THE TATE CONSTRUCTION

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The goal of these notes is to introduce the Tate construction, prove that it is monoidal and construct the Tate diagonal. This mostly follows the paper *On topological cyclic homology* by Nikolaus and Scholze.

1. A useful analogy

In this brief section, we explain the Tate construction in an algebraic setting. Hopefully, having this simple situation in mind will make the rest of the talk clearer.

For an abelian group A with an action of a finite group G, there is a norm map

$$\mathrm{Nm} : A \to A^G$$

that sends an element x of A to the sum $\sum_{g \in G} gx$. It is clear that this map factors through the coinvariants A_G. We define the algebraic Tate construction to be the cokernel of this map :

$$A^{tG} := \mathrm{coker}(\mathrm{Nm} : A \to A^G)$$

This defines a functor from abelian group with an action of G to abelian groups. This functor is lax monoidal.

Here is an easy situation where we can compute what that is. If G acts trivially on A, then it is easy to see that the norm map is just the multiplicaiton by the order of G and thus $A^{tG} = A/|G|$.

From now on p always denotes a prime number and C_p is the cyclic group of order p.

The Tate diagonal is a map of abelian group $A \to (A^{\otimes p})^{tC_p}$ whose existence is the content of the following proposition.

Proposition 1.1. *The (non-additive) diagonal map $A \to (A^{\otimes p})^{C_p}$ sending a to the p-fold tensor product $a \otimes \ldots \otimes a$ composed with the quotient map $(A^{\otimes p})^{C_p} \to (A^{\otimes p})^{tC_p}$ is a morphism of abelian group.*

Proof. We do the case $p = 2$. The general case is similar. In that case we have

$$(a+b) \otimes (a+b) = a \otimes a + b \otimes b + a \otimes b + b \otimes a = a \otimes a + b \otimes b + \mathrm{Nm}(a \otimes b)$$

Therefore if we mod out by the image of the norm map, the diagonal is additive. □

Now assume that A has the structure of a commutative ring. This implies that we have a C_p-equivariant multiplication map

$$A^{\otimes p} \to A$$

We can hit this map with the Tate construction for the C_p action and precompose with the Tate diagonal $A \to (A^{\otimes p})^{tC_p}$. We end up with a map $A \to A^{tC_p} = A/p$. This map is described by the following proposition.

Proposition 1.2. *This map coincides with the quotient map $A \to A/p$ composed with the Frobenius map for the $\mathbb{F}_p$-algebra A/p.*

Proof. Easy. □

In the context of spectra these statements admit generalizations. The Tate construction will be the cofiber of an analogue of the norm map. It is defined in a similar way but is much more technical to define because we can't talk about elements in a spectrum. It will be also a lax monoidal functor. We will also construct an analogue of the Tate diagonal. If A is a commutative ring spectrum we will thus get a spectral Frobenius map $A \to A^{tC_p}$ defined as in the algebraic case. In good cases, the target can be identified with the p-completion of A (a sepctral analogue of quotienting by p).

2. The Norm map

We start with a very general situation. We consider a commutative ring spectrum S, an S-algebra R and an augmentation map $R \to S$ that splits the unit map $S \to R$. In practice, S will be the sphere spectrum and R will be the spherical group algebra $S[G]$ for G a finite group.

In this general situation, we have a functor from $\mathbf{Mod}_R$ to $\mathbf{Mod}_S$ defined by the formula

$$F : M \mapsto \mathsf{Hom}_R(S, M)$$

This functor is exact but only preserves filtered colimits if S is compact as an R-module. We would like to find the best approximation to this functor on the left by a functor that preserves all colimits. This is possible because the inclusion

$$\mathrm{Fun}^L(\mathbf{Mod}_R, \mathbf{Mod}_S) \subset \mathrm{Fun}^{ex}(\mathbf{Mod}_R, \mathbf{Mod}_S)$$

has a right adjoint[1] (indeed, the colimit preserving functors are stable under colimits). This right adjoint sends an exact functor to the left Kan extension of its restriction to compact objects of $\mathbf{Mod}_R$. In order to understand this object better, we recall the following result (Morita theory for ring spectra).

Theorem 2.1. *Given two ring spectra A and B and an object P of ${}_A\mathbf{Mod}_B$, we can construct the functor $F_P(M) \mapsto M \otimes_A P$ from A-modules to B-modules. This produces an equivalence of categories*

$${}_A\mathbf{Mod}_B \to \mathrm{Fun}^L(\mathbf{Mod}_A, \mathbf{Mod}_B)$$

Proof. The inverse functor sends F to $F(A)$ with the obvious right B-action and where the left A-action comes from the equivalence of rings $A \simeq \mathrm{End}_A(A)$. □

Given an arbitrary exact functor $F : \mathbf{Mod}_R \to \mathbf{Mod}_S$, its universal approximation by a colimit preserving functor must coincide with F on compact R-modules and in particular on R, therefore by the previous theorem it must be given by $M \mapsto M \otimes_R \mathsf{Hom}_R(S, R)$. In particular, we get a natural transformation of functors from R-modules to S-modules

$$M \otimes_R \mathsf{Hom}_R(S, R) \to \mathsf{Hom}_R(S, M)$$

that we can call the norm map.

Now we specialize it to the situation where S is the sphere spectrum and $R = S[G] = \Sigma^\infty_+ G$ where G is a finite group. In this situation the functor $M \mapsto \mathsf{Hom}_{S[G]}(S, M)$ can be denoted $M \mapsto M^{hG}$. The reason for this notation is that there is an equivalence of categories

$$\mathbf{Mod}_{S[G]} \simeq \mathbf{Sp}^{BG}$$

where $\mathbf{Sp}^{BG}$ denotes the categories of functors from BG (the category with a unique objet whose monoid of endomorphisms is G). Through this equivalence of categories, the functor

[1] There is a set theoretic issue here. The category of exact functors is not locally small, the way to fix this is to consider $\mathrm{Fun}^{ex,\kappa}(\mathbf{Mod}_R, \mathbf{Mod}_S)$ instead of $\mathrm{Fun}^{ex}(\mathbf{Mod}_R, \mathbf{Mod}_S)$. This is the category of exact functors that commute with κ-filtered colimits. This category is locally small and all the exact functors of interest will live in that category if we take κ large enough. This issue will appear at other places in these notes but we will not mention it anymore.

$M \mapsto \mathsf{Hom}_{S[G]}(S, M)$ is exactly the fixed point functor $M \mapsto M^{hG}$. (We use the notation M^{hG} instead of M^G because we will also talk about genuine G-spectra in this seminar and for those the notation M^G has a different meaning.)

The goal is now understand what this object $\mathsf{Hom}_{S[G]}(S, S[G]) = S[G]^{hG}$ is. In order to do this we first observe that the forgetful functor $\mathbf{Mod}_{S[G]} \to \mathbf{Mod}_S$ has a right adjoint. This right adjoint is given by $X \mapsto \mathsf{Hom}(S[G], X) = \prod_G X$. One easily observes that this right adjoint is canonically equivalent to $X \mapsto X \otimes S[G] = \oplus_G X$ (this uses the fact that finite products and coproducts coincide in spectra). On the other hand, the limit functor $M \mapsto M^{hG}$ is right adjoint to the functor sending a spectrum X to X with the trivial G-action. It follows that the functor $X \mapsto (X \otimes S[G])^{hG}$ is right adjoint to the identity functor of the category of spectra and therefore is the indentity functor.

For future reference, we record this fact.

Proposition 2.2. *The functor from spectra to spectra $X \mapsto (X \otimes S[G])^{hG}$ is naturally equivalent to the identity functor.*

In particular applying this to $X = S$, we find an equivalent $S[G]^{hG} = S$. The tensor product $X \otimes_{S[G]} S$ can be identified with the orbit spectrum through the equivalence $\mathbf{Sp}^{BG} \simeq \mathbf{Mod}_{S[G]}$ and the norm map takes the simplified form

$$\mathrm{Nm}_G : X_{hG} \to X^{hG}$$

Definition 2.3. We define the Tate construction to be the exactfunctor $\mathbf{Sp}^{BG} \to \mathbf{Sp}$ sending X to the cofiber of $\mathrm{Nm}_G(X)$.

Remark 2.4. Note that our construction of the norm map can be applied to G any topological group, the source of this map in general is the functor $X \mapsto X \otimes_{S[G]} \mathsf{Hom}_{S[G]}(S, S[G])$. This can be quite complicated to understand but in the case where G is a compact Lie group, the G-spectrum $\mathsf{Hom}_{S[G]}(S, S[G])$ turns out to be the one point compactification of the Lie algebra of G equipped with the adjoint action of G. In particular, for $G = S^1$, we get a norm map

$$\Sigma X_{hS^1} \to X^{hS^1}$$

whose cofiber is by definition the S^1-Tate construction.

3. Monoidality of the Tate construction

We first recall some facts about Verdier quotient. Given a small stable ∞-category C and a stable full subcategory D, we define C/D by the following pushout square in the category of stable ∞-categories.

$$\begin{tikzcd} D \arrow[r] \arrow[d] & C \arrow[d] \\ 0 \arrow[r] & C/D \end{tikzcd}$$

Proposition 3.1. *Let E be a stable ∞-category. Then the quotient map $p : C \mapsto C/D$ induces a fully faithful functor*

$$p^* : \mathrm{Fun}^{ex}(C/D, E) \to \mathrm{Fun}^{ex}(C, E)$$

whose essential image is spanned by the functors $C \to E$ that vanish on objects of D.

Moreover, if E is a presentable stable ∞-category, the inclusion

$$\mathrm{Fun}^{ex}(C/D, E) \to \mathrm{Fun}^{ex}(C, E)$$

has a left adjoint $p_!$.

Proof. The first claim follows from the definition and the fact that a pullback of a fully faithful map is fully faithful. The second claim follows from the fact that the category of functors $C \to E$ that vanish on D is stable under limits and colimits. $\square$

In particular, in the case where E is presentable, given an arbitrary exact functor $F : C \to E$, there is an initial functor $G : C \to E$ with a map from F and that vanish on D.

This proposition can be refined in the presence of monoidal structures. In that case we want to take the Verdier quotient by a tensor ideal so we first recall the relevant definition.

Definition 3.2. Let C be a symmetric monoidal category. A full subcategory I of C is said to be a tensor ideal if for any object U of I and X of C, the tensor product $U \otimes X$ is an object of I.

We now have a lax monoidal refinement of Proposition 3.1

Proposition 3.3. *Let C be a symmetric monoidal stable ∞-category and I be a full stable subcategory that is also an ideal of C. Then the Verdier quotient C/I has a symmetric monoidal structure that makes the quotient map $p : C \to C/I$ into a symmetric monoidal functor. For any other symmetric monoidal stable ∞-category, the induced functor*

$$p^* : \mathrm{Fun}^{ex,lax}(C/I, E) \to \mathrm{Fun}^{ex,lax}(C, E)$$

is fully faithful with essential image spanned by the lax monoidal functors $C \to E$ that vanish on I. If E is moreover a presentable symmetric monoidal stable ∞-category, the functor p^* has a left adjoint $p_!^{\otimes}$. Finally if F is a lax monoidal functor $C \to E$, then $p_!^{\otimes} F$ is naturally equivalent to $p_! F$.

Now, we consider the full stable subcategory $\mathbf{Sp}^{BG}_{ind}$. This is the smallest stable subcategory of $\mathbf{Sp}^{BG}$ that contains the induced spectra (that is the objects of the from $X \otimes S[G]$ with X a spectrum). Note that not all maps between induced spectra are induced so the category $\mathbf{Sp}^{BG}_{ind}$ contains many G-spectra that are not induced.

Proposition 3.4. *The subcategory $\mathbf{Sp}^{BG}_{ind}$ is an ideal of $\mathbf{Sp}^{BG}$.*

Proof. First we observe that if $U = V \otimes S[G]$ is an induced spectrum, then $X \otimes U = X \otimes V \otimes S[G]$ is an induced spectrum for all X (since $\mathbf{Sp}^{BG}$ is generated under colimits by $S[G]$ it suffices to prove it when $X = S[G]$ in which case it is easy). Therefore, the full subcategory of $\mathbf{Sp}^{BG}$ spanned by objects U such that $U \otimes X$ is in $\mathbf{Sp}^{BG}_{ind}$ for all X is a stable subcategory of $\mathbf{Sp}^{BG}$ that contains all the induced spectra. Thus it must contain $\mathbf{Sp}^{BG}_{ind}$. This means precisely that $\mathbf{Sp}^{BG}_{ind}$ is an ideal. $\square$

Now, we consider the functor $(-)^{hG} : \mathbf{Sp}^{BG} \to \mathbf{Sp}$. This functor is strong monoidal. Therefore by Proposition 3.3, there is an initial lax monoidal functor H with a map $(-)^{hG} \to H$ that vanishes on $\mathbf{Sp}^{BG}_{ind}$. We want to identify this functor with the Tate construction. We first observe that the Tate construction vanishes on $\mathbf{Sp}^{BG}_{ind}$. Indeed, for a spectrum of the form $X \otimes S[G]$, the Tate construction is the cofiber of the norm map

$$(X \otimes S[G])_{hG} \to (X \otimes S[G])^{hG}$$

both the source and the target are equivalent to X (this is obvious for the source and for the target this is Proposition 2.2) and the norm map can easily be checked to be the identity. Thus the Tate construction vanishes on induced spectra. Since it is an exact functor, it vanishes on $\mathbf{Sp}^{BG}_{ind}$. By the last fact of Proposition 3.3, we are reduced to proving that, for any exact functor $K : \mathbf{Sp}^{BG} \to \mathbf{Sp}$ that vanishes on $\mathbf{Sp}^{BG}_{ind}$, the induced map

$$\mathrm{Map}((-)^{tG}, K) \to \mathrm{Map}((-)^{hG}, K)$$

is an equivalence (here Map denotes the mapping spaces in $\mathrm{Fun}^{ex}(\mathbf{Sp}^{BG}_{ind}, \mathbf{Sp})$). By definition of the Tate construction, the cofiber of this map is $\mathrm{Map}((-)_{hG}, K)$. Now, we observe that the functor $(-)_{hG}$ is colimit preserving. In the second section we mentioned that the inclusion $\mathrm{Fun}^L(\mathbf{Sp}^{BG}, \mathbf{Sp}) \subset \mathbf{Fun}^{ex}(\mathbf{Sp}^{BG}, \mathbf{Sp})$ has a right adjoint that sends F to $X \mapsto X \otimes_{S[G]} F(S[G])$. In particular, this right adjoint sends K to 0 since K vanishes on $S[G]$. This shows that $\mathrm{Map}((-)_{hG}, K) = 0$ which is what we wanted.

4. The Tate diagonal

Here p is a prime number and C_p is the cyclic group of order p. We consider the functor $X \mapsto X^{tC_p}$ from spectra to spectra. We would like to construct a natural transformation from the identity to this functor.

Proposition 4.1. *Let X be a spectrum, the functor $X \mapsto X^{tC_p}$ is a lax monoidal exact functor from spectra to spectra.*

Proof. The fact that it is lax monoidal is the content of the previous section. The exactness is left to the reader. ☐

The identity functor from spectra to spectra is represented in the enriched sense by the spehre spectrum. The functor $X \mapsto X^{tC_p}$ is naturally an $\mathbf{Sp}$-enriched functor (as is any exact functor). Therefore, by the spectrally enriched Yoneda lemma, the set of homotopy classes of $\mathbf{Sp}$-enriched maps from the identity functor to $X \mapsto (X^{\otimes p})^{tC_p}$ is the set $\pi_0((S^{\otimes} p)^{tC_p})$. The Segal conjecture (which is a theorem) for the group C_p implies that $(S^{\otimes p})^{tC_p}$ is the p-completed sphere. In particular, the set of homotopy classes of $\mathbf{Sp}$-enriched maps from the identity functor to $X \mapsto (X^{\otimes p})^{tC_p}$ is $\mathbb{Z}_p$. The map corresponding to $1 \in \mathbb{Z}_p$ is by definition the Tate diagonal.

There is a different approach that does not use the Segal conjecture and gives the additional information that the Tate diagonal is a lax monoidal natural transformation.

Let C be a small stable symmetric monoidal ∞-category. The category of exact functors $C \to \mathbf{Sp}$ can be given a Day symmetric monoidal structure. The tensor product is given by the following formula

$$(F \otimes G)(U) = \operatorname*{colim}_{X \otimes Y \to U} F(X) \otimes F(Y)$$

A lax monoidal exact functor $F : C \to \mathbf{Sp}$ is then exactly a commutative monoid with respect to this tensor product. As in any symmetric monoidal category, the unit is the initial object in commutative algebras in $\mathrm{Fun}(C, \mathbf{Sp})$. In particular, there is a unique lax monoidal natural transformation from the unit of $\mathrm{Fun}^{ex}(\mathbf{Sp}, \mathbf{Sp})$ to $X \mapsto (X^{\otimes p})^{tC_p}$. This map will be the Tate diagonal once we have identified the unit of $\mathrm{Fun}^{ex}(\mathbf{Sp}, \mathbf{Sp})$ with the identity functor.