TD08: RIEMANN TENSOR

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

CHIH-KANG HUANG

Exercise 1. Let E and E' be two smooth vector bundles over a manifold M. Let $F : \Gamma(E) \to \Gamma(E')$ be a $C^\infty(M)$-linear map.

(1) Show that there exists a unique $f : E \to E'$ which is a bundle map over M and satisfies:
$$\forall s \in \Gamma(E), \forall x \in M,\ F(s)(x) = f(s(x)). \tag{1}$$
(2) Check that f defines an element of $\Gamma(E^* \otimes E')$.
(3) Conversely, check that a section $f \in \Gamma(E^* \otimes E')$ defines a unique $C^\infty(M)$-linear map $F : \Gamma(E) \to \Gamma(E')$ satisfying (1).
(4) Conclusion.

Remark: Similarly, $\Gamma\left(\bigwedge^k T^*M \otimes_{\mathbf{R}} E\right)$ is isomorphic to $\Omega^k(M,E) := \Omega^k(M) \otimes_{C^\infty(M)} \Gamma(E)$, the space of k-forms with values in E.

Exercise 2. Let ∇ be a connection on a smooth vector bundle $E \to M$. We extend $\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E)$ by the Leibniz rule:
$$\forall \alpha \in \Omega^k(M), \forall s \in \Gamma(E), \forall x \in M, \quad \nabla_x(\alpha \otimes s) = d_x\alpha \otimes s(x) + (-1)^k \alpha_x \wedge \nabla_x s$$
and by $\mathbf{R}$-linearity.

(1) Show that $\nabla \circ \nabla$ is $C^\infty(M)$-linear from $\Omega^0(M,E)$ to $\Omega^2(M,E)$.
(2) Show that, by using the previous exercise, $\nabla \circ \nabla$ defines a section $R \in \Omega^2(M, End(E))$. We call R the *curvature* of (E, ∇).
(3) Let $(x_1, \dots, x_n)$ be local coordinates on M and $(e_1, \dots, e_r)$ be a local frame for E defined on the same open set U. We denote by (Γ_{ij}^k) the Christoffel symbols of ∇ in these coordinates. We also define (R_{ijk}^l) by:
$$\forall i,j \in \{1,\dots,n\},\ \forall k \in \{1,\dots,r\},\ R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) e_k = \sum_{l=1}^r R_{ijk}^l e_l.$$
Check that, for any $i,j \in \{1,\dots,n\}$ and any $k,l \in \{1,\dots,r\}$, we have
$$R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial x_i} - \frac{\partial \Gamma_{ik}^l}{\partial x_j} + \sum_{m=1}^r \Gamma_{jk}^m \Gamma_{im}^l - \sum_{m=1}^r \Gamma_{ik}^m \Gamma_{jm}^l.$$
(4) Check that for any $X, Y \in \Gamma(TM)$ and any $s \in \Gamma(E)$, we have
$$\nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s = R(X,Y)s.$$

Exercise 3 (Riemann tensors in dimension 2 and 3)**.** Let (M,g) be a Riemannian n-manifold. Let R be a $(4,0)$-tensor defined by
$$\mathrm{R}(X,Y,W,Z) = g(R(X,Y)W,Z), \forall X,Y,W,Z \in \Gamma(TM),$$
where R denotes the Riemann curvature tensor on (M,g). We also denote the Ricci tensor by Ric and the scalar curvature by S.

(1) Recall briefly the definitions of Ric and S. Show that in the case $n = 2$, R is given by the scalar curvature $\mathbf{S}$ and that, in terms of local coordinates, we can write
$$\mathrm{R}_{ijkl} = \frac{\mathrm{S}}{2}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right), \forall i,j,k,l \in \{1,2\}.$$

(2) Show that in $n = 3$, R is given by the Ricci tensor:

$\mathrm{R}_{ijkl} = f(\mathrm{Ric}_{ik})g_{jl} - f(\mathrm{Ric}_{il})g_{jk} + f(\mathrm{Ric}_{jl})g_{ik} - f(\mathrm{Ric}_{jk})g_{il}, \forall i, j, k, l \in \{1, 2, 3\}$

where $f(\mathrm{Ric}_{ij}) = \mathrm{Ric}_{ij} - \frac{1}{4}\mathrm{S}g_{ij}$.