TD11: GAUSS-BONNET'S THEOREM

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

CHIH-KANG HUANG

Solution of exercise 1.

(1) Yes (try to for instance, locally deform the unit sphere to get a saddle point.)

(2) Thanks to Gauss-Bonnet's theorem, we have $\chi(\mathbf{T}^2) = 0 = \int_{\mathbf{T}^2} K_g dv_g$. If K_g did not vanish, then by continuity of $K_g : \mathbf{T}^2 \to \mathbf{R}$, we would have $\int_{\mathbf{T}^2} K_g dv_g \neq 0$.

(3) Yes, for instance, a plane $P := f^{-1}(0)$ with $f(x,y,z) = z$. Since $\nabla^2 f = D^2 f = 0$, we get $\mathrm{II}_p \equiv 0$ and $K_p = 0$ for every $p \in P$.

Other than planes? Yes a cylinder for example. Let $f(x,y,z) = x^2 + y^2 - 1$ and $\mathcal{C} = f^{-1}(0)$. For $p = (x,y,z)$, we have $d_p f = 2x\,dx + 2y\,dy$ and $\nabla_p^2 f = 2(dx \otimes dx + dy \otimes dy)$.

Let $p \in \mathcal{C}$, then we have $T_p\mathcal{C} = \mathrm{Ker} d_p f = (x,y,0)^\perp = \mathrm{spam}(e_1, e_2)$ with $e_1 = (-y, x, 0)$ and $e_2 = (0,0,1)$. Hence we have $\nabla_p^2 f = \begin{pmatrix} \star & 0 \\ 0 & 0 \end{pmatrix}$ in (e_1, e_2).

Moreover, thanks to the previous exercise in TD10, using the fact that $\mathrm{II}_p = -\frac{\nabla_p^2 f}{\|\nabla_p f\|^2} \nabla_p f$. we get that

$$\det(\nabla_p^2 f) = 0 \text{ iff } \det(\mathrm{II}_p f) = 0 \text{ iff } K_p = 0 \, \forall p \in \mathcal{C}.$$

Remark: One can show that the cylinder is locally isometric to a plane.

(4) Let $N : M \to \mathbf{S}^2$ be a unit (outer) normal vector. We have $d_x N : T_x M \to T_{N(x)}\mathbf{S}^2$ with $N^\perp = T_x M = T_{N(x)}\mathbf{S}^2$ and $K(x) = \det(dN)$.

If $K \equiv 0$, then for every $x \in M$, $\det(d_x N) = 0$ and thus x is critical of N. Hence the set of critival values of N is $N(M)$ and by Sard's theorem , we have $N(M) \neq \mathbf{S}^2$, this contradicts the fact that N is surjective. Indeed, for every $v \in \mathbf{S}^2$, by compacity of M, there exists a smallest t such that $(tv + v^\perp) \cap M \neq \emptyset$. Let $x \in (tv + v^\perp) \cap M$ then by the definition of t, $T_x M = v^\perp$ and $N(x) = v$. (Draw a picture to convince yourself!)

(5) No, as before, for every $v \in \mathbf{S}^2$, consider the smallest t such that $(tv + v^\perp) \cap M \neq \emptyset$.

Let $x \in (tv + v^\perp) \cap M$. Then $v^\perp = T_x M$. Let $f : \mathbf{R}^3 \to \mathbf{R}$ such that locally around x, we have $M = f^{-1}(0)$. Up to translation and rotations, we can assume $x = 0$ and $d_0 f = dz$. We have $K(0) < 0$ and $det(\nabla_0^2 f_{|T_0 M}) < 0$. Therefore, in the propriate basis of $T_0 M = \mathbf{R}^2 \times \{0\}$ we have $\nabla_0^2 f_{|T_0 M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $f(x,y,0) = \frac{1}{2}(x^2 - y^2) + o(\|(x,y)\|^2)$, which is in contradiction with the fact that $(x,y) \mapsto f(x,y,0)$ has constant sign around $(0,0)$ (since locally M only intersects the $T_x M$ at x).

Solution of exercise 2.

(1) 1.

(2) Since M is compact, there exists $p \in M$ such that p is at maximal distance from the origin, that is, $\|p\| = \max_{x \in M} \|x\|$. Up to a rotation and a homothetic transformation, one can assume that $p = (1,0,0)$. In a neighbourhood U of x in $\mathbf{R}^3$, we have $M = f^{-1}(0)$ for some submersion $f : \mathbf{R}^3 \to \mathbf{R}$. Up to multiply f with a constant, one can assume that $\nabla_x f = e_1$.

Let N denote the normal vector on $U \cap M$. the matrix of the differential d_pN in the basis (e_2, e_3) is given by

$$d_pN = \begin{pmatrix} \frac{\partial^2 f}{\partial X_2^2}(p) & \frac{\partial^2 f}{\partial X_2 \partial X_3}(p) \\ \frac{\partial^2 f}{\partial X_2 \partial X_3}(p) & \frac{\partial^2 f}{\partial X_3^2}(p) \end{pmatrix}.$$

We also have $K_p = \det(d_pN)$. In order to prove that $K_p > 0$, one only needs to show that the quadratic form $Q(h_2, h_3) = \frac{\partial^2 f}{\partial X_2^2}(p)h_2^2 + 2\frac{\partial^2 f}{\partial X_2 \partial X_3}(p) + \frac{\partial^2 f}{\partial X_3^2}(p)h_3^2$ is definite positive.

Indeed, for h_2 and h_3 small enough and et $h_1 = \frac{-(h_2^2+h_3^2)}{4}$ such that $(1+h) \notin B(0,1)$. In particular, thanks to the definition of x, we have $M \subset B(0,1)$ and thus $f(1+h) \geq 0$. Finally, using the expansion of f, we get that

$$Q(h_2, h_3) \geq \frac{h_2^2 + h_3^2}{4}.$$

This shows that Q is definite positive.

(3) Thanks to Gauss-Bonnet's theorem, we have

$$\int_M K_g dv_g \leq 0$$

since M is not diffeomorphic to $\mathbf{S}^2$. Therefore the curvature K of M takes negative values at some point. Moreover, thanks to the previous question, K takes non-negative values as well. Hence, by the connectness of M, K vanishes at some point of M.