TD02: SOLUTION OF EXERCISE 2.3

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

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Exercise 1. What are smooth curves endowed with an intrinsic metric and without boundary up to isometries?

Solution:

Let (M, g) be a curve without boundary. Since an isometry is also a diffeomorphism, M is either diffeomorphic to $\mathbf{R}$ (if M is not compact) or to the circle $\mathbf{S}^1$ (If M is compact).

(1) First case: We assume that M is diffeomorphic to $\mathbf{R}$. Let $\varphi : \mathbf{R} \to M$ denote the diffeomorphism. In the following, we construct a parametrization of M by arc-length. Let $\gamma : \mathbf{R} \to \mathbf{R}$ be defined by

$$\gamma : t \mapsto \int_0^t \sqrt{g_s\left(d_s\varphi(1), d_s\varphi(1)\right)}\, ds = \int_0^t \|\varphi'(s)\|_{g_s}\, ds.$$

Since γ' is smooth and positive, γ induces a diffeomorphism from $\mathbf{R}$ to its image $I = \gamma(\mathbf{R})$ which is an open interval of $\mathbf{R}$. Let $\psi := \varphi \circ \gamma^{-1} : I \to M$. It remains to show that ψ is an isometry from (I, g_0) to (M, g). Indeed, for any $t \in I$ and $(\alpha, \beta) \in (T_t\mathbf{R})^2 \simeq \mathbf{R}^2$, we have

$$\begin{aligned}
(\psi^* g)_t(\alpha, \beta) &= g_{\psi(t)}\left(d_t\psi(\alpha), d_t\psi(\beta)\right) = \alpha\beta\, g_{\psi(t)}\left(d_t\psi(1), d_t\psi(1)\right) \\
&= \alpha\beta\, g_{\psi(t)}\left(d_{\gamma^{-1}(t)}\varphi((\gamma^{-1})'(t)), d_{\gamma^{-1}(t)}\varphi((\gamma^{-1})'(t))\right) \\
&= \alpha\beta\left((\gamma^{-1})'(t)\right)^2 g_{\varphi\circ\gamma^{-1}(t)}\left(d_{\gamma^{-1}(t)}\varphi(1), d_{\gamma^{-1}(t)}\varphi(1)\right) \\
&= \alpha\beta\left(\frac{1}{\gamma'(\gamma^{-1}(t))}\right)^2 \gamma'(\gamma^{-1}(t))^2 \\
&= \alpha\beta = (g_0)_t(\alpha, \beta).
\end{aligned}$$

Thus (M, g) is isometric to (I, g_0).

(2) Second case: Now assume that M is diffeomorphic to $\mathbf{S}^1$ and let $\varphi : \mathbf{S}^1 \to M$ be a diffeomorphism. We identify $\mathbf{S}^1$ to the unit circle in $\mathbf{C}$ and let $\tilde\varphi : \mathbf{R} \to M$ be defined by $\tilde\varphi(t) = \varphi(e^{it})$. Let $L := \int_0^{2\pi} \|\tilde\varphi'(t)\|_{g_{\tilde\varphi(t)}}\, dt$ denotes the total length of M. Indeed, we have that

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{g_{\varphi(e^{i\theta})}\left(d_{e^{i\theta}}\varphi\left(\frac{\partial}{\partial\theta}\right), d_{e^{i\theta}}\varphi\left(\frac{\partial}{\partial\theta}\right)\right)}\, d\theta \\
&= \int_0^{2\pi} \sqrt{(\varphi^* g)_{e^{i\theta}}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right)}\, d\theta = \int_{\mathbf{S}^1} \mathrm{vol}_{\varphi^* g} = \int_M \mathrm{vol}_g.
\end{aligned}$$

Let us now prove that (M, g) is isometric to $(\mathbf{R}/L\mathbf{Z}, g_0)$ where by abuse of notation, g_0 is the induced quotient metric by $(\mathbf{R}, g_0)$.

Let $\gamma : \mathbf{R} \to \mathbf{R}$ be defined by

$$\gamma : t \mapsto \int_0^t \|\tilde\varphi'(s)\|_{g_s}\, ds,$$

Since γ' is smooth and positive and $\lim_{t\to\pm+\infty} \gamma(t) = \pm\infty$, γ is a diffeomorphism from $\mathbf{R}$ to $\mathbf{R}$. Set $\tilde\psi := \tilde\varphi \circ \gamma^{-1}$, since $\tilde\varphi$ is 2π-periodic, $\tilde\psi$ is L-periodic and thus induces a smooth map $\psi : \mathbf{R}/L\mathbf{Z} \to M$. By direct computations, one can show that ψ is a smooth bijection. Moreover, locally in charts ψ coincides with $\tilde\psi = \tilde\varphi \circ \gamma^{-1}$, hence its differential is not zero, and ψ is a diffeomorphism. Let $s \in \mathbf{R}$ such that $x = \overline{s}$, then by the same computations as in the case of $\mathbf{R}$, one can show that $(\psi^* g)_x = g_{\tilde{\psi}(s)}\left(d_s\tilde{\psi}\cdot, d_s\tilde{\psi}\cdot\right) = (g_0)_x$.

Conclusion: A curve without boundary is isometric to (I, g_0) where I is an open interval of $\mathbf{R}$, or is isometric to $(\mathbf{R}/L\mathbf{Z}, g_0)$ if compact of length $L > 0$.