \times \hat{X})$ of commutative and gerby deformations of the pair $(X \times \hat{X}, E)$.

The semiregularity theorem (generalized for twisted sheaves) interprets the latter subspace as the tangent space to the 9-dimensional moduli space of polarized abelian sixfolds of Weil type deformation equivalent to $(X \times \hat{X}, \eta_P, h_P)$. So, at least to first order, these abelian sixfolds of Weil type are a geometric incarnation of the generalized (including commutative, non-commutative, and gerby) deformations of X , along which the object F_1 deforms.

Upcoming work of Bayer-Macri-Pertucci-Perry-... elaborates on this for the analogous case of abelian surfaces and abelian fourfolds of split Weil type. The latter appear as the identity components of the derived categories of non-commutative deformations of an abelian surface, realized geometrically as equivariant categories of sheaves on a carefully chosen family of K3-surfaces admitting a suitable involution of their derived category. See also the related [M2, Rem. 8.5.5].

- State [M2, Assumption 9.1.1]. State without proof¹² [M2, Lemma 9.1.4] that the assumption is satisfied for a G -equivariant collection $\{C_i\}_{i=1}^{d+1}$ in X of translates of the Abel-Jacobi image.
- State [M2, Assumption 9.2.1]
- In Section 9 of [M2] there is a change of notation and F_1 denotes F_1^\vee in [M2, Sec. 8]. In order to avoid confusion, we will use the notation $K_1 := F_1^\vee$ and $K_2 = F_2$, where F_i are as above and as in [M2, Sec. 8]. State (without proof) [M2, Prop. 9.2.2] replacing the notation F_i by K_i .
- State [M2, Lemma 9.2.4]. Include only Steps 0 and 1 of the proof so that we get the flavor of the proof.

14. A $[\mathbb{Z}/(d+1)\mathbb{Z}]^2$ -EQUIVARIANT SHEAF OVER $X \times \hat{X}$

The first part is a continuation of Lecture 13.

- See the comment in the previous lecture plan about the change of notation in [M2, Sec. 9].
- Present the strategy of the proof of [M2, Prop. 9.2.2] assuming the vanishing criteria (i) and (ii) of the cohomologies $H^i(X, K_1 \otimes \tau_x^*(K_2) \otimes L)$. The strategy consists of the part of the proof preceding the proof of part (1) of the proposition.
- Prove that the sheaf \mathcal{E} in [M2, Prop. 9.2.2] has rank $8d$. This involves only the paragraph containing the short exact sequence [M2, (9.2.9)]. Do not include the proof of [M2, Lemma 11.0.1].
- Explain that the surfaces $\tilde{\Theta}_{i,j} \subset X \times \hat{X}$ consist of points (x, L) , where C_i meets $\tau_{-x}(\Sigma_j)$ and the line bundle $\mathcal{O}X(\Theta + \tau_{-x}(\Theta)) \otimes L^{-1}$ restricts to the canonical line bundle of $C_i \cup \tau_{-x}(\Sigma_j)$. This causes the failure of vanishing of cohomologies needed for local freeness of \mathcal{E} at that point. The fact that \mathcal{E} is reflexive is not used in the paper.

The proof of the algebraicity of the Weil classes on abelian sixfolds of Weil type is completed in [M2, Section 9.3]. There are two remaining issues. One technical issue is to show that if a sheaf E is semiregular, then the associated twisted sheaf with trivial determinant line-bundle is semiregular as well. We will not elaborate on this technical issue. Granted this, if we could apply [M2, Lemma 8.3.4] (reviewed in Lecture 12) to prove that the sheaf \mathcal{E} of [M2, Prop. 9.2.2] is semi-regular, then we would be done. However, $\text{ob}_{\mathcal{E}} : \text{HT}^2(X \times \hat{X}) \rightarrow \text{Ext}^2(\mathcal{E}, \mathcal{E})$ is not surjective. In fact, $\dim(\text{Ext}^2(\mathcal{E}, \mathcal{E}))$ grows quadratically as a function of d as $d \rightarrow \infty$ (by [M2, Lemma 8.3.2], reviewed in Lecture 12, and Serre's duality), while the dimension of the domain of $\text{ob}_{\mathcal{E}}$ is fixed. The solution is to replace \mathcal{E} with a closely related sheaf, which is equivariant with respect to a group \bar{G} of translations isomorphic to

¹²The proof of Lemma 9.1.4 relies on [M2, Lemma 9.1.2], and the latter involves an argument surprisingly rich with the classical geometry of rank 2 vector bundles with trivial determinant on curves and their relation to divisors in the linear system 2Θ on the Jacobian and, in genus 3, the Coble quartic.

$[\mathbb{Z}/(d+1)\mathbb{Z}]^2$, so that it descends to a sheaf $\bar{\mathcal{E}}$ over $(X \times \hat{X})/\bar{G}$ with a surjective $\text{ob}_{\bar{\mathcal{E}}}$. The idea was already employed in Diagram (8) above to simplify a proof, but its application below is essential.

- Let G_1 and G_2 be cyclic subgroups of X of order $d+1$, such that $G_1 \cap G_2 = (0)$. Choose the collections $\{C_i\}_{i=1}^{d+1}$ to be G_1 -equivariant and the collection $\{\Sigma_i\}_{i=1}^{d+1}$ to be G_2 -equivariant. State, without proof, [M2, Lemma 9.3.1]. It states that genericity [M2, Assumption 9.2.1] holds in this case.
- Set

$$\begin{aligned} E' &:= \mathcal{I}_{\bigcup_{i=1}^{d+1} \Sigma_i} \boxtimes \mathcal{I}_{\bigcup_{i=1}^{d+1} C_i}, \\ \Theta \boxtimes \Theta &:= \pi_1^* \mathcal{O}_X(\Theta) \otimes \pi_2^* \mathcal{O}_X(\Theta), \\ \mathcal{G} &:= \Phi(E' \otimes [\Theta \boxtimes \Theta])[-3], \\ \tilde{\Phi} &:= \Phi \circ ([\Theta \boxtimes \Theta] \otimes (\bullet)) : D^b(X \times X) \rightarrow D^b(X \times \hat{X}). \end{aligned}$$

Let $G \subset \text{Aut}(D^b(X \times \hat{X}))$ be the image of $G_2 \times G_1 \subset \text{Aut}(D^b(X \times X))$ via conjugation by $\tilde{\Phi}$. Note that E' is $G_2 \times G_1$ -equivariant. Hence \mathcal{G} is G -equivariant in the sense¹³ of [BO]. State and prove [M2, Lemma 9.3.2] about the equality of the image of $\text{ob}_{\mathcal{G}} : \text{HH}^2(X \times \hat{X}) \rightarrow \text{Hom}(\mathcal{G}, \mathcal{G}[2])$ and $\text{Hom}(\mathcal{G}, \mathcal{G}[2])^G$.

- State and prove [M2, Lemma 9.3.3]. The group $G_2 \times G_1$ is contained in the identity component of $D^b(X \times X)$. Hence, G is contained in the identity component $(X \times \hat{X}) \times \text{Pic}^0(X \times \hat{X})$ of $\text{Aut}(D^b(X \times \hat{X}))$. The Lemma states that G projects injectively into the factor $X \times \hat{X}$, acting on $\text{Aut}(D^b(X \times \hat{X}))$ via pushforward by translation automorphisms of $X \times \hat{X}$. We get a subgroup $\bar{G} \subset X \times \hat{X}$ isomorphic to $G_2 \times G_1$.
- State and prove [M2, Lemma 9.3.5]. It states that when d is even, we can tensor \mathcal{G} by a suitable power a of $D := \det(\mathcal{G})$, so that $\mathcal{G} \otimes D^a$ is \bar{G} -equivariant. Mention the statement of [M2, Footnote 25] that $\mathbb{Q}(\sqrt{4d}) = \mathbb{Q}(\sqrt{d})$, so all imaginary quadratic number fields are considered even if we assume that d is even.
- Set $Y := [X \times \hat{X}]/\bar{G}$ and denote the quotient morphism by $q : X \times \hat{X} \rightarrow Y$. Set $\mathcal{E} := (\mathcal{G}^1)^*$, as in [M2, Prop. 9.2.2]. Then $\mathcal{E} \otimes D^{-a}$ is G -equivariant and it descents to a coherent sheaf $\bar{\mathcal{E}}$ over Y . Construction [M2, 7.3.3] (discussed in Lecture 5) applied to $\bar{\mathcal{E}}$ produces a twisted sheaf \mathcal{B} , with respect to a Brauer class θ in $H^2(Y, \mu_r)$, $r = 8d$, with trivial determinant, such that $q^* \mathcal{B}$ is a $q^* \theta$ -twisted sheaf related to \mathcal{E} by Construction [M2, 7.3.3]. The sheaf \mathcal{B} is constructed in [M2, Lemma 9.3.6] for odd d as well, but we do not need it.
- Conclude, that $\text{ob}_{\bar{\mathcal{E}}}$ is surjective, by [M2, Lemma 9.3.2] discussed above. Hence, $\bar{\mathcal{E}}$ is semiregular, by [M2, Lemma 8.3.4].

15. THE CM-FIELDS CASE

We review the more general open problem of the algebraicity of the Weil classes on abelian varieties of Weil type with complex multiplication by a CM-field. A CM-field is a totally complex quadratic extension K of a totally real number field F . The case $F = \mathbb{Q}$ is the case of imaginary quadratic number fields. The basic definitions of the Weil classes are in [DM, Sec. 4]. A more detailed exposition is found in [CS, Sec. 11.5.4, 11.5.5]. The Weil classes appear in $H^d(A, \mathbb{Q})$, where $d = \dim_K(H^1(A, \mathbb{Q}))$, and are no longer in the middle dimension, when $[K : \mathbb{Q}] > 2$.

The general strategy for proving the algebraicity of the Weil classes, outlined in Lecture 7, is extended to the CM-field case in [M3] (see also [M4] for a concise summary). We start with an abelian variety X with real multiplication $\hat{\eta} : F \rightarrow \text{End}_{\mathbb{Q}}(X)$ and construct complex multiplication $\eta : K \rightarrow \text{End}_{\mathbb{Q}}(X \times \hat{X})$ in a manner analogous to the one in Lecture 8. The secants in $\mathbb{P}(H^{ev}(X, \mathbb{C}))$ are no longer lines, but rather linear subspaces of dimension $2^{[F:\mathbb{Q}]} - 1$. The challenge is to find semiregular secant sheaves with Chern characters in these secant spaces.

In a recent paper Milne points out that if the Weil classes are proven to be algebraic (over complex algebraic varieties), important conjectures in arithmetic algebraic geometry would follow, and a theory of abelian motives over fields of finite characteristic would be available [Mi3, Footnote 3] (the footnote

¹³Do not spend time elaborating on group actions on derived categories $D^b(M)$ more general than those arising from groups of automorphisms of M . Although the G -action is such, below we very quickly return to the special case of groups of automorphisms of $M = X \times \hat{X}$.

does not appear in the current archive version of this paper arXiv:2508.09972.v2). The importance of the Weil classes stems from a result of André that proves the Hodge conjecture for abelian varieties of CM-type (whose special Mumford-Tate group is a torus, different from Weil-type) assuming the algebraicity of the Weil classes on abelian varieties of split Weil-type [A] (surveyed in [CS, Sec. 11.5.6]). Abelian varieties of CM-type are rigid, corresponding to special points in Shimura varieties. Reduction mod p relates them to abelian varieties over finite fields.

- Define abelian varieties of CM-type following [CS, Sec. 11.5.1].
- Present the construction of the Weil classes on an abelian variety of split Weil type with complex multiplication by a CM-field, following [CS, Sec. 11.5.4 and 11.5.5].
- Present the proof of André's theorem following [CS, Sec. 11.5.6].

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