Surfaces of section
for geodesic flows of closed surfaces

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Joint work with:

- Gonzalo Contreras
- Gonzalo Contreras, Gerhard Knieper, Benjamin Schulz

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X nowhere vanishing vector field,
$\phi_t : N \to N$ flow of X

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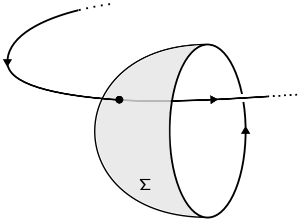

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A global surface of section is a compact immersed surface $\Sigma \looparrowright N$ such that:

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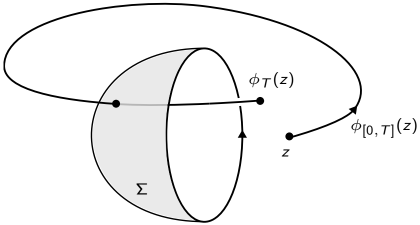
- for some $T > 0$, any orbit segment $\phi_{[0,T]}(z)$ intersects Σ.



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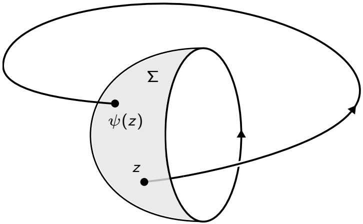
First return map: $\psi : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma)$



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  $\phi_X^t(\dot\gamma(0)) = \dot\gamma(t)$, where γ is a geodesic with $\|\dot\gamma\|_g \equiv 1$

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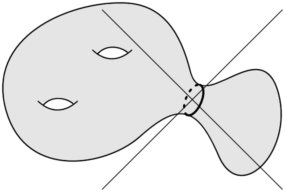
Theorem (Contreras-Mazzucchelli-Knieper-Schulz)
If (M, g) has no contractible simple closed geodesics without conjugate points, there there exists a global surface of section of genus one and $8G - 4$ boundary components



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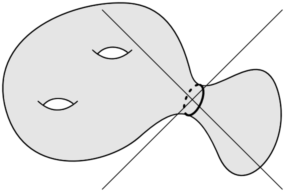

Remark. There are no contractible simple closed geodesics provided

$$\max K_g \leq \frac{2\pi}{\mathrm{area}(M, g)}$$

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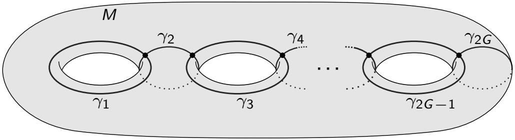
- $\Gamma = \gamma_1 \cup \ldots \cup \gamma_{2G}$



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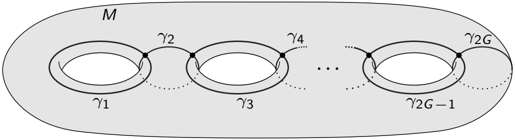


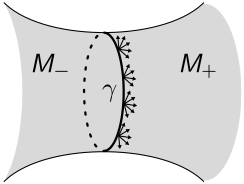
- No geodesic ray is trapped in $M \setminus \Gamma$

  (otherwise $M \setminus \Gamma$ would contain a simple closed geodesic without conjugate points)

Global surfaces of section of geodesic flows

- Birkhoff annuli of a simple closed geodesic γ:

$$A_+(\gamma) := \{ v \in SM|_\gamma \mid v \text{ points inside } M_+ \}$$



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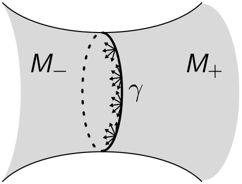

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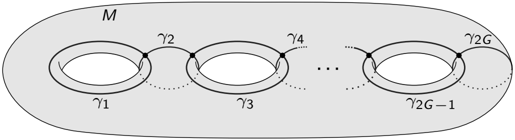

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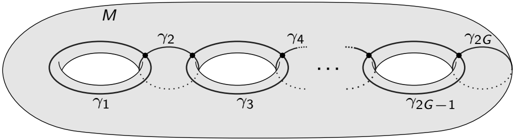
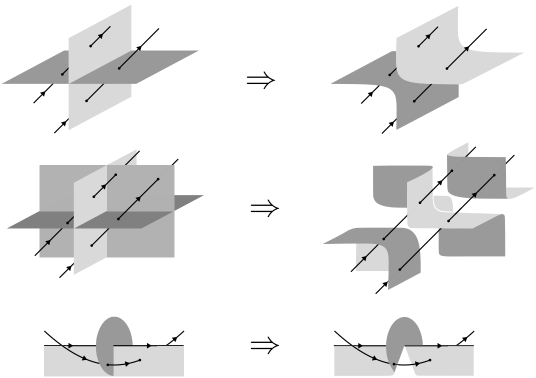


Σ is almost global surface of section, except that is has self-intersections.

Global surfaces of section of geodesic flows

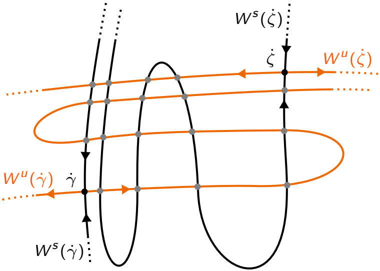
- (Fried) Resolve self-intersections of Σ with surgery:



$\square$

A weak Kupka-Smale condition

We require all the contractible simple closed geodesics without conjugate points γ, ζ to be hyperbolic, and $W^s(\dot\gamma) \pitchfork W^u(\dot\zeta)$:



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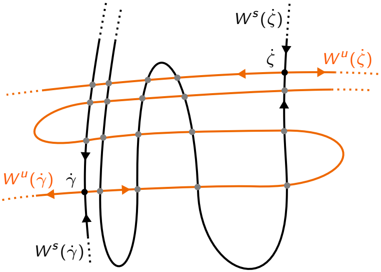

Theorem (Contreras-Paternain)
Weak Kupka-Smale holds for a C^∞-generic Riemannian metric.

Main theorem

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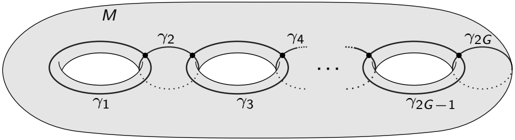

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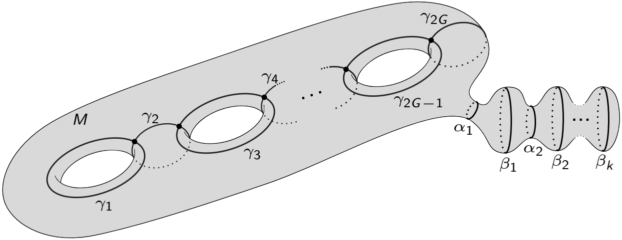

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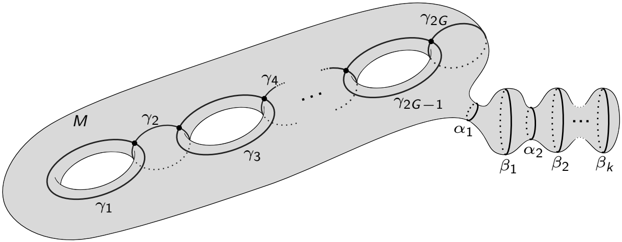

$A = \alpha_1 \cup \ldots \cup \alpha_k, \quad B = \beta_1 \cup \ldots \cup \beta_k$

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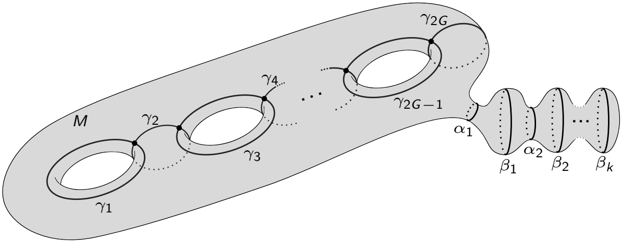

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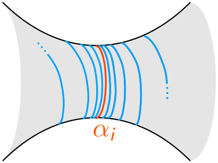
No complete geodesic is contained in $M \setminus (\Gamma \cup A \cup B)$

Geodesic rays in $M \setminus (\Gamma \cup A \cup B)$ are asymptotic to some α_i

Main theorem

- Some α_i has homoclinics on both sides:



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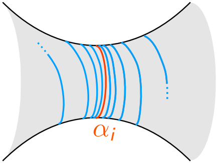

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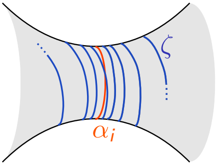

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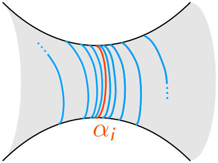

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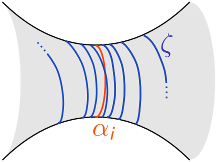

No geodesic ray in $M \setminus (\Gamma \cup A \cup B \cup \zeta)$ is asymptotic to α_i.

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- We obtained a finite collection of closed geodesics Z such that $M \setminus (\Gamma \cup A \cup B \cup Z)$ does not contain geodesic rays.
- Build a global surface of section by doing surgery on the Birkhoff annuli of $\Gamma \cup A \cup B \cup Z$.

□

Application: characterization of Anosov Reeb flows

Theorem (Contreras-Mazzucchelli). *Let X be the Reeb vector field of a closed contact 3-manifold such that:*

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- $W^u(\gamma_1) \pitchfork W^s(\gamma_2)$ *for all closed Reeb orbits* $\gamma_1, \gamma_2 \subset \mathrm{Per}(X)$.

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This corollary extends a theorem of Contreras-Oliveira for S^2, which extended a theorem of Herman for positively curved S^2, which in turn was first claimed (with a slightly wrong statement and an incomplete proof) by Poincaré in 1905.

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Corollary^2. *The geodesic flow of a closed Riemannian surface is C^2-structurally stable if and only if it is Anosov.*

Thank you for your attention!