TD11: GAUSS-BONNET'S THEOREM

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

CHIH-KANG HUANG

Exercise 1.

(1) Is there a metric g on the 2-sphere $\mathbf{S}^2$ such that the Gauss curvature K takes negative values at some point?
(2) Is there a metric g on the torus $\mathbf{T}^2$ such that K does not vanish?
(3) Is there a 2-dimensional submanifold M without boundary of $\mathbf{R}^3$ such that K vanishes everywhere on M?
(4) Same question assuming that M is compact without boundary (*Hint:* One can consider Sard's theorem: the set of critical values of a smooth map between manifolds is negligible.)
(5) Is there a compact surface M without boundary in $\mathbf{R}^3$ such that K is negative everywhere?

Exercise 2 (More Gauss curvature)**.**

(1) Recall the Gauss curvature of the unit sphere $\mathbf{S}^2$ embedded in $\mathbf{R}^3$.
(2) Let $M \subset \mathbf{R}^3$ be a compact connected oriented surface. Show that there exists $p \in M$ such that the Gauss curvature at p is positive: $K_p > 0$.
(3) Show that any compact connected oriented surface in $\mathbf{R}^3$ which is not diffeomorphic to the sphere $\mathbf{S}^2$ has its Gauss curvature vanishing at some point.