TD07: COMPLETE RIEMANNIAN MANIFOLDS AND RIEMANN TENSORS

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

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Exercise 1 (Completeness of submanifolds)**.** Let (M, g) be a complete Riemannian manifold. Show that a submanifold N of M is complete if, and only if N is a closed subset of M.

Exercise 2. Let (M, g) be a complete Riemannian manifold. Let G be a group of isometries acting on M without fixed points. We endow G with the uniform topology (topology of uniform convergence on compact sets) and we assume that G is discrete.

(1) Show that the orbit space, denoted by $N := G \setminus M$, endowed with a quotient distance d_N (to be defined), is a complete Riemannian manifold.
(2) Compute the injectivity radius of N at any point $p \in M$ in terms of the injectivity radius of M and d_N.

Exercise 3. Let E and E' be two smooth vector bundles over a manifold M. Let $F : \Gamma(E) \to \Gamma(E')$ be a $C^\infty(M)$-linear map.

(1) Show that there exists a unique $f : E \to E'$ which is a bundle map over M and satisfies:
$$F(s)(x) = f(s(x)), \qquad \forall s \in \Gamma(E),\ x \in M. \tag{$*$}$$
(2) Check that f defines an element of $\Gamma(E^* \otimes E')$.
(3) Conversely, check that a section $f \in \Gamma(E^* \otimes E')$ defines a unique $C^\infty(M)$-linear map $F : \Gamma(E) \to \Gamma(E')$ satisfying $(*)$.

Similarly, $\Gamma\left(\bigwedge^k T^*M \otimes_{\mathbf{R}} E\right)$ is isomorphic to $\Omega^k(M, E) := \Omega^k(M) \otimes_{C^\infty(M)} \Gamma(E)$, the space of k-forms with values in E.