SPECTRA, AN OVERVIEW

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1. Homology, cohomology and spectra

1.1. **Representability.** First of all lets discuss about representability of functors. Suppose $F : \mathcal{D}^{\mathsf{op}} \to \mathsf{Set}$, we say that F is representable if there is $Y \in \mathcal{D}$ and a natural isomorphism $\mathcal{D}(-, Y) \to F$. In case $\mathcal{D}$ is presentable, the functor F is representable if it carries colimits to limits. But what happens if $\mathcal{D}$ is not a presentable category ? The example of such a situation that will be of interest to us is if $\mathcal{D}$ is given as the homotopy category of a presentable ∞-category $\mathcal{C}$. Indeed such a category need not admit colimits.

The key point is that under some assumptions on $\mathcal{C}$, we have a generalization of the Brown's representability theorem.

1.1.1. **Théorème** (Brown Representablity)**.** *Let $\mathcal{C}$ be a presentable ∞-category containing a set of objects $\{S_\alpha\}_{\alpha \in A}$ with the following properties,*

(1) Each object S_α is a cogroup object of the homotopy category $\mathsf{Ho}(\mathcal{C})$.

(2) Each object S_α is compact.

(3) $\mathcal{C}$ is generated under small colimits by $\{S_\alpha\}$.

Then a functor $F : \mathsf{Ho}(\mathcal{C})^{\mathsf{op}} \to \mathsf{Set}$ is representable if and only if it satisfy the following,

(1) F carries coproducts in $\mathsf{Ho}(\mathcal{C})$ to products in Set.

(2) For any pushout square,

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C' \arrow[d] \\ D \arrow[r] & D', \end{tikzcd}$$

the induced map $F(D') \to F(C') \times_{F(C)} F(D)$ is surjective.

In the case where $\mathcal{C}$ is the ∞-category of pointed spaces we get back the classical Brown representability. Indeed the ∞-category of spaces is generated under colimits by the single point, which is a cogroup object in the homotopy category of spaces.

1.1.2. **Définition** ([2, def. 1.4.1.6])**.** Let $\mathcal{C}$ be a pointed ∞-category which admts small colimits and let $\Sigma : \mathcal{C} \to \mathcal{C}$ denote the suspension functor. A (reduced ?) cohomology theory on $\mathcal{C}$ is a sequence of functors $\{H^n : \mathsf{Ho}(\mathcal{C})^{\mathsf{op}} \to \mathsf{Set}, n \in \mathbf{Z}\}$ together with natural isomorphisms $\delta^n : H^n \simeq H^{n+1} \circ \Sigma$ satisfying the following conditions :

(1) For every collection of objects $\{C_\alpha\}$ the canonical map
$$H^n\left(\coprod C_\alpha\right) \to \prod H^n(C_\alpha)$$
is a bijection. In particular $H^n(*)$ is a singleton for any zero object $*$ and any $n \in \mathbf{Z}$. The canonical map $C \to *$ yields a 0 element in $H^n(C)$

(2) Suppose we are given a cofiber sequence
$$C' \to C \to C''$$
in the ∞-category $\mathcal{C}$. Then we get a sequence
$$H^n(C'') \to H^n(C) \to H^n(C')$$
exact at $H^n(C)$.

Blabla

We can remark here that the first axiom gives us a way to define an abelian group structure on the H^n for any $n \in \mathbf{Z}$. Indeed, $\Sigma^2 X$ is an abelian cogroup object in $\mathsf{Ho}(\mathcal{C})$ since $\mathsf{Ho}(\mathcal{C})(\Sigma^2 X, Y) \simeq \pi_0 \mathsf{Map}_{\mathcal{C}}(\Sigma^2 X, Y) \simeq \pi_2(\mathsf{Map}_{\mathcal{C}}(X, Y))$ is an abelian group for any $Y \in \mathcal{C}$. Then, because
$$H^n(X) \simeq H^{n+1}(\Sigma X) \simeq H^{n+2}(\Sigma^2 X)$$
and because H^n is contravariant, $H^n(X)$ is an abelian group object in the categroy of sets.

Also we can build a long exact sequence from the second axiom. Just remark that if we have a cofiber sequence $C' \to C \to C''$, we can build two other cofiber sequences $C \to C'' \to \Sigma C'$ and $C'' \to \Sigma C' \to \Sigma C$. Just consider the following diagram,
$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & C'' \arrow[r] \arrow[d] & \Sigma C' \arrow[d] \\ & \bullet \arrow[r] & \Sigma C, \end{tikzcd}$$
where all the small squares are pushouts, and use the natural isomorphism to build the connecting homomorphism,
$$H^n(C') \to H^{n+1}(\Sigma C') \to H^{n+1}(C'')$$
where the last map is induced by the canonical map $C'' \to \Sigma C'$ that appears already in $C \to C'' \to \Sigma C'$.

Exactness at $H^n(C)$ is given by the second axiom, exactness at $H^n(C'')$ is given by appliying th second axiom to the cofiber sequence $C \to C'' \to \Sigma C'$ and similarly for exactness at $H^n(C')$.

Now we remark that if $\{(H^n, \delta_n), n \in \mathbf{Z}\}$ is a cohomology theory in $\mathcal{C}$, then for any $n \in \mathbf{Z}$, H^n satisfies the necessary and sufficient conditions of theorem 1.1.1 to be a representable functor.

Indeed the first condition is just the first axiom of cohomology theories. For the second condition, suppose we are given a pushout square

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "e"'] & C' \arrow[d, "h"] \\ D \arrow[r, "g"] & D', \end{tikzcd}$$

We have to show that the induced map $H^n(D') \to H^n(D) \times_{H^n(C)} H^n(D')$ is surjective. Setting $E \simeq \mathrm{cofib}(f) \simeq \mathrm{cofib}(g)$, we get the following,

$$\begin{tikzcd} H^n(E) \arrow[r] \arrow[d, "\phi"'] & H^n(C) \arrow[r] \arrow[d] & H^n(C') \arrow[r] \arrow[d, "h"] & H^{n+1}(E) \arrow[d, "\psi"] \\ H^n(E) \arrow[r] & H^n(D) \arrow[r, "g"] & H^n(D') \arrow[r] & H^{n+1}(E), \end{tikzcd}$$

where ϕ and ψ are isomorphisms. This gives via the algebraic version of Mayer-Vietoris, a sequence,

$$\cdots \to H^n(C) \xrightarrow{(f,e)} H^n(D) \oplus H^n(C') \xrightarrow{g-h} H^n(D') \to H^{n+1}(C) \to \cdots$$

that is exact at $H^n(D) \oplus H^n(C')$, meaning that (f,e) is onto on the kernel of $g-h$ which is given by $H^n(D) \times_{H^n(D)} H^n(C')$. This proves the following,

1.1.3. **Corollaire.** *Let $\mathcal{C}$ be a presentable pointed ∞-category. Assume $\mathcal{C}$ is generated under colimits by compact objects which are cogroup onjects of the homotopy category $\mathsf{Ho}(\mathcal{C})$, and let $\{H^n, \delta^n\}$ be a cohomology theory on $\mathcal{C}$. Then for every integer n, the functor H^n is representable by an object $E(n) \in \mathcal{C}$.*

Here the δ^n give the family $E(n)$ some structure, namely the natural isomorphims and the adjunction $(\Sigma_{\mathcal{C}} \dashv \Omega_{\mathcal{C}})$ induce a family of canonical isomorphisms $E(n) \simeq \Omega E(n+1)$ in the homotopy category $\mathsf{Ho}(\mathcal{C})$.

Then E can be seen as an object of the homotopy limit $\mathrm{Sp}\,\mathcal{C}$ of the tower,

$$\cdots \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}. \tag{1.1.0.1}$$

Let's call an object of that homotopy limit a Ω-spectrum.

1.2. **Homology and excisive reduced functors.** Now we turn to the study of the category $\mathrm{Sp}\,\mathcal{C}$ but before that lets discuss some homology theory. One key feature of homology theory is the Mayer-Vietoris sequence,

$$\cdots \to H_{n+1}(A \cup B, \mathbf{Z}) \to H_n(A \cap B) \to H^n(A) \oplus H^n(B) \to H^n(A \cup B) \to \cdots$$

This is true since singular homology can be defined as the homology of the singular complex given by the Moore complex of the simplicial set $\mathrm{Sing}(X)_\bullet$, and the construction $X \mapsto \mathbf{Z}\mathrm{Sing}(X)_\bullet$ carries homotopy pushouts to homotopy pullbacks.

1.2.1. **Définition.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor betwen ∞-categories.

(1) If $\mathcal{C}$ admits pushouts, we will say that F is excisive if it carries pushout squares to pullback squares.

(2) If $\mathcal{C}$ admits a final object $*$, we will say F is reduced if $F(*)$ is a final object of $\mathcal{D}$.

Denote $\mathrm{Exc}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the excisive functors and denote $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ the fullsubcategory spanned by the reduced excisive functors.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functorn suppose $\mathcal{C}$ is pointed. If $\mathcal{C}$ is stable, then F is reduced and excisive if and only if it is right exact ([2, prop. 1.1.3.4]). If instead $\mathcal{D}$ is stable, F is reduced and excisive if and only if it is right exact. If both are exact, F is reduced and excisive if and only if F is exact.

Let $\mathcal{S}^{\mathrm{fin}}$ denote the smallest full subcategory of $\mathcal{S}$ which contains the final object $*$ and is stable under finite colimits. This will be called the ∞-category of finite spaces and $\mathcal{S}^{\mathrm{fin}}_*$ the finite pointed object of $\mathcal{S}^{\mathrm{fin}}$.

Remark that $\mathcal{S}^{\mathrm{fin}}_*$ is characterized by the following universal property. For any $\mathcal{D}$ wich admits finite colimits, evaluation at $*$ induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{D}) \to \mathcal{D}$. Where $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{D})$ denotes the full subcategory spanned by the right exact functors, i.e. the functors that commute with finite colimits.

1.2.2. **Définition.** Let $\mathcal{C}$ be a ∞-category which admits finite limits. A spectrum object of $\mathcal{C}$ is a reduced, excisive functor $X : \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{C}$. Lets note $\mathrm{Sp}\,\mathcal{C} = \mathrm{Exc}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$ the full subcategory spanned by the spectrum objects of $\mathcal{C}$.

Remark here that there is a canonical isomorphism $\mathrm{Sp}(\mathrm{Fun}(K, \mathcal{C})) \simeq \mathrm{Fun}(K, \mathrm{Sp}(\mathcal{C}))$.

1.3. **Stability.** We are now able to prove the stability of the category $\mathrm{Sp}(\mathcal{C})$. First of all let's remind the definition of stability.

1.3.1. **Définition.** Let $\mathcal{C}$ be an ∞-category. We say that $\mathcal{C}$ is stable if it satisfies the following conditions,

(1) There is a zero object in $\mathcal{C}$.

(2) Every morphism in $\mathcal{C}$ admits a cofiber and a fiber.

(3) A sequence $A \to B \to C$ in $\mathcal{C}$ is a fiber sequence if and only if it is a cofiber sequence.

1.3.2. **Lemme.** *Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits, and let $\mathcal{D}$ be an ∞-category which admits finite limits. Then the ∞-category $Exc_*(\mathcal{C}, \mathcal{D})$ is pointed and amdits finite limits.*

1.3.3. **Proposition** ([2, prop. 1.4.2.11])**.** *Let $\mathcal{C}$ be a pointed ∞-category which admits finite limits and colimits. then,*

(1) If the suspension functor is fully faithful, then every pushout square is a pullback square.

(2) If the loop functor is fully faithfull, then every pullback square is a pushout square.

(3) If the loop functor is an equivalence, $\mathcal{C}$ is stable.

1.3.4. **Proposition.** *If $\mathcal{C}$ is pointed and admits finite colimits and $\mathcal{D}$ admits finite limits, $Exc_*(\mathcal{C}, \mathcal{D})$ is stable*

Démonstration. Using characterization (3) of proposition 1.3.3. □

1.3.5. **Corollaire.** *Suppose $\mathcal{C}$ admits finite colimits, then $\mathrm{Sp}(\mathcal{C})$ is stable.*

1.4. **Universal property of $\mathrm{Sp}\,\mathcal{C}$ and the Ω-tower.** Lets identify $\mathrm{Sp}(\mathcal{C})$ with the homotopy limit of the tower 1.1.0.1. First of all lets define $\Omega^\infty : \mathrm{Sp}\,\mathcal{C} \to \mathcal{C}$ as the evaluation at S^0 functor.

1.4.1. **Théorème** ([2, prop. 1.4.2.21])**.** *Let $\mathcal{D}$ admit finite limits. then the folloing are equivalent.*

(1) $\mathcal{D}$ is stable

(2) $\Omega^\infty : \mathrm{Sp}\,\mathcal{D} \to \mathcal{D}$ is an equivalnce of ∞-categories.

1.4.2. **Proposition.** *Let $\mathcal{C}$ be a pointed which admits finite colimits and $\mathcal{D}$ which admits finite limits. Then composition with the functor $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ induces an equivalence of ∞-categories*

$$\theta : \mathrm{Exc}_*(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_*(\mathcal{C}, \mathcal{D}).$$

1.4.3. **Corollaire** (Universal property of $\mathrm{Sp}(\mathcal{D})$, [2, coro. 1.4.2.23])**.** *Let $\mathcal{C}$ be a stable ∞-category, let $\mathcal{D}$ admit finite limits,* $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$ *and* $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Sp}(\mathcal{D}))$ *denote the full subcategories spanned by the left exact functors. Then composition with $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ induces an equivalence of ∞-categories,*

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$$

This can be pictured as the following diagram,

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[dr, dashed, "\exists!"'] & \mathcal{D} \\ & \mathrm{Sp}(\mathcal{D}) \arrow[u, "\Omega^\infty"'] \end{tikzcd}$$

1.4.4. **Proposition** ([2, prop. 1.4.2.24])**.** *Let $\mathcal{C}$ be pointed and admitting finite limits. Then $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ commutes with $\Omega : \mathcal{C} \to \mathcal{C}$ and it can be lifted to an equivalence,*

$$\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \xrightarrow{\sim} \underleftarrow{\mathrm{holim}} \left\{ \cdots \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \right\}$$

In other words, Spectra are Ω-spectra.

Let's illustrate the theorem. Given a functor $E : \mathcal{S}_*^{\mathrm{fin}} \to \mathcal{C}$, excisive and reduced. Note that in $\mathcal{S}_*^{\mathrm{fin}}$ we have the following pushout square,

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & S^1, \end{tikzcd}$$

So under E that is an excisive and reduced functor we get the following pullback square,

$$\begin{tikzcd} E(S^0) \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & E(S^1), \end{tikzcd}$$

We already defined $\Omega^\infty(E) \simeq E(S^0)$ and so we get that $E(S^0) \simeq \Omega(E(S^1))$. Setting $\{E(n), n \in \mathbf{Z}\}$, to be the family $E(n) \simeq E(S^n)$, we get that E is, under Ω^∞ an Ω-spectrum, $E(n) \simeq \Omega E(n+1)$.

Let's make here the remark that $\mathrm{Sp}(\mathcal{C})_* \simeq \mathrm{Sp}(\mathcal{C}_*)$, and so we can replace $\mathcal{C}$ by $\mathcal{C}_*$ in the tower of theorem 1.4.4.

1.5. **Another Universal property of the category of spectra.** Let's recall the definition of presentable ∞-categories.

1.5.1. **Définition.** (1) A ∞-category is accessible if it has a set of compact objects that span the ∞-category under filtered colimits.

(2) A ∞-category is presentable if it is accessible and admits finite colimits.

In the sable setting, presentability can be formulated in a particularly simple way given the following theorem.

1.5.2. **Théorème.** *(1) A stable ∞-category admits small colimits if and only if it admits small coproducts.*

(2) an exact functor between stable ∞-categories preserves small colimits if and only if it preserves small limits.

(3) An object in a stable ∞-category is compact if for any map landing in a coproduct, it factors up to homotopy through a finite subcoproduct.

1.5.3. **Corollaire** ([2, coro. 1.4.4.2])**.** *Let $\mathcal{C}$ be a stable ∞-category. Then $\mathcal{C}$ is presentable if and only if the following are satisfied,*

(1) The ∞-category $\mathcal{C}$ admits small coproducts,

(2) The homotopy category $\mathsf{Ho}(\mathcal{C})$ is locally small,

(3) There is a compact generator $X \in \mathcal{C}$.

Then we can move to state that $\mathrm{Sp}(\mathcal{C})$ is presentable given $\mathcal{C}$ is.

1.5.4. **Proposition.** *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories, and suppose $\mathcal{D}$ is stable.*

(1) The ∞-category $\mathrm{Sp}(\mathcal{C})$ is presentable.

(2) The functor $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\Sigma^\infty_+ : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$.

(3) A functor $G : \mathcal{D} \to \mathrm{Sp}(\mathcal{C})$ admits a left adjoint if and only if $\Omega^\infty \circ G$ admits a left adjoint.

Now we can give the universal property of Σ^∞_+ in the realm of presentable ∞-categories and left adjoint functors. Let's denote $\mathrm{LFun}(\mathcal{C}, \mathcal{D})$ the full subcategory spanned by functors which admit right adjoints (i.e. that are left adjoints) given $\mathcal{C}$ and $\mathcal{D}$ are presentable ∞-categories.

1.5.5. **Corollaire.** *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable ∞-categories, and suppose that $\mathcal{D}$ is stable. Then composition with Σ^∞_+ induces an equivalence,*

$$\mathrm{LFun}(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{LFun}(\mathcal{C}, \mathcal{D}).$$

This can be pictured as the following diagram,

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "\Sigma^\infty_+"'] & \mathcal{D} \\ \mathrm{Sp}(\mathcal{D}), \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

given the functor $\mathcal{C} \to \mathcal{D}$ is left adjoint.

1.6. **Connective Spectra.**

2. Smash product is monoidal

Lurie [2]

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