TD10: CURVATURES, PART II
M1 - DIFFERENTIAL GEOMETRY, 2019-2020

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Solutions of exercise 1.

(1) Since $dV = \det(g_{ij})^{1/2} dx^1 \wedge \cdots \wedge dx^n$ and

$$\begin{aligned}\det(g_{ij})(x) &= 1 - \frac{1}{3}\mathrm{tr}\left(\left(\sum_{kl} R_{iklj}(0)x^k x^l\right)_{ij}\right) + O(\|x\|^3)\\ &= 1 - \frac{1}{3}\sum_{i,k,l} R_{ikli}(0)x^k x^l + O(\|x\|^3)\\ &= 1 - \frac{1}{3}\sum_{k,l} \mathrm{Ric}_{kl}(0)x^k x^l + O(\|x\|^3).\\ &= 1 - \frac{1}{3}\mathrm{Ric}_p(X,X) + O(\|x\|^3),\end{aligned}$$

where $X = \sum x^i \frac{\partial}{\partial x_i}$. Thus we have $dV = \left(1 - \frac{1}{6}\mathrm{Ric}_p(X,X) + O(\|x\|^3)\right) dx^1 \wedge \cdots \wedge dx^n$.
Remark: $\mathrm{Ric}_p(X,X)$ controls "$dV - dV_{eucl}$" to the first order in the direction X (in normal coordinates).

(2) Using the exponential chart at p, we can write

$$\begin{aligned}\mathrm{Vol}(B_M(p,r)) &= \int_{B_M(p,r)} dV\\ &= \int_{B_{\mathbf{R}^n}(0,r)} \sqrt{\det g_{ij}}\, dx^1 \wedge \cdots \wedge dx^n\\ &= \int_{B_{\mathbf{R}^n}(0,r)} \left(1 - \frac{1}{6}Ric_p(x,x) + O(\|x\|^3)\right) dx^1 \wedge \cdots \wedge dx^n\\ &= \mathrm{Vol}(B_{\mathbf{R}^n}(0,r)) - \frac{1}{6}\int_{B_{\mathbf{R}^n}(0,r)} \mathrm{Ric}_p(x,x)\,dx + \int_{B_{\mathbf{R}^n}(0,r)} O(\|x\|^3)\,dx.\end{aligned}$$

We have

$$\int_{B_{\mathbf{R}^n}(0,r)} \|x\|^3\,dx = \int_0^r \rho^{n+2}\,d\rho\,\mathrm{Vol}(\mathbf{S}^{n-1}) = O(r^{n+3}) = \mathrm{Vol}(B_{\mathbf{R}^n}(0,r))O(r^3) \text{ and}$$

$$\int_{B_{\mathbf{R}^n}(0,r)} \mathrm{Ric}_p(x,x)\,dx = \sum_{k,l} \mathrm{Ric}_{kl}(0)\int_{B_{\mathbf{R}^n}(0,r)} x^k x^l\,dx.$$

If $k \neq l$, by symmetry of the domain, we get $\int_{B_{\mathbf{R}^n}(0,r)} x^k x^l\,dx = 0$.
If $k = l$, we have $\int_{B_{\mathbf{R}^n}(0,r)} x^k x^k\,dx = \int_{B_{\mathbf{R}^n}(0,r)} (x^k)^2\,dx$ which is independent of k by symmetry. Moreover, we have

$$\begin{aligned}\frac{1}{n}\int_{B_{\mathbf{R}^n}(0,r)} (x^k)^2\,dx &= \int_{B_{\mathbf{R}^n}(0,r)} \|x\|^2\,dx = \mathrm{Vol}(\mathbf{S}^{n-1})\int_0^r \rho^{n+1}\,d\rho\\ &= \mathrm{Vol}(B_{\mathbf{R}^n}(0,r))\frac{r^2}{n+2}\end{aligned}$$

Hence we get

$$\begin{aligned}\mathrm{Vol}(B_M(0,r)) &= \mathrm{Vol}(B_{\mathbf{R}^n}(0,r))\left(1-\frac{1}{6}\left(\sum_{k,l} Ric_{kl}(0)\delta_{kl}\right)\frac{r^n}{n+2}+O(r^3)\right)\\ &= \mathrm{Vol}(B_{\mathbf{R}^n}(0,r))\left(1-\frac{1}{6}S_p\frac{r^n}{n+2}+O(r^3)\right)\end{aligned}$$

where S_p is the scalar curvature of M at p.

Solutions of exercise 2.

(1) Set $f' = \frac{\partial f}{\partial r}$ and $f'' = \frac{\partial^2 f}{\partial r^2}$. By using the formula seen in the course:

$$R(X,Y,Z,W) = K_g\left(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right)$$

with $X = Z = e_r$ and $Y = W = e_\theta$, we have

$$\begin{aligned}-K_g &= R(X,Y,X,Y)\\ &= g(\nabla_X\nabla_Y X,Y) - g(\nabla_Y\nabla_X X,Y) - g(\nabla_{[X,Y]}X,Y),\\ &= g(\nabla_X(\frac{f'}{f}Y),Y) - 0 - g\left(\nabla_{-\frac{f'}{f}Y}X,Y\right)\\ &= \frac{ff''-(f')^2}{f^2} + \frac{f'}{f}g(\nabla_X Y,Y) + \frac{f'}{f}g(\nabla_Y X,Y)\\ &= \frac{ff''-(f')^2}{f^2} + \frac{(f')^2}{f^2} = \frac{f''}{f},\end{aligned}$$

which gives $K_g = -\frac{f''}{f}$.

(2) Since $C(p,r)$ is the image of the circle of radius r on the tangent space T_pM by the exponential map at p, in polar coordinates, thanks to the expansion of G around p, we have

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{1\le k,l\le 2} R_{iklj}(0)x^k x^l + o(\|x\|^2),\ \forall x \in C_{\mathbf{R}^2}(0,r).$$

In particular, taking $i = j = 2$, we obtain that

$$\frac{f^2}{r^2} = 1 - \frac{r^2}{3}K_p + o(r^2).$$

Therefore, by computation, we get that

$$L(C(p,r)) = \int_0^{2\pi} f(r,\theta)\,d\theta = 2\pi r - \frac{\pi}{3}K_p r^3 + o(r^3).$$

Solutions of exercise 3.

(1) Let $X, Y \in \Gamma(TZ)$ defined on a neighbourhood of $x \in Z$. We can extend X and Y locally into vector fields on a neighbourhood of x. We have $df(Y) \equiv 0$ since $TZ = \mathrm{Ker}(df)$. We can then write

$$\begin{aligned}0 = d(df(Y))(X) &= (\nabla_X df)(Y) + df(\nabla_X Y)\\ &= \nabla^2 f(X,Y) + df\left((\nabla_X Y)^\perp\right)\\ &= \nabla^2 f(X,Y) + df \circ \mathrm{II}(X,Y),\end{aligned}$$

which gives the result.

(2) For any $x \in Z$ and $X, Y \in T_xZ$, we have

$$\mathrm{II}_x(X,Y) \in (T_xZ)^\perp.$$

Since $(T_xZ)^\perp = \mathrm{Span}(\mathrm{grad}_x f)$, we get $\mathrm{II}_x(X,Y) = \alpha\, \mathrm{grad}_x f$ for some $\alpha \in \mathbf{R}$. Moreover, we have

$$\begin{aligned}\alpha\|\nabla_x f\|^2 = \alpha\|\mathrm{grad}_x f\|^2 &= g_x\left(\mathrm{grad}_x f, \mathrm{II}_x(X,Y)\right)\\ &:= d_x f(\mathrm{II}_x(X,Y)) = -\nabla_x^2 f(X,Y).\end{aligned}$$

Therefore, we obtain

$$\mathrm{II}_x = -\frac{\nabla_x^2 f_{|T_xZ}}{\|\nabla_x f\|^2}\nabla_x f.$$