TD08: RIEMANN TENSOR
M1 - DIFFERENTIAL GEOMETRY, 2019-2020

CHIH-KANG HUANG

Solutions of Exercise 1.

(1) Let $y \in E_x$. There exists $s \in \Gamma(E)$ s.t. $s(x) = y$. We define $f(y) = (F(s))(x) \in E'_x$. It remains to show that f is well-defined. (And thus $f_{|E_x} : E_x \to E'_x$ satisfies (1)). Let $(e_1, \ldots, e_k)$ be a local frame for E, defined on a neighborhood U of x. Let $\chi : M \to [0,1]$ smooth s.t. $\overline{\text{supp}}(\chi) \subset U$ and $\chi(x) = 1$. By $C^\infty(M)$-linearity, we have $F(\chi s) = \chi F(s)$. We can also write $s = \sum \varphi_i e_i$ on U with $\varphi_x(x) = y_i$ and $y = \sum y_i e_i(x)$. Therefore, we get that

$$\begin{aligned} F(s)(x) = F(\chi s)(x) &= F\left(\sum \varphi_i \chi e_i\right)(x) \\ &= \sum \varphi_x(x) F(\chi e_i)(x) = \sum y_i F(\chi e_i)(x) = \sum y_i F(e_i)(x), \end{aligned}$$

which depends only on $y = s(x)$ and is linear in y. Thus f is well-defined and $f_{|E_x} \in End(E_x, E'_x)$.

(2) $x \mapsto f_{|E_x}$ is a section of $End(E_x, E'_x) \simeq E^* \otimes E'$.
Let $(e_1, \ldots, e_k)$ be a local frame of E over U and $(e'_1, \ldots, e'_l)$ be a local frame of E' over U' then $((e^j)^* \otimes e'_i)$ form a local frame for $E^* \otimes E'$ and we have

$$f \circ e_j = F(e_j) := \sum F_{ij} e'_i.$$

Hence $f = \sum F_{ij} e^j \otimes e'_i$ where $(F_{ij})_{1 \leq i \leq k}$ are the components of $F(e_j)$ thus smooth, which gives $f \in \Gamma(E^* \otimes E')$.

(3) Let $f \in \Gamma(E^* \otimes E') \simeq \Gamma(End(E, E'))$. We set

$$\begin{aligned} F : \Gamma(E) &\to \Gamma(E') \\ s &\mapsto f \circ s. \end{aligned}$$

One can check that F is $C^\infty(M)$-linear and satisfies (1).

(4) Thanks to the previous questions, we have defined a canonical isomorphism of $C^\infty(M)$-modules between $Hom(\Gamma(E), \Gamma(E'))$ and $\Gamma(E^* \otimes E)$.

Solutions of Exercise 2.

(1) Let $f \in C^\infty(M)$ and $s \in \Gamma(E)$. By Leibniz's rule, we have

$$\nabla_x(fs) = d_x f \otimes s(x) + f(x)\nabla_x s.$$

In a local frame $(e_1, \ldots, e_k)$ over $U \ni x$, we can write $\nabla s = \sum_{i=1}^k \alpha_i \otimes e_i$ with $\alpha_i \in \Omega^1(U)$. We have that

$$\begin{aligned} \nabla_x(\nabla(fs)) &= \nabla_x(df \otimes s + f\nabla s) \\ &= d_x(df) \otimes s(x) - d_x f \wedge \nabla_x s + \sum d_x(f\alpha_i) \otimes e_i(x) - f\alpha_i \wedge \nabla_x e_i \\ &= -d_x f \wedge \nabla_x s + \sum d_x(f\alpha_i) \otimes e_i(x) - f\alpha_i \wedge \nabla_x e_i \end{aligned}$$

Since $d_x(f\alpha_i) = d_x f \wedge \alpha_i + f d\alpha_i$, we get by computation that

$$\nabla_x(\nabla(fs)) = f \sum d\alpha_i \otimes e_i - \alpha_i \wedge \nabla_x e_i = f(x)\nabla_x(\nabla s).$$

Thus $\nabla \circ \nabla(fs) = f\nabla \circ \nabla s$.

(2) $\nabla^2 : \Gamma(E) \simeq \Omega^0(M,E) \to \Omega^2(M,E) \simeq \Gamma(\bigwedge^2 T^*M \otimes E)$ is $C^\infty(M)$-linear. By Exercise 1, there exists a unique $R \in \Gamma(End(E, \bigwedge^2 T^*M \otimes E)$ s.t

$$(\nabla^2(s))(x) = R(s(x)),\ \forall s \in \Gamma(E).$$

Since we have

$$End(E, \bigwedge^2 T^*M \otimes End(E)) \simeq \bigwedge^2 T^*M \otimes E \otimes E^* \simeq \bigwedge^2 T^*M \otimes End(E).$$

We obtain $R \in \Gamma(\bigwedge^2 T^*M \otimes End(E)) \simeq \Omega^2(M) \otimes \Gamma(End(E)) =: \Omega^2(M, End(E))$.

(3) Thanks to the fact that

$$\nabla e_k = \sum_j \sum_l \Gamma^l_{jk} dx^j \otimes e_l,$$

we can write

$$\begin{aligned} R(\cdot,\cdot)e_k := \nabla(\nabla e_k) &= \sum_{j,l} \nabla\left(\Gamma^l_{jk} dx^j \otimes e_l\right) \\ &= \sum_{j,l} d\left(\Gamma^l_{jk} dx^j\right) \otimes e_l - \sum_{j,m} \Gamma^m_{jk} dx^j \wedge \nabla e_m \\ &= \sum_{i,j,l} \frac{\partial \Gamma^l_{jk}}{\partial x_i} dx^i \wedge dx^j \otimes e_l - \sum_{j,m} \Gamma^m_{jk} dx^j \wedge \sum_{i,l} \Gamma^l_{im} dx^i \otimes e_l \\ &= \sum_{i,j,l} \left(\frac{\partial \Gamma^l_{jk}}{\partial x_i} + \sum_m \Gamma^m_{jk} \Gamma^l_{im} \right) dx^i \wedge dx^j \otimes e_l. \end{aligned}$$

Hence we get

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) e_k = \sum_l \left(\frac{\partial \Gamma^l_{jk}}{\partial x_i} - \frac{\partial \Gamma^l_{ik}}{\partial x_j} + \sum_{m=1}^r \Gamma^m_{jk}\Gamma^l_{im} - \sum_{m=1}^r \Gamma^m_{ik}\Gamma^l_{jm} \right) e_l.$$

Solutions of Exercise 3. *Hint:* This is a relatively classical exercise. Using the symmetric and skew-symmetric properties, one can first show that the family of tensor coefficients (R_{ijkl}) is of rank $\frac{1}{12}n^2(n^2-1)$.