TD03: HYPERBOLIC SPACES AND CONFORMAL MAPS

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

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Exercise 1 (Hyperbolic spaces)**.** Let $B = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + \cdots + dx^n \otimes dx^n$ denote the standard Lorentz form on $\mathbf{R}^{n+1}$. Let $\mathcal{H}^n$ denote the set $\{x \in \mathbf{R}^{n+1} \mid B(x,x) = -1, x_0 > 0\}$. We also denote by $\mathbf{H}^n$ the half-space $\{x \in \mathbf{R}^n \mid x_n > 0\}$ and by $\mathbf{D}^n$ the unit open ball in $\mathbf{R}^n$. Prove that the following Riemannian manifolds are isometric to one another:

- $\mathcal{H}^n$ endowed with the restriction of B,
- $\mathbf{H}^n$ with the metric $\frac{1}{|x_n|^2} \sum_{i=1}^n dx^i \otimes dx^i$,
- $\mathbf{D}^n$ with the metric $\frac{4}{(1-\|x\|^2)^2} \sum_{i=1}^n dx^i \otimes dx^i$

Exercise 2 (Hyperbolic half-plane and Poincaré disc)**.** In this exercise, we simply denote by $\mathbf{H}$ the hyperbolic half-plane $\mathbf{H}^2$ and by $\mathbf{D}$ the hyperbolic disc $\mathbf{D}^2$ (also called *Poincaré disc*).

(1) Check that conformal diffeomorphisms of $\mathbf{D}$ (resp. $\mathbf{H}$) preserving the orientation are biholomorphisms.
(2) Describe the conformal diffeomorphisms of $\mathbf{D}$ (resp. $\mathbf{H}$). Which one are isometries?

Exercise 3 (Connections on $\mathbf{R}$)**.** Let g be a Riemannian metric on $\mathbf{R}$ defined by $g_x := e^{-x^2}(g_0)_x$ for any $x \in \mathbf{R}$. Let ∇_0 (resp. ∇) denote the Levi-Civita connection of $T\mathbf{R}$ associated with g_0 (resp. g). Compute $\nabla_0 \frac{\partial}{\partial x}$ and $\nabla \frac{\partial}{\partial x}$.