## Ex 1 :

- « Reminder » , $(X,x)$ pointed top. space , $n \geq 1$

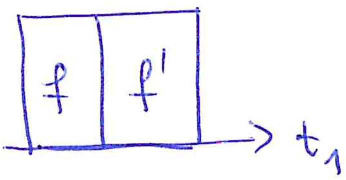
Then, $\pi_n(X,x) = [(I^n, \partial I^n), (X,x)]$

$$= \{ f : I^n \longrightarrow X \mid f(\partial I^n) = \{x\} \} / \sim$$

$f \sim f'$ if there is $H : I^n \times I \longrightarrow X$

such that $H(-,0) = f$
$H(-,1) = g$
$\forall s \in \partial I^n, \forall t \in I$, $H(s,t) = x$.

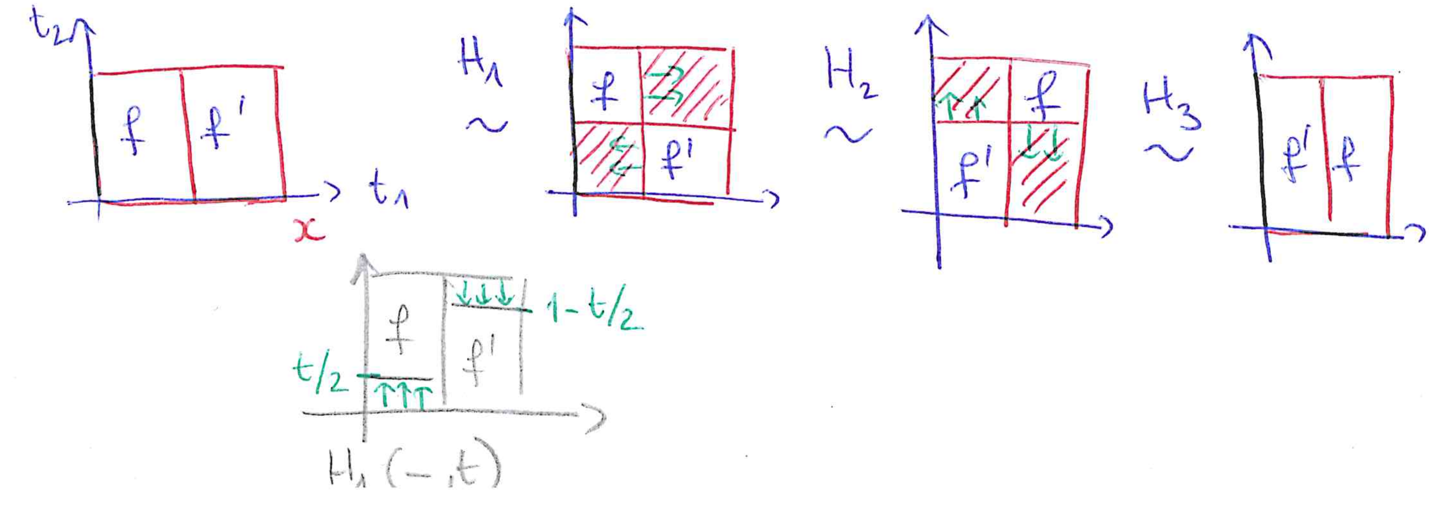
We set
$$[f]+[f'] = \left[ \begin{array}{ccl} I^n & \longrightarrow & X \\ (t_1, t_2, \dots, t_n) & \longmapsto & \begin{cases} f(2t_1, t_2, \dots, t_n) & \text{if } t_1 \leq 1/2 \\ f(2t_1 - 1, t_2, \dots, t_n) & \text{if } t_1 \geq 1/2 \end{cases} \end{array} \right]$$




which defines a group law on $\pi_n(X,x)$.

Claim: if $n \geq 2$, this group law is commutative.

Proof 1 : We find homotopies :



We give a formula for $H_1$ (exercise: find formulas for $H_2$ & $H_3$)

$$H_1 : I^n \times I \longrightarrow X$$

$$(t_1, \dots, t_n, t) \longmapsto \begin{cases} f\left(2t_1, \dfrac{t_2 - t/2}{1-t/2}, t_3, \dots, t_n\right) & \text{if } \left[\begin{array}{l} 0 \le t_1 \le 1/2 \\ t/2 \le t_2 \end{array}\right. \\ f'\left(2t_1 - 1, \dfrac{t_2}{1-t/2}, \dots\right) & \text{if } \left[\begin{array}{l} t_1 \ge 1/2 \\ t_2 \le 1 - t/2 \end{array}\right. \\ x & \text{otherwise} \end{cases}$$

Proof 2: Eckman - Hilton principle:

$X$ set endowed with $* : X \times X \longrightarrow X$ and $\cdot : X \times X \longrightarrow X$ such that each operation has a unit (denoted $1_*$ and $1_\cdot$),

and $\forall x, x', y, y' \in X$, $\quad (x \cdot x') * (y \cdot y') \overset{(1)}{=} (x * y) \cdot (x' * y')$

Then $* = \cdot$ & $\cdot$ defines a com. grp law on X.

Denote $\quad x * y = (x \quad y)$

$$x \cdot y = \begin{pmatrix} x \\ y \end{pmatrix}$$

(1) states:

$$\left(\begin{pmatrix} x \\ x' \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}\right) = \begin{pmatrix} (x \quad y) \\ (x' \quad y') \end{pmatrix}$$

a) $1_\cdot = 1_*$ : we have $1_\cdot = (1_\cdot \quad 1_*) = \begin{pmatrix} (1_\cdot \quad 1_*) \\ (1_* \quad 1_\cdot) \end{pmatrix} = \left(\begin{pmatrix} 1_\cdot \\ 1_* \end{pmatrix} \begin{pmatrix} 1_* \\ 1_\cdot \end{pmatrix}\right)$

$$= (1_* \quad 1_*) = 1_*$$

Denote $1 = 1_\cdot = 1_*$

b) $x * y = (x \quad y) = \left(\begin{pmatrix} x \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ y \end{pmatrix}\right) = \begin{pmatrix} (x \quad 1) \\ (1 \quad y) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = x \cdot y$

Thus $* = \cdot$

$$x \cdot y = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (1 \quad x) \\ (y \quad 1) \end{pmatrix} = \left(\begin{pmatrix} 1 \\ y \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}\right) = (y \quad x) = y * x$$

thus $* = \cdot$ is commutative.

(1) $\Rightarrow$ associativity. Thus $*$ is a com. grp law over X.

Apply EH-principle to

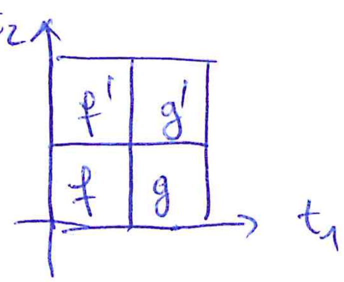
$$* : \pi_n(X,x) \times \pi_n(X,x) \longrightarrow \pi_n(X,x)$$

$$([f],[f']) \longmapsto \left[\begin{array}{ccc} I^n & \longrightarrow & X \\ (t_1,t_2,-) & \longmapsto & \begin{cases} f(2t_1,t_2,-) & \text{if } t_1 \leq 1/2 \\ f(2t_1-1,t_2,-) & \text{if } t_1 \geq 1/2 \end{cases} \end{array}\right]$$

$$\cdot : \pi_n(X,x) \times \pi_n(X,x) \longrightarrow \pi_n(X,x)$$

$$([f],[f']) \longmapsto \left[\begin{array}{ccc} I^n & \longrightarrow & X \\ (t_1,t_2,-) & \longmapsto & \begin{cases} f(t_1,2t_2,-) & \text{if } t_2 \leq 1/2 \\ f(t_1,2t_2-1,-) & \text{if } t_2 \geq 1/2 \end{cases} \end{array}\right]$$

We have $(f \cdot f') * (g \cdot g') = (f * g) \cdot (f' * g')$




<u>Ex 2:</u> $p : (X', x') \longrightarrow (X, x)$ a covering.

i.e. - $p$ is surjective

- $\forall t \in X$, $\exists U \ni x$ open, ~~homeo~~ $F$ a discrete top space,

$$p^{-1}(U) \longrightarrow U \times F \text{ a homeo.}$$

Such that

$$\begin{tikzcd} p^{-1}(U) \arrow[rr] \arrow[dr, "p"'] & \circlearrowleft & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

1. $\sigma : I^n \longrightarrow X$

<u>Uniqueness</u>: Let $\tilde\sigma, \tilde\sigma' : I^n \longrightarrow X'$ be such that

$$\begin{tikzcd} I^n \arrow[r, "\tilde\sigma"] \arrow[dr, "\sigma"'] & X' \arrow[d, "p"] \arrow[dl, phantom, "\circlearrowleft"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} I^n \arrow[r, "\tilde\sigma'"] \arrow[dr, "\sigma"'] & X' \arrow[d, "p"] \arrow[dl, phantom, "\circlearrowleft"] \\ & X \end{tikzcd}$$

& such that $\exists t_0 \in I^n, \tilde\sigma(t_0) = \tilde\sigma'(t_0)$.

we prove that $\tilde\sigma = \tilde\sigma'$.

~~Lebesgue's lemma~~ $\cancel{\Rightarrow \exists m \geq 1, \forall 0 \leq k_1, \dots, k_n < m,}$

$\cancel{\sigma\left(\left[\frac{k_1}{m}, \frac{k_1+1}{m}\right] \times \dots \times \left[\frac{k_n}{m}, \frac{k_n+1}{m}\right]\right) \subseteq U}$ ~~open with a homeo~~ $\cancel{p^{-1}(U) \xrightarrow{\simeq} U \times F}$

<u>Lebesgue's lemma</u> $\Rightarrow \exists r > 0, \forall t \in I^n, \sigma(B(t,r)) \subseteq U$ open

with a homeo
$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\simeq"] \arrow[dr, "p"'] & U \times F \arrow[d, "pr_1"] \arrow[dl, phantom, "\circlearrowleft"] \\ & U \end{tikzcd}$$

<u>Claim</u>: if $t_1 \in \underbrace{\{t \in I^n \mid \tilde\sigma(t) = \tilde\sigma'(t)\}}_{= Z}$, then $B(t_1, r) \subseteq Z$.

Indeed, if $t_1 \in Z$, $\exists U$ open $\ni \sigma(B(t_1, r))$ + $\varphi : p^{-1}(U) \xrightarrow{\simeq} U \times F = \coprod_{f \in F} U_f$, $U_f \cong U$

Let $f$ be the element of $F$ such that
$$\varphi(\tilde\sigma(t_1)) = \varphi(\tilde\sigma'(t_1)) \in U_f$$

we have
$$\begin{tikzcd} B(t_1, r) \arrow[r, "\varphi \circ \tilde\sigma"] \arrow[dr, "\sigma"'] & U \times F \arrow[d] \\ & U \end{tikzcd}$$
$\varphi(\tilde\sigma(B(t_1, r)))$ is connected and thus $\subseteq U_f$

same for $\varphi(\tilde\sigma'(B(t_1, r)))$.

since $U_f \longrightarrow U$ is a homeo, $\varphi \circ \tilde\sigma|_{B(t_1, r)} = \varphi \circ \tilde\sigma'|_{B(t_1, r)}$

$\Rightarrow \tilde\sigma|_{B(t_1, r)} = \tilde\sigma'|_{B(t_1, r)} \Rightarrow B(t_1, r) \subseteq Z$

Claim $\Rightarrow Z = I^n$.

Existence :

Lebesgue's lemma $\Rightarrow \exists m \geq 1, \forall 0 \leq k_1, \dots, k_n < m,$

$$\sigma\Bigg(\underbrace{\left[\frac{k_1}{m}, \frac{k_1+1}{m}\right] \times \dots \times \left[\frac{k_n}{m}, \frac{k_n+1}{m}\right]}_{C_{k_1,\dots,k_n}}\Bigg) \subseteq U \text{ open}$$

w.a homeo
$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\simeq"] \arrow[dr, "p"'] & U \times F \arrow[d, "pr_1"] \\ & U \end{tikzcd}$$

We proceed by induction on $k_1, \dots, k_n$ to find a map

$$\tilde{\sigma} : \bigcup_{(l_1,\dots,l_n) \leq_{lex} (k_1,\dots,k_n)} C_{l_1,\dots,l_n} \longrightarrow X'$$

lexicographic order

that lifts $\sigma\Big|_{\bigcup C_{l_1,\dots,l_n}}$.

* If $k_1 = \dots = k_n = 0$, chose $f \in F$. This gives a map

$$\theta_f : U \xrightarrow{} U \times F \xrightarrow{\simeq} p^{-1}(U)$$
$$u \longmapsto (u, f)$$

let $\tilde{\sigma} : (C_{0,\dots,0} \longrightarrow X') = \theta_f \circ \sigma$.

$$\begin{tikzcd} C_{0,\dots,0} \arrow[r] \arrow[dr, "\sigma"'] & p^{-1}(U) \arrow[d, "p"'] \\ & U \arrow[u, bend right, "\theta_f"'] \end{tikzcd}$$

* If $\tilde{\sigma}$ is defined on $\bigcup_{(l_1,\dots,l_n) <_{lex} (k_1,\dots,k_n)} C_{l_1,\dots,l_n}$,

$$\left\{\left(\frac{k_1}{m}, \dots, \frac{k_n}{m}\right)\right\} = \bigcap_{\substack{i=1 \\ s.t.\ k_i > 0}}^{n} C_{k_1,\dots,k_i-1,\dots,k_n}.$$

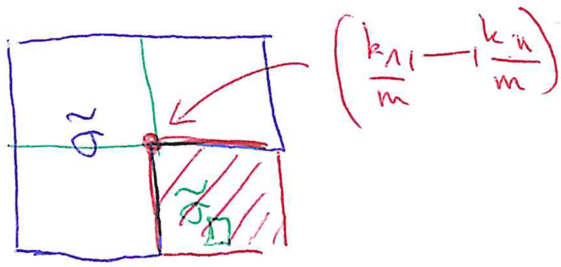
Chose a lift $\tilde{\sigma}_{\square}\Big|_{C_{\dots}}$ such that $\tilde{\sigma}_{\square}\left(\frac{k_1}{m}, \dots, \frac{k_n}{m}\right) = \tilde{\sigma}\left(\frac{k_1}{m}, \dots, \frac{k_n}{m}\right)$

To show that $\tilde{\sigma}_{\square}$ extends $\tilde{\sigma}$, we have to show that

$$\tilde{\sigma}_{\square}\Big|_{(C_{k_1,\dots,k_i-1,\dots,k_n})\cap C_{k_1,\dots,k_n}} = \tilde{\sigma}\Big|_{C_{k_1,\dots,k_i-1,\dots,k_n}\cap C_{k_1,\dots,k_n}}$$

if $k_i > 0$ $\quad 1 \le i \le n$.

This follows from uniqueness, since $C_{k_1,\dots,k_i-1,\dots,k_n} \cap C_{k_1,\dots,k_n} \cong I^{n-1}$.




2. If $n \geq 2$, $\partial I^n$ is connected, $p^{-1}(x) \cong \{x\} \times F$ with $F$ discrete

thus, ~~we can assume~~ $\tilde{\sigma}(\partial I^n)$ is a point.

~~3~~. Furthermore, by construction of $\tilde{\sigma}$, we can assume that $\tilde{\sigma}(\partial I^n) = \{x'\}$.

3. <u>Surjectivity</u>: let $[\sigma] \in \pi_n(X,x)$

let $\tilde{\sigma} : (I^n, \partial I^n) \longrightarrow (X', x')$ be a lifting of $\sigma$

$$\begin{tikzcd} (I^n,\partial I^n) \arrow[r, "\tilde{\sigma}"] \arrow[dr, "\sigma"'] & (X',x') \arrow[d] \\ & (X,x) \end{tikzcd}$$

then $p_*([\tilde{\sigma}]) = [\sigma]$.

Injectivity: Let $[\sigma]\in\pi_n(X',x')$ be such that $p_*([\sigma])=0$.

then, $p\circ\sigma \sim_{htp} \begin{pmatrix} I^n \longrightarrow X \\ t\longmapsto x\end{pmatrix}$.

Let $H : I^n\times I \longrightarrow X$ be such that $\begin{cases} H(-,0)=p\circ\sigma \\ H(-,1)=x \\ \forall s\in\partial I^n, \forall t\in I, \\ \qquad H(s,t)=x \end{cases}$

$x$

$p\circ\sigma$

H lifts to $\tilde H : I^n\times I \longrightarrow X'$ be ~~such that~~ a lifting of H

such that $\tilde H(0,-,0)=x'$.

Then, ~~$\partial I^n\times\{0\}$~~ $A=\partial(I^n\times I)\setminus\overline{(I^n\times\{0\})}$ is connected ~~&~~

~~contains~~ & $H\big|_A = x$

thus, $\tilde H\big|_A = x'$ and $\begin{cases} \tilde H(-,0)=\sigma \\ \tilde H(-,1)=x' \\ \forall s\in\partial I^n, \forall t\in I, \tilde H(s,t)=x \end{cases}$

$\Rightarrow$ $\tilde H$ is a homotopy $\sigma\sim\begin{pmatrix} I^n\longrightarrow X' \\ *\longmapsto x'\end{pmatrix}$.

$\Rightarrow$ $[\sigma]=0$.

4. $(\mathbb{R},0) \longrightarrow (S^1,1)$ is a covering. $\pi_1(S^1,1)=\mathbb{Z}$

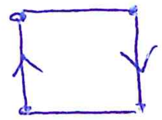
$t \longmapsto e^{2i\pi t}$ $\pi_0(S^1,1)=*$

Thus, $0 = \pi_n(\mathbb{R},0) \cong \pi_n(S^1,1)$ if $n \geq 2$.

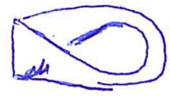
5. $\pi_1(\mathbb{T}^n,0) \cong (\pi_1(S^1,1))^n \cong \mathbb{Z}^n$ $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n$

$m \geq 2$ $\pi_m(\mathbb{T}^n,0) \cong \pi_m(\mathbb{R}^n,0) = 0$ $\mathbb{R}^n \xrightarrow{\pi} \mathbb{T}^n$ is a covering

$\pi_0(\mathbb{T}^n,0) = *$.



Ex 4: $M = [0,1]^2 / ((x,0) \sim (1-x,1),\ x \in [0,1])$



$$S^1 \cong [0,1]/_{0\sim 1}$$

$i : [0,1]/_{0\sim 1} \longrightarrow M$ $r : M \longrightarrow [0,1]/_{0\sim 1}$

$t \longmapsto (t,1/2)$ $(x,y) \longmapsto x$

* $i \circ r = id_{S^1}$

* $H : M \times I \longrightarrow M$ is an homotopy

$((x,y),t) \longmapsto (x, ty + (1-t)1/2)$ $id_M \sim r \circ i$

Thus, $M$ is homotopy equivalent to $S^1$.

Ex 7 of ~~section~~ sheet 1 :

1. $S^n \setminus \{pt\}$ is contractible.

2. Take such a loop $\gamma$. $\bar{\gamma} : t \longmapsto \gamma(1-t)$

then $\gamma\bar{\gamma} \sim$ cst loop and $\gamma\bar{\gamma}(I) = S^n$.

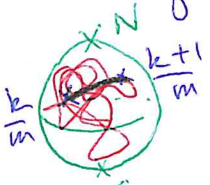
3. $n \geq 2$ Lebesgue's lemma:
$\exists m > 0$, $\gamma\left(\left[\frac{k}{m}, \frac{k+1}{m}\right]\right) \subseteq S^n \setminus N$ ← north pole
$\forall 0 \leq k < m$ or $S^n \setminus S$ ← south pole


$$S^n \setminus N \underset{\text{homeo}}{\overset{\varphi}{\xrightarrow{\sim}}} \mathbb{R}^n$$

$\underbrace{\varphi \circ \gamma}_{\gamma'}\Big|_{\left[\frac{k}{m}, \frac{k+1}{m}\right]}$ is homotopic relative to $\left\{\frac{k}{m}, \frac{k+1}{m}\right\}$ to the path ~~$t \longmapsto t\gamma'\left(\frac{k+1}{m}\right) + (1-t)\gamma'\left(\frac{k}{m}\right)$~~

$$\frac{k+t}{m} \longmapsto (1-t)\gamma'\left(\frac{k}{m}\right) + t\gamma'\left(\frac{k+1}{m}\right) = \gamma''\left(\frac{k+t}{m}\right)$$

through




$$H_k : \left[\frac{k}{m}, \frac{k+1}{m}\right] \times I \longrightarrow \mathbb{R}^n$$
$$\left(\frac{k+t}{m}, s\right) \longmapsto \gamma'\left(\frac{k+t}{m}\right)(1-s) + s\gamma''\left(\frac{k+t}{m}\right)$$

$\varphi^{-1} \circ \gamma''$ ~~$\left(\frac{k+t}{m}\right)$~~ has a nowhere dense image in $S^n$.

4. Join the $H_k$ together to get that $\gamma$ is homotopic to a path w. is nowhere dense in $S^n$ and thus not surjective.

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Ex6 (sheet 2) : Same ideas ~~with~~ $\Longrightarrow$ $\pi_k(S^n) = 0$ if $k < n$.

Ex 5 : $\pi_n(S^n) \neq 0$ because $S^n \xrightarrow{id} S^n$ is not homotopic to a constant map.

$\pi(S^2)$ … !! Homotopy groups are difficult to compute!!