TD12: (A BIT OF) JACOBI FIELDS

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

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Solution of exercise 1. Set $V = \frac{\partial}{\partial \theta}$. It is clear that V is the variation of the family of geodesics $\gamma_\theta(r) = (r, \theta)$. Therefore, thanks to the Proposition 1, $W(r) = f(r, \theta)e_\theta$ is a Jacobi field along $r \to (r, 0)$. By injecting the expression of $W(r)$ in the Jacobi equation, we get

$$\frac{\partial^2 f}{\partial r^2} + Kf = 0,$$

which gives the result.

Solution of exercise 2. Set $T(t) = \gamma_0'(t)$

(1) We have

$$\nabla_T W(0) = (\nabla_W \gamma_0')(0) = v'(0) = w.$$

(2) Using the exponention chart at p, since

$$d_{rv} \exp_p(w) = \frac{\partial}{\partial s} \gamma_s(r)_{|s=0} = W(r)$$

, we get

$$\begin{aligned} g_r(w) &= \exp_p^*(g)(w) \\ &= g(d_{rv} \exp_p(w)) \\ &= |W(r)|^2. \end{aligned}$$

(3) Set $W' = nabla_T W$, $W'' = \nabla_T \nabla_T W$, etc.... The Jacobi equation can be written as $W'' + R(W, T)T = 0$. By direct computions, we have

$$\begin{aligned} S'(0) &= 0 \\ S''(0) &= 2|w|^2 \\ S^{(3)}(0) &= 6g(W''(0), W'(0)) \\ &= -6g(R(W, T)T(0), w) = 0 \\ S^{(4)}(0) &= 8g(W^{(3)}(0), W'(0)) + 6g(W''(0), W''(0)) \\ . &= 8g(W^{(3)}(0), W'(0)), \end{aligned}$$

since $W''(0) = R(W, T)T(0) = 0$. Moreover, we have

$$\begin{aligned} W'''(0) &= -\nabla_T(R(W, T)T)(0) \\ &= -((\nabla_T R)(W, T)T)(0) - (R(\nabla_T W, T)T)(0) - (R(W, \nabla_T T)T)(0) - (R(W, T)\nabla_T T)(0) \\ &= -R(W'(0), T(0))T(0) \\ &= -R(w, v)v, \end{aligned}$$

which gives $S^4(0) = -8g(R(w, v)v, w) = R(w, v, v, w)$ (covariant version). Finally, thanks to Taylor's formula, we obtain

$$g_r(w) = S(r) = |w|^2 r^2 - \frac{8}{4!} R(w, v, v, w) r^4 + o(r^4).$$

(4) Let $x = \sum_k x_k \frac{\partial}{\partial x_k}$ be a point of T_pM such that $x = rv$ and $y = \sum_i y_i \frac{\partial}{x_i}$ a vector of T_pM with $y = rw$ where $w \in T_v\mathbf{S}^{n-1}$, then we have

$$\begin{aligned}\exp_p^* g_x(y,y) &= |w|^2 r^2 - \frac{8}{4!} R(w,v,v,w) r^4 + o(r^4). \\ &= \langle y, y\rangle - \frac{1}{3}\sum_{i,j,k,l} R_{iklj}(0) y_i x_k x_l y_k) + o(r^4),\end{aligned}$$

which implies

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}\sum_{k,l} R_{iklj}(0) x_k x_l + o(|x|^2).$$