TD09: CURVATURES, PART I
M1 - DIFFERENTIAL GEOMETRY, 2019-2020

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Exercise 1. Compute the Riemann, Ricci and scalar curvatures of the following Riemmannian manifolds with their standard metric:

- Euclidean space $\mathbf{R}^n$,
- Unit n-sphere $\mathbf{S}^n$,
- Flat torus $\mathbf{T}^n$,
- Poincaré's Disc $(\mathbf{D}, g_{\mathbf{D}})$.

Exercise 2. Let N denote the North pole of $\mathbf{S}^2$ and $\rho \in (0, \pi)$.

(1) Compute the volume of the geodesic ball $B_{\mathbf{S}^2}(N, \rho)$. Compare it to the volume of the Euclidean ball of radius ρ in $\mathbf{R}^2$.
(2) Compute the length of the circle $C_{\mathbf{S}^2}(N, \rho)$. Also give its expansion at $\rho = 0$.
(3) Let γ_1 and γ_2 be two geodesics parametrized by arc length on $\mathbf{S}^2$ such that $\gamma_i(0) = N$. We denote $v_i = \gamma_i'(0)$, compute the distance between $\gamma_1(t)$ and $\gamma_2(t)$ for $t \in (-\pi, \pi)$. Compare the distance to its Euclidean analogue.

Exercise 3 (Gauss curvatures of homothetic/conformal metrics)**.** Let M be a 2-manifold and g_1, g_2 two Riemmannian metrics on M.

(1) Show that if there exists $\lambda > 0$ such that $g_2 = \lambda^2 g_1$, then we have $K_2 = \lambda^{-2}\lambda_1$ where K_1 and K_2 respectly denotes the Gauss curvature of (M, g_1) and (M, g_2).
(2) (Liouville's equation) Show that if g_1 and g_2 are conformal: $g_2 = e^{2f} g_1$ for some smooth function f on M, then we have
$$K_2 = e^{-2f}(K + \Delta f) \text{ on } M.$$