Relative homology groups; Mayer Vietoris exact sequence

Exercice 1. Some properties of relative homology groups.
Let (X, A) be a topological pair.

1. Show that $H_0(X, A) = 0$ if and only if A meets each path-component of X.
2. Show that $H_1(X, A) = 0$ if and only if the canonical map $H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.
3. Assume that A is a single point space. Compute the relative homology groups $H_n(X, A)$ in terms of the homology groups of X.
4. Assume that $i : A \to X$ is a retract, show that the map $H_n(A) \to H_n(X)$ is injective.

Exercice 2. Suspensions and cones.
Let X be a topological space. The *cone over* X is the topological space $CX := X \times [0, 1]/X \times \{1\}$. The *suspension of* X is the topological space $\Sigma X := CX/X \times \{0\}$

1. Show that CX is contractible.
2. Compute the homology groups of ΣX in terms of the homology groups of X.
3. Let $n \geqslant 0$ be an integer. What is the suspension of the sphere S^n ?
4. Compute the homology groups of S^n.

Exercice 3. Show that the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is free and find a basis.

Exercice 4. A description of relative homology groups.
Let (X, A) be a topological pair. Consider the topological space

$$X \cup CA = (X \sqcup CA)/(a \sim (a, 0) \mid a \in A).$$

Let p be the image of $(a, 1)$ (which is the same for any $a \in A$) in $X \cup CA$.

1. Let $n \geqslant 1$ be an integer. Let B^n be the closed unit ball. What is $B^n \cup C(\partial B^n)$?
2. Show that the map of pairs $(X \cup CA, p) \to (X \cup CA, CA)$ induces an isomorphism of homology groups.
3. Show that the map of pairs $(X, A) \to (X \cup CA, CA)$ induces an ismorphism of homology groups.
4. Deduce that for all n,
$$H_n(X, A) \simeq \widetilde{H}_n(X \cup CA).$$
5. Deduce another way to compute the homology groups of the sphere S^n.

Exercice 5. Long exact sequence of the triple.
Let (X, A, B) be a topological triple.

1. Show that the following sequence is exact:
$$\cdots \to H_n(A, B) \overset{(i_*)_n}{\to} H_n(X, B) \overset{(j_*)_n}{\to} H_n(X, A) \overset{d_n}{\to} H_{n-1}(A, B) \to \cdots$$
where $i\colon (A, B) \to (X, B)$ and $j\colon (X, B) \to (X, A)$ are the inclusions and d_n is the composition
$$H_n(X, A) \overset{\delta_n}{\to} H_{n-1}(A) \overset{(k_*)_{n-1}}{\longrightarrow} H_{n-1}(A, B)$$
where δ_n is the connecting map of the exact sequence of the pair (X, A) and $k\colon (A, \varnothing) \to (A, B)$ is the inclusion.

2. Deduce the exact sequence of the pair in reduced homology.

Exercice 6. Homology groups of the parachute.

Let $\Delta^2 = [e_0, e_1, e_2]$ be the standard 2-simplex. Compute the homology groups of $\Delta^2/\{e_0, e_1, e_2\}$.

Exercice 7. Wedge sum of spaces.

Let (X, x) and (Y, y) be pointed topological spaces. We define their wedge sum $X \vee Y = X \sqcup Y/(x \sim y)$.

Assume that x and y have contractible neighborhoods respectively in X and Y. Compute the homology groups of $X \wedge Y$.

Exercice 8. A counterexample.

Show that the torus $T^2 = S^1 \times S^1$ and the space $S^1 \vee S^1 \vee S^2$ have the same homology groups but not the same fundamental group.

Exercice 9. Relative homology of good pairs.

1. Let (X, A) be a topological pair. Assume that A is a *good pair i.e.* that there is a neighborhood V of A which deformation retracts onto A.

   Let $q : (X, A) \to (X/A, A/A)$ be the quotient map. Let $n \geqslant 0$ be an integer. Show that in the commutative diagram:

$$\begin{array}{ccccc}
H_n(X,A) & \longrightarrow & H_n(X,V) & \longleftarrow & H_n(X-A, V-A) \\
\downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} & & \downarrow{\scriptstyle q_*} \\
H_n(X/A, A/A) & \longrightarrow & H_n(X/A, V/A) & & H_n(X/A - A/A, V/A - A/A)
\end{array}$$

   all the maps are isomorphisms.

   Deduce that the map $q_* : H_n(X, A) \to \widetilde{H}_n(X/A)$ is an isomorphism.

2. Compute once again the homology groups of the sphere S^n.

Exercice 10. Homology groups of some pathological topological spaces.

Compute the homology groups of the line with 2 origins. Same question with the closure in $\mathbb{R}^2$ of the graph of the function $(x > 0) \mapsto \sin(1/x)$.