TD10: CURVATURES, PART II

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

CHIH-KANG HUANG

Exercise 1. Let (M, g) be a Riemannian manifold and $p \in M$. In geodesic normal coordinates $x^1, \dots, x^n$ centered at p, we have the expansion of g around p is given by

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{1 \le k,l \le n} R_{iklj}(0) x^k x^l + O(\|x\|^3),$$

where $(g_{ij}), (\Gamma_{ij}^k)$ and (R_{ijkl}) are respectly the matrix of the metric, Christoffel symbols and the components of the Riemann tensor in these coordinates.

(1) Show that the expansion of the Riemannian volume dV around 0 in these coordinates is given by

$$dV = \left(1 - \frac{1}{6}\mathrm{Ric}_p(X, X) + O(\|x\|^3)\right) dx^1 \wedge \cdots \wedge dx^n,$$

where $X = \sum_i x^i \frac{\partial}{\partial x_i}$ is the radial vector field.

(2) Give a two-term expansion for the volume of the geodesic ball $B(p, r)$ of center p and radius r as $r \to 0$.

Exercise 2. Let (M, g) be a Riemannian 2-manifold.

(1) Show that if $g = dr^2 + f(r, \theta)^2 \, d\theta^2$ written in polar coordinates, then the Gauss curvature $K = -\frac{1}{f}\frac{\partial^2 f}{\partial r^2}$.

(2) Let $p \in M$ and $C(p, r)$ denote the geodesic circle of center p and radius r. Show that

$$L(C(p, r)) = 2\pi r - \frac{\pi}{3} K_p r^3 + o(r^3),$$

where K_p is the Gauss curvature at p.

Exercise 3. Let (M, g) be a Riemannian manifold and let $f : M \to \mathbf{R}$ be a smooth map. We assume that f vanishes transversally, that is, $d_x f \neq 0$ if $f(x) = 0$. Let us denote by Z the hypersurface $f^{-1}(0)$ and by II its second fundamental form.

(1) Show that for any $x \in Z$, we have $d_x f \circ \mathrm{II}_x = -\nabla_x^2 f_{|T_x Z}$.

(2) Express II only in terms of f.