TD12: (A BIT OF) JACOBI FIELDS

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

CHIH-KANG HUANG

Definition. Let $s \mapsto \gamma_s$ be a family of geodesics. Set $T = \gamma_0'(t)$ the vector field tangent to γ_0 and V *the variation* of (γ_s), the vector field along γ_0 defined by

$$V(t) := \frac{\partial \gamma_s(t)}{\partial s}_{|s=0}.$$

Then V verifies the *Jacobi equation*

$$\nabla_T^2 V + R(V,T)T = 0. \tag{1}$$

Conversely, for any vector field V along γ_0 verifying (1), there exists a family of geodesics γ_s such that V is the variation of γ_s. In this case, we say that V is a *Jacobi field* along γ_0.

Exercise 1 (Application to TD10 Exercise 2-1)**.** Let (M,g) be a Riemannian manifold of dimension 2 such that $g = dr^2 + f(r,\theta)^2\, d\theta^2$ in polar coordinates. Show that the vector field $V = \frac{\partial}{\partial\theta}$ defines a Jacobi field along each line passing through origin, then (re)deduce that the Gauss curvature $K = -\frac{1}{f}\frac{\partial^2 f}{\partial r^2}$.

Exercise 2 (Expansion of metric in normal coordinates)**.** Let (M,g) be a Riemannian manifold and $p \in M$. Let $\mathbf{S}^{n-1}$ denote the unit sphere of T_pM. In polar coordinates $(r,v) \in \mathbf{R}_+ \times \mathbf{S}^{n-1} \mapsto rv \in T_pM$, we write $g = dr^2 + g_r$

(1) For every $v \in \mathbf{S}^{n-1}$ and $w \in T_v\mathbf{S}^{n-1}$, let γ be a path on $\mathbf{S}^{n-1}$ such that $\gamma(0) = v$ and $\gamma'(0) = w$. Let $\gamma_s(t) = \exp_p(t\gamma(s))$ and W be the corresponding variation of γ_s. Show that W is a Jacobi field along γ with initial conditions $W(0) = 0$ and $\nabla_v W(0) = w$.

(2) Check that, thanks to the definition of W, we have $g_r(w) = \|W(r)\|_g^2$.

(3) By computing successive derivatives of the function $S : r \mapsto \|W(r)\|_g^2$, show that we have

$$g_r(w) = |w|^2 r^2 - \frac{1}{3}\left((R(w,v,v,w)\right) r^4 + o(r^4).$$

(4) Deduce that in Cartesian coordinates $(x_1, \dots, x_n)$ on T_pM, we have

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{k,l} R_{iklj}(0) x_k x_l + o\left(|x|^2\right).$$