TD05: GEODESICS

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

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Exercise 1. Let $E \to M$ be a vector bundle over a manifold M. Let $x_0 \in M$.

(1) Let ∇ be a connection on E. For any $y \in E_{x_0}$ construct a smooth section $s \in \Gamma(E)$ such that $s(x_0) = y$ and ∇s vanishes at x_0?

(2) Conversely, let s be a smooth section of the vector bundle $E \to M$, does there always exist a connection ∇ on E such that ∇s vanishes on a neighborhood of x_0? (*Hint*: you may deal separately with the cases in which $s(x_0) = 0$ and $s(x_0) \neq 0$)

Exercise 2 (Normal coordinates)**.** Let (M^n, g) be a Riemannian manifold and let $p \in M$. Show that in any normal coordinates $x = (x_1, \ldots, x_n)$ centered at p, we have that

$$g_{ij}(x) = \mathrm{Id}_n + O(\|x\|^2),\ g_{ij,\,k}(0) = 0 \text{ and } \Gamma_{ij}^k(x) = O(\|x\|)$$

for any $i, j, k \in \{1, \ldots, n\}$.

Exercise 3 (Difference tensor and geodesics)**.** Let ∇ and ∇' be two linear connections on M. The *difference tensor* between ∇ and ∇' is a $(2,1)$-tensor field A defined by:

$$A(X, Y) := \nabla'_X Y - \nabla_X Y, \quad \forall X, Y \in \Gamma(TM).$$

(1) Show that ∇ and ∇' have the same torsion tensor if and only if their difference tensor is symmetric, i.e., $A(X, Y) = A(Y, X)$.

(2) Show that ∇ and ∇' determine the same geodesics if and only if their difference tensor is antisymmetric, i.e., $A(X, Y) = -A(Y, X)$.

Exercise 4. Let $f : M \to N$ be a diffeomorphism between two connected Riemannian manifolds (M, g) and (N, h).

(1) Show that if f is an isometry, then f is geodesic-preserving. We say that f is a *geodesic map*.

(2) Assume that f is a conformal map, is f necessarily a geodesic map?

(3) Assume that f is a conformal geodesic map, show that f is a *scaled isometry*, i.e, $g = \lambda f^* h$ for some positive constant $\lambda > 0$.