TD04: CONNECTIONS ON VECTOR BUNDLES

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

CHIH-KANG HUANG

Exercise 1 (Warm up)**.** Given two local coordinates $(x_1, \dots, x_i, \dots, x_n)$ and $(x'_1, \dots, x'_j, \dots, x'_n)$ on a open set U of M such that $x_i = a_{ij} x'^j$ for some matrix of functions $A = (a_{ij})_{1 \le i,j \le n}$. Let ∇ be a linear connection on M. Let Γ_{ij}^k and $(\Gamma')_{ij}^k$ denote the Christoffel symbols of ∇ with respect to these two coordinates. Express Γ_{ij}^k in terms of $(\Gamma')_{ij}^k$ and A.

Exercise 2. Let g and g' be two conformal metrics on M. Let ∇ and ∇' denote respectly the Levi-Civita connection of g and g'. Show that we have

$$\nabla'_X Y = \nabla_X Y + d\lambda(X)Y + d\lambda(Y)X - g(X,Y)\nabla\lambda,$$

where $\lambda : M \to \mathbf{R}$ is smooth map satisfying $g' = e^{2\lambda} g$.

Exercise 3 (Hessian and symmetric connection)**.** Let M be a n-manifold and let $f : M \to \mathbf{R}$ be a smooth map.

(1) Is there an intrinsic notion of second differential of f that would read as $D^2(f \circ \varphi^{-1})$ in any local chart (U, φ)?

(2) Let ∇ be a linear connection on M. Show that ∇ is symmetric if and only if its Christoffel symbols in any local chart are symmetric, that is, $\Gamma_{ij}^k = \Gamma_{ji}^k$ for any $i, j, k \in \{1, \dots, n\}$.

(3) We define the *covariant Hessian* $\nabla^2 f$ of f by:

$$\nabla^2 f(X,Y) := (\nabla_X(df)) \cdot Y, \, \forall X, Y \in \Gamma(TM).$$

Show that $\nabla^2 f(X,Y) = X \cdot (Y \cdot f) - (\nabla_X Y) \cdot f$.

(4) Show that ∇ is symmetric if and only if $\nabla^2 f$ is a symmetric $(2,0)$-tensor field for any $f \in C^\infty(M)$.

Exercise 4 (Computations of Christoffel symbols)**.**

(1) Let (M, g) be a Riemannian n-manifold. We denote by $(x_1, \dots, x_n)$ local coordinates on a open subset U of M and by $G = (g_{ij})_{1 \le i,j \le n}$ the matrix of g in these coordinates. Show that we have, for any $i, j, k \in \{1, \dots, n\}$,

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^n g^{kl} \left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right)$$

where $G^{-1} := (g^{kl})_{1 \le k,l \le n}$ is the invertible matrix of G.

(2) Consider the hyperbolic half-plane $\mathbf{H}^2 := \{(x,y) \in \mathbf{R}^2 \mid y > 0\}$ endowed with the metric

$$g_{(x,y)} = \frac{1}{y^2} \left(dx \otimes dx + dy \otimes dy \right).$$

Let ∇ denote the associated Levi-Civita connection. Compute $\nabla \frac{\partial}{\partial x}$ and $\nabla \frac{\partial}{\partial y}$.

(3) Consider the Poincaré's disc $\mathbf{D}^2 := \{(x,y) \in \mathbf{R}^2 \mid y > 0\}$ endowed with the metric

$$g_{(x,y)} = \frac{4}{1 - (x^2 + y^2)} \left(dx \otimes dx + dy \otimes dy \right).$$

Let ∇ denote the associated Levi-Civita connection. Compute the covariant derivatives $\nabla \frac{\partial}{\partial r}$ and $\nabla \frac{\partial}{\partial \theta}$ associated with the polar coordinates.