TD05 - Hyperbolic spaces, connections

Exercise 1 (Hyperbolic half-plane and Poincaré disc)**.** In this exercise, we simply denote by $\mathbb{H}$ the hyperbolic half-plane $\mathbb{H}^2$ and by $\mathbb{D}$ the hyperbolic disc $\mathbb{D}^2$ (also called *Poincaré disc*).

1. Check that conformal diffeomorphisms of $\mathbb{D}$ (resp. $\mathbb{H}$) preserving the orientation are biholomorphisms.
2. Describe the conformal diffeomorphisms of $\mathbb{D}$ (resp. $\mathbb{H}$). Which one are isometries?

Exercise 2 (Warm-up)**.** Let g_0 denote the Euclidean metric of $\mathbb{R}$, we define another Riemannian metric g on $\mathbb{R}$ by $g_x := e^{-x^2}(g_0)_x$ for any $x \in \mathbb{R}$. Let ∇_0 (resp. ∇) denote the Levi–Civita connection on $T\mathbb{R}$ associated with g_0 (resp. g). Compute $(\nabla_0)\frac{\partial}{\partial x}$ and $\nabla\frac{\partial}{\partial x}$.

Exercise 3 (Dual connection)**.** Let ∇ be a connection on a vector bundle $E \to M$. We still denote by ∇ the induces connections on bundles of the form $E \otimes \cdots \otimes E \otimes E^* \otimes \cdots \otimes E^* \to M$.

1. Compute $d(\alpha(X))$ where $\alpha \in \Gamma(E^*)$ and $X \in \Gamma(E)$.
2. Compute $\nabla \operatorname{Id}$ where $\operatorname{Id} : x \mapsto \operatorname{Id}_{E_x}$ is a section of $\operatorname{End}(E) \to M$.