TD09: CURVATURES, PART I

M1 - DIFFERENTIAL GEOMETRY, 2019-2020

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Solution of exercise 1. Recall that

$$R^l_{ijk} = \frac{\partial \Gamma^l_{jk}}{\partial x_i} - \frac{\partial \Gamma^l_{ik}}{\partial x_j} + \sum_m \Gamma^m_{jk}\Gamma^l_{im} - \Gamma^m_{ik}\Gamma^l_{jm}).$$

(1) Since $\Gamma^k_{ij} \equiv 0$, $R \equiv \mathrm{Ric} \equiv S \equiv 0$.

(2) Let $p \in \mathbf{S}^n$ and $(e_1, \dots, e_n)$ an orthonormal basis of $T_p\mathbf{S}^n$. Without lost of generality, one can assume that $(p, e_1, \dots, e_n)$ is the canonical basis in $\mathbf{R}^{n+1}$. Let

$$F : (x_1, \dots, x_n) \mapsto \left(\sqrt{1 - \|x\|^2}, x_1, \dots, x_n\right)$$

be local coordinates such that $e_i = \frac{\partial}{\partial x_i}(0)$. By direct computation, we get that

$$g_{ij}(x) = \delta_{ij} + x_i x_j + O\left(\|x\|^4\right), \ \frac{\partial g_{ij}}{\partial x_k}(0) = 0 \text{ and } \frac{\partial^2 g_{ij}}{\partial x_k \partial x_l}(0) = \delta_{ki}\delta_{lj} + \delta_{il}\delta_{jk}.$$

Moreover, using

$$\Gamma^k_{ij}(x) = \frac{1}{2}\sum_m g^{km}\left(\frac{\partial g_{im}}{\partial x_j} + \frac{\partial g_{jm}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_m}\right), \tag{1}$$

we get that

$$R_{ijkl}(0) = \frac{\partial \Gamma^l_{jk}}{\partial x_i}(0) - \frac{\partial \Gamma^l_{ik}}{\partial x_j}(0) = \delta_{il} - \delta_{ik}\delta_{jl}.$$

Consequently, we obtain that

$$R_p = \sum_{i<j, k<j} R_{ijkl}(0) dx^i \wedge dx^j \otimes dx^k \zeta dx^l = -\sum_{i<j} dx^i \wedge dx^j \otimes dx^i \wedge dx^j.$$

Since $Ric_{jk}(0) = \sum_i R^i_{ijk}(0) = (n-1)\delta_{jk}$, we also have $Ric_p = (n-1)g$ and $S(p) = n(n-1)$.

(3) Same as in $\mathbf{R}^n$, since the flat torus is locally isometric to $\mathbf{R}^n$.

(4) By direct computation and the fact that $\forall x \in \mathbf{D}$, $g_{ij}(x) = \delta_{ij}\left(1 + \frac{\|x\|^2}{2} + O(\|x\|^4)\right)$i, we get that $R_{ijkl}(0) = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$, $Ric = -g_{\mathbf{D}}$ and $S = -2$.

Solution of exercise 2.

(1) Using spherical coordinates , we have

$$\mathrm{Vol}(B(N, \rho)) = \int_{\theta=0}^{l}\int_{\varphi=0}^{2\pi} \sin\theta \, d\theta d\varphi = 2\pi(1 - \cos\rho).$$

Moreover we have

$$\mathrm{Vol}(B(N, \rho)) = \mathrm{Vol}(B_{\mathbf{R}^2}(0, \rho)) = -2\pi\left(\cos\rho - (1 - \rho^2/2\right) < 0$$

for $\rho > 0$ small enought. Hence $\mathrm{Vol}(B_{\mathbf{S}^2}(N, \rho)) < \mathrm{Vol}\left(B_{\mathbf{R}^2}(0, \rho)\right)$.

(2) Let $\gamma : [0, 2\pi] \to \mathbf{R}^3$ with $\gamma(\theta) = (\sin\rho\cos\theta, \sin\rho\sin\theta, \cos\rho)$ be a parametrization of $C(N, \rho)$. We have $L(C(N, \rho)) = 2\pi\sin\rho$.

(3) Set $\gamma_i(t) = \exp_N(tv_i) = \cos tp + \sin tv_i$ and $\alpha = d(\gamma_1(t), \gamma_2(t))$. Since α is also the angle between $\overline{O\gamma_1(t)}$ and $\overline{O\gamma_2(t)}$ (draw the corresponding figure!) We have $\cos(\alpha) = \langle\gamma_1(t), \gamma_2(t)\rangle$ and thus $\alpha = \arccos(1 - (\sin t)^2(1 - \langle v_1, v_2\rangle)$. One can also show that $d_{\mathbf{S}^2}(\gamma_1(t), \gamma_2(t)) \leq d_{\mathbf{R}^2}(\tilde{\gamma}_1(t), \tilde{\gamma}_2(t))$.

Solution of exercise 3.

(1) For convenience reasons, we write $g = g_2$ and $\tilde{g} = g_1$. We then have $g = \lambda^2 \tilde{g}$. Thanks to the previous formulae on Riemann tensor and Christoffel symbols (1), we deduce that $\Gamma_{ij}^k = \tilde{\Gamma}_{ij}^k$, $R_{ijk}^l = \tilde{R}_{ijk}^l$ and $R_{ijkl} = \lambda^2 \tilde{R}_{ijkl}$. Let $p \in M$ and (X, Y) a basis of T_pM. We can write

$$K_p = \frac{R(X,Y,Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} = \frac{\frac{1}{\lambda^2}\tilde{R}(X,Y,Y,X)}{\frac{1}{\lambda^4}\left[\tilde{g}(X,X)\tilde{g}(Y,Y) - \tilde{g}(X,Y)^2\right]} = \lambda^2 \tilde{K}_p.$$

Remark: One can also recall that the associated Levi-Civita connexions are identics, thus have the same Christoffel symbols and Riemann tensors, thanks to the exercise in previous TD.

(2) *Hint:* For any (X, Y) orthonormal basis for the metric g, we can show that $|\nabla f|^2 = (\nabla_X f)^2 + (\nabla_Y f)^2$ and

$$-\nabla^2_{X,X} f - \nabla^2_{Y,Y} f = -\text{trace}(\nabla^2 f)$$

which gives the Laplcian of f. The result can be obtained in an analogue way as in the previous question.