ON THE MORSE INDICES OF FREDHOLM QUADRATIC FORMS

Let $\mathbb{E}$ be a separable Hilbert space, $P : \mathbb{E} \to \mathbb{E}$ a positive-definite self-adjoint isomorphism, and $K : \mathbb{E} \to \mathbb{E}$ a compact self-adjoint linear operator. We consider the quadratic form $Q : \mathbb{E} \to \mathbb{R}$ given by

$$Q(v) = \langle (P + K)v, v\rangle.$$

We define its nullity $\mathrm{nul}(Q)$ as the dimension of the kernel of $P + K$, and its Morse index $\mathrm{ind}(Q)$ as the maximal dimension of a vector subspace $\mathbb{E}^- \subset \mathbb{E}$ such that $Q(v) < 0$ for all non-zero $v \in \mathbb{E}^-$.

Proposition 0.1. *The indices* $\mathrm{ind}(Q)$ *and* $\mathrm{nul}(Q)$ *are both finite.*

Proof. The operator $P + K$ is Fredholm, since it is a compact perturbation of the isomorphism P (more generally, compact perturbations of Fredholm operators are Fredholm). In particular, the nullity $\mathrm{nul}(Q) = \dim\ker(P + K)$ is finite.

We denote by $\mathbb{E}_\lambda$ the eigenspace of K corresponding to the eigenvalue λ, i.e.

$$\mathbb{E}_\lambda = \ker(K - \lambda I).$$

The spectral theorem for compact self-adjoint operators tells us that, for $\lambda \neq 0$, each eigenspace $\mathbb{E}_\lambda$ is finite dimensional. Moreover, either K has finitely many eigenvalues, or its eigenvalues form an infinite sequence converging to zero. Now, set $c := \|P^{-1/2}\|^{-2} > 0$, and notice that $\langle Pv, v\rangle \geq c\|v\|^2$. We define the finite dimensional vector subspace

$$\mathbb{F} := \bigoplus_{\lambda < -c} \mathbb{E}_\lambda,$$

and its orthogonal projector $\pi : \mathbb{E} \to \mathbb{F}$. We claim that $\mathrm{ind}(Q) \leq \dim(\mathbb{F})$. Indeed, assume by contradiction that there exist linearly independent vectors $v_1, ..., v_n \in \mathbb{E}$ such that $n > \dim(\mathbb{F})$ and

$$Q(v) < 0, \qquad \forall v \in \mathrm{span}\{v_1, ..., v_n\} \setminus \{0\}. \tag{0.1}$$

If we project these vectors to $\mathbb{F}$, we obtain a linearly dependent set. Therefore we can find $\lambda_1, ..., \lambda_n \in \mathbb{R}$ such that the vector $v_0 := \lambda_1 v_1 + ... + \lambda_n v_n$ is non-zero and belongs to $\ker(\pi)$. This kernel is the orthogonal complement of $\mathbb{F}$, which is

$$\mathbb{F}^\perp = \ker(\pi) = \bigoplus_{\lambda \geq -c} \mathbb{E}_\lambda.$$

However, notice that $\langle Kv, v\rangle \geq -c\|v\|^2$ for all $v \in \mathbb{F}^\perp$. This implies

$$Q(v_0) = \langle Pv_0, v_0\rangle + \langle Kv_0, v_0\rangle \geq 0,$$

which contradicts (0.1). ■