NOTES FOR THE GROUPE DE TRAVAIL "HIGHER ALGEBRA AND GEOMETRY" 2017-2018 THH

1. Talk 1: Hochschild Homology and Circle Actions. (Marco)

Let k be a base field and R a commutative k-algebra. The goal of today's talk is to illustrate the following statement:

- The Hochschild homology complex $\mathrm{HH}(R/k)$ carries an homotopy action of the circle S^1;

One way to defined the Hochschild complex $\mathrm{HH}(R/k)$ directly is via the derived tensor product

$$(*) \quad \mathrm{HH}(R/k) := R \otimes^{\mathbb{L}}_{R \otimes^{\mathbb{L}}_k R} R$$

This fancy formula makes it easy to make the circle appear: in the ∞-category $\mathrm{CAlg}(\mathrm{Mod}_k)$, the tensor product corresponds to the homotopy pushout [HA, 5.3.3.3] and we find

$$\mathrm{HH}(R/k) := R \coprod^{h}_{R \coprod^h_k R} R \simeq R \otimes_k (* \coprod^{h}_{* \coprod *} *) \simeq R \otimes_k S^1$$

This description presents Hochschild $\mathrm{HH}(R/k)$ as the free algebra with S^1-action.

Task today: review some alternative ways to make the circle appear. Usually one finds in the literature (Ex: Loday) the groups $\mathrm{HH}_n(M, R/k)$ defined as the homology groups of the normalized chain complex associated to the simplicial object in k-modules given by $(M, R)_\bullet =$

$$\begin{tikzcd}
\mathscr{L} \quad \dots \; M \otimes R^{\otimes 3} \arrow[r, shift left=9, "\partial_3"] \arrow[r, leftarrow, shift left=6, "\epsilon_2"] \arrow[r, shift left=3, "\partial_2"] \arrow[r, leftarrow, "\epsilon_1"] \arrow[r, shift right=3, "\partial_1"] \arrow[r, leftarrow, shift right=6, "\epsilon_0"] \arrow[r, shift right=9, "\partial_0"'] & M \otimes R \otimes R \arrow[r, shift left=6, "\partial_2"] \arrow[r, leftarrow, shift left=3, "\epsilon_1"] \arrow[r, "\partial_1"] \arrow[r, leftarrow, shift right=3, "\epsilon_0"] \arrow[r, shift right=6, "\partial_0"'] & M \otimes R \arrow[r, shift left=4, "\partial_1"] \arrow[r, leftarrow, "\epsilon_0"] \arrow[r, shift right=4, "\partial_0"'] & M
\end{tikzcd}$$

with

- $\partial_0(m \otimes r_1 \otimes \cdots r_n) = m.r_1 \otimes r_2 \otimes \cdots r_n$
- $\partial_i(m \otimes r_1 \otimes \cdots r_n) = m \otimes r_2 \otimes \cdots \otimes r_i.r_{i+1} \otimes \cdots \otimes r_n$
- $\partial_n(m \otimes r_1 \otimes \cdots r_n) = r_n.m \otimes r_1 \otimes \cdots r_{n-1}$

Remark 1.1. Don't care about degeneracies because the inclusion $\Delta^{op}_{inj} \subseteq \Delta^{op}$ is cofinal.

One must explain why $(*)$ coincides with this definition:

Proposition 1.2. *We have an equivalence in the ∞-category* Mod_k

$$\mathrm{HH}(R/k) \simeq |(R,R)_\bullet|$$

Proof. Here's a sketch: the r.h.s is the homotopy colimit of the simplicial object $(R,R)_\bullet$. For the l.h.s one considers the bar-resolution of R as an $R \otimes R^{op}$-module given by the augmented simplicial object

$$\begin{tikzcd}
\mathscr{L} \quad \ldots\ R^{\otimes 5} \arrow[r, shift left=9, "\partial_3"] \arrow[r, shift left=3, "\partial_2"] \arrow[r, shift right=3, "\partial_1"] \arrow[r, shift right=9, "\partial_0"'] & R^{\otimes 4} \arrow[l, shift left=6, "\epsilon_2"'] \arrow[l, "\epsilon_1"'] \arrow[l, shift right=6, "\epsilon_0"'] \arrow[r, shift left=6, "\partial_2"] \arrow[r, "\partial_1"] \arrow[r, shift right=6, "\partial_0"'] & R^{\otimes 3} \arrow[l, shift left=3, "\epsilon_1"'] \arrow[l, shift right=3, "\epsilon_0"'] \arrow[r, shift left=3, "\partial_1"] \arrow[r, shift right=3, "\partial_0"'] & R^{\otimes 2} \arrow[l, "\epsilon_0"'] \arrow[r] & R
\end{tikzcd}$$

which exhibits

$$\mathrm{colim}_{n \in \Delta} R^{n+2} \simeq R$$

Finally we have

$$l.h.s = R \otimes_{R \otimes R} (\mathrm{colim}_{n \in \Delta} R^{n+2} \simeq R) \simeq \mathrm{colim}_{n\in\Delta} R \otimes_{R\otimes R} (R^{n+2}) \simeq \mathrm{colim}_{n\in\Delta} R^{n+1}$$

□

Warning 1.3. The l.h.s of Prop 1 has two advantages: it carries a canonical algebra structure and a canonical S^1-action. The goal of today's lecture is to explain how to make the S^1-action appear on the r.h.s. The fact that it also carries an algebra structure and that the equivalence of Prop. 1 is compatible with both S^1 and algebra structures will be address in future talks.

Last week we saw that the simplicial object $(R,R)_\bullet$ carries extra symmetries, namely, at each level $[n]$ we have an action of the cyclic group $C_{n+1} \simeq \mathbb{Z}(n+1)\mathbb{Z}$ and the face and degeneracy maps are compatible with these actions. Namely, the cyclic permutation τ_n acts by

$$\tau_n(r_0 \otimes r_1 \otimes \ldots \otimes r_n) = (-1)^n r_n \otimes r_0 \otimes \ldots \otimes r_{n-1}$$

and we have the following compatibilities $(**)$

- $\tau^n \delta_i = \delta_{i-1}\tau_{n-1}$ for $1 \le i \le n$;
- $\tau_n^{n+1} = Id$
- $\tau_n \delta_0 = \delta_n$
- $\tau^n \sigma_i = \sigma_{i-1}\tau_{n-1}$ for $1 \le i \le n$;

- $\tau_n\sigma_0 = \sigma_n\tau_{n+1}^2$

One way to axiomatize the situation is to formally add extra symmetries to the category Δ, by designing a new category Λ (the cyclic category) with an inclusion $i : \Delta \to \Lambda$ such that it contains the same objects, namely finite non-empty ordered sets $[n]$ and morphisms are freely generated by faces δ_i, degeneracies σ_j and cyclic permutations $\tau_n : [n] \to [n]$, and imposing the relations $(**)$.

The observation that $(R, R)_\bullet$ possesses the extra symmetries can now be rephrased by saying that the simplicial object $(R, R)_\bullet : \Delta^{op} \to \mathrm{Mod}_k$ in fact comes from a functor $\Lambda^{op} \to \mathrm{Mod}_k$ by composition with the inclusion $\Delta^{op} \to \Lambda^{op}$

Remark 1.4. Notice that:

(a) $Aut_\Lambda([n]) = C_{n+1}$ is the cyclic group of order $n + 1$;

(b) Any morphism in $f \in \Lambda$ can be written uniquely as the composite $u \circ g$ where $u \in \Delta$ and $g \in C_n$. In particular, $Hom_\Lambda([n], [m]) = Hom_\Delta([n], [m]) \times C_{n+1}$ as sets; Notice in particular that contrary to the situation in the category of sets, nothing in the cyclic relations forces $\epsilon_0.t = \epsilon_0$ for $\epsilon_0 : [2] \to [0]$.

(c) Using (b), the structure of the category Λ encodes a simplicial set $C_\bullet$ defined by $[n] \mapsto Aut_\Lambda([n]) = C_{n+1}$ and sending $u : [m] \to [n]$ in Δ to the map $u^* : Aut_\Lambda([n]) \to Aut_\Lambda([m])$ defined by the property (b): given $f : [n] \to [n]$ in Λ we form $f \circ u$ and we know that there exists a unique $\tilde{f} : [m] \to [m]$ in $Aut_\Lambda([m])$ and $v : [m] \to [n]$ in Δ such that $f \circ u = v \circ \tilde{f}$. We then set $u^*(f) = \tilde{f}$. This forms a simplicial set. The simplicial set $C_\bullet$ is a minimal model for the simplicial set encoding the circle $\Delta[1]/\partial\Delta[1]$.

(d) The category Λ is equivalent to Λ^{op} by sending $[n]$ to $[n]$, faces to degeneracies and vice-versa. This is possible because one can use the cyclic permutation τ_n to define a new degeneracy $\sigma_{n+1} := \sigma_0 \circ \tau_n^{-1}$.

The fact that $(R, R)_\bullet$ comes from a cyclic object is crucial to understand why the geometric realization $|(R, R)_\bullet|$ carries a circle action. Namely, the extra symmetries provide additional information to the geometric realization.

Remark 1.5. It is crucial to this story to remark that the homotopy colimit of a cyclic object F is very different from the homotopy colimit of its underlying simplicial object, ie, in general,

$$|F|_{cyc} := \mathrm{colim}_{\Lambda^{op}} F \neq \mathrm{colim}_{\Delta^{op}} F_{|\Delta^{op}} =: |F \circ i|_{\Delta^{op}}$$

In other words, the map $i : \Delta^{op} \to \Lambda^{op}$ is not cofinal. It is somehow the goal of this talk to explain this.

Now we illustrate the mechanism responsible for the appearance of the circle action. Fix an ∞-category $\mathscr{C}$. Let K be an ∞-category (actually we could take any simplicial set here) and let us consider $\mathrm{Fun}(K, \mathscr{C})$. In mind you should keep the examples $K = N(\Delta^{op})$ or $K = N(\Lambda^{op})$. Recall that the colimit functor

$$\mathrm{Fun}(K, \mathscr{C}) \to \mathscr{C}$$

is obtained as a left-Kan extension along the projection $\pi_K : K \to *$. Let us write $(\pi_K)_! := \mathrm{hocolim}_K$. As π_K sends all morphisms in K to equivalences, it factors through the ∞-groupoid completion of K i.e., the ∞-categorical localization of K along the class W_{all} consisting of all its morphisms, $L(K) := K[W_{all}^{-1}]_\infty$.

$$\begin{tikzcd}
K \arrow[dr, "l"] \arrow[dd, "\pi_K"'] & \\
& L(K) \arrow[dl, "\pi_{L(K)}"] \\
* &
\end{tikzcd}$$

Remark 1.6. Notice that if we see K as a simplicial set, $L(K)$ can be obtained as a Kan fibrant replacement of K for the standard Quillen model structure.

In particular,the colimit functor $\mathrm{Fun}(K, \mathscr{C}) \to \mathscr{C}$ presents itself as composition of left Kan extensions

$$\mathrm{Fun}(K, \mathscr{C}) \xrightarrow{l_!} \mathrm{Fun}(L(K), \mathscr{C}) \xrightarrow{(\pi_{L(K)})_!} C$$

Whenever K is connected, by [Boardman-Vogt, May Theorem for iterated loop spaces] $L(K)$ is of the form $B\, \Omega_k L(K)$ where $\Omega_k L(K) \simeq \mathrm{Aut}_{L(K)}(k)$ for k the choice of an object in K. In this case, the colimit functor factors as

$$\begin{tikzcd}
\mathrm{Fun}(K, \mathscr{C}) \arrow[rr, "l_!"] \arrow[dr, "hocolim_K"'] & & \mathrm{Fun}(B\,\mathrm{Aut}_{L(K)}(k), \mathscr{C}) \arrow[dl, "(\pi_{L(K)})_!"] \\
& \mathscr{C} &
\end{tikzcd}$$

Notice that $\mathrm{Fun}(B\,\mathrm{Aut}_{L(K)}(k), \mathscr{C})$ corresponds to the category of objects in $\mathscr{C}$ endowed with an action of the loop group $\mathrm{Aut}_{L(K)}(k)$. The object being acted is obtain by the restriction map

$$e^* : \mathrm{Fun}(B\,\mathrm{Aut}_{L(K)}(k), \mathscr{C}) \to \mathscr{C}$$

given by composition along the inclusion of the canonical point $e : * \to B\,\mathrm{Aut}_{L(K)}(k)$.

Moreover, the left Kan extension along the projection $\pi_{L(K)} : B\,\mathrm{Aut}_{L(K)}(k), \mathscr{C}) \to *$ is by definition, the construction of homotopy orbits. In other words:

- For any K-diagram F, the object $X = e^*(l_!(F))$ carries an action of the loop group $\mathrm{Aut}_{L(K)}(k)$ such that

$$hocolim_K\, F \simeq X_{h\,\mathrm{Aut}_{L(K)}(k)}$$

Let us now bring this discussion to $K = N(\Delta^{op})$ and $K = N(\Lambda^{op})$.

When we apply this to $K = N(\Delta^{op})$,$L(K)$ recovers the classical classifying space of Δ. We recall that

Lemma 1.7. *The classifying space $L(N(\Delta^{op}))$ is contractible $*$.*

Proof. Following [Quillen Higher Algebraic K-theory I, Corollary I p.8] this follows because Δ has a final object $[0]$. ☐

In other words, in this case nothing happens.

However, when we apply this to $K = N(\Lambda^{op})$, the interesting stuff appears:

Proposition 1.8. *We have:*

- *(Connes) $Aut_{L(N(\Lambda^{op}))}([0]) \simeq S^1$ and $L(N(\Lambda^{op})) \simeq K(\mathbb{Z}, 2) \simeq BS^1 \simeq \mathbb{CP}^\infty$*
- *The diagram*

$$\begin{tikzcd}
\mathrm{Fun}(N(\Delta^{op}), \mathscr{C}) \arrow[r, "|-|_{\Delta^{op}}"] & \mathscr{C} \\
\mathrm{Fun}(N(\Lambda^{op}), \mathscr{C}) \arrow[u, "-\circ i"] \arrow[r, "l_!"] & \mathrm{Fun}(L(N(\Lambda^{op})) = B\,S^1, \mathscr{C}) \arrow[u, "-\circ e"']
\end{tikzcd}$$

commutes. In other words, if F is a cyclic object in $\mathscr{C}$ then the geometric realization of its underlying simplicial object $F_{|\Lambda^{op}} := F \circ i$ carries an action of S^1.

Proof. For the first identification the steps are:

(i) $N(\Lambda^{op})$ is connected as a simplicial set because every two objects can be related by at least one morphim;

(ii) the space $Hom_\Lambda(0, 0)$ consists of a single element so that $L(N(\Lambda^{op}))$ is simply connected;

(iii) We compute the cohomology ring $H^*(L(N(\Lambda^{op})), \mathbb{Z})$ and show that it is equal to the polynomial algebra $\mathbb{Z}[\sigma]$ with

$$\sigma \in H^2(L(N(\Lambda^{op})), \mathbb{Z}) \simeq [L(N(\Lambda^{op})), \mathbb{Z}), K(\mathbb{Z}, 2) \simeq \mathbb{CP}^\infty]$$

.

This is the key computation done by Connes. The idea is more or less the following: first, one remarks that as explained that by the Grothendieck constructionf for spaces, (see also Quillen's algebraic K-theory paper Prop. 1 page 6) the cohomology groups $H^*(L(N(\Lambda^{op})), L)$ with coefficients in a local system L can be computed by looking at the functor associated to L, $L' : L(N(\Lambda^{op})) \to \mathrm{Mod}_{\mathbb{Z}}$, taking its homotopy limit and computing its homology groups:

$$H^*(L(N(\Lambda^{op})), L) \simeq H^*(\mathrm{holim}_{\Lambda^{op}}(L'))$$

Notice that from the adjunction defining limits, one has

$$\mathrm{holim}_{\Lambda^{op}}(L') \simeq \mathrm{RHom}_{\mathrm{Fun}(\Lambda^{op},\mathrm{Mod}_{\mathbb{Z}})}(\underline{\mathbb{Z}}, L')$$

where $\underline{\mathbb{Z}}$ is the constant cyclic module $\Lambda^{op} \to \mathrm{Mod}_{\mathbb{Z}}$ with value $\mathbb{Z}$. Another way to see this is to prescribe an explicit resolution of the constant cyclic module $\underline{\mathbb{Z}}$ via the Yoneda's map: notice that for a general category C, it is a consequence of the Yoneda's lemma that the colimit of the Yoneda diagram $C \to \mathrm{PSh}(C)$ is the constant presheaf of spaces equal to the point $*$. Using the same principle one checks that

$$\underline{\mathbb{Z}} \simeq \mathrm{colim}_{\Lambda}\, j_{[n]}$$

where $j : \Lambda \to \mathrm{Fun}(\Lambda^{op}, \mathrm{Mod}_{\mathbb{Z}})$ is $\mathbb{Z}$-linearization of the Yoneda inclusion. In particular, we find

$$H^*(L(N(\Lambda^{op})), \mathbb{Z}) \simeq H^*(\mathrm{RHom}_{\mathrm{Fun}(\Lambda^{op},\mathrm{Mod}_{\mathbb{Z}})}(\underline{\mathbb{Z}}, \underline{\mathbb{Z}}))$$

To conclude, the computation needs an explicit model for the functor $\Lambda \times \Lambda^{op} \to \mathrm{Mod}_Z$ associated to j. Check Connes paper [Cohomologie cyclique et foncteurs Extn, Section 4] for this.

(iv) The map corresponding to σ is an equivalence on each H_n and by Whithead's Theorem, it is in this case, a weak-equivalence of topological spaces.

The second claim follows from the fact that the commutative diagram

$$\begin{tikzcd} N(\Delta^{op}) \arrow[r, "i"] \arrow[d, "\pi_\Delta"] & N(\Lambda^{op}) \arrow[d, "\pi_\Lambda"] \\ * \simeq L(N(\Delta^{op})) \arrow[r, "\tilde{i}"] & L(N(\Lambda^{op})) \end{tikzcd}$$

is adjointable in the sense that $\tilde{i}^*(\pi_\Lambda)_! \simeq (\pi_\Delta)_! i^*$. This can check using the right adjoints, as for the right kan extensions it is easy to check directly that

$$i_*\pi_\Delta^* \simeq \pi_\Lambda^* \tilde{i}_*$$

□

Remark 1.9. There is a simpler way to show that $Aut_{L(N(\Lambda^{op}))}([0]) \simeq S^1$ that avoids the use of an explicit resolution for the Yoneda map $\Lambda \to \mathrm{PSh}(\Lambda)$. Indeed, it follows from the remark 1.4 that the underlying simplicial object of the cyclic set $j[0] : \Lambda^{op} \to \mathrm{N}(\mathrm{Sets})$ is already a simplicial model of the circle $\Delta^1/\partial^0\Delta^1$. Indeed, if $i : \Delta^{op} \to \Lambda^{op}$ is the inclusion, the factorization property explained in the Remark 1.4 a) gives an isomorphism of simplicial sets $i^*(j[0]) \simeq \Delta[0] \times C_\bullet \simeq C_\bullet$ where $C_\bullet$ is the simplicial set of Remark 1.4 b) which is a model for the circle. Using this, we conclude the computation of the classifying space using the fact that all the maps $j[n] \to j[m]$ become equivalences after the Kan extensions to $L(\Lambda^{op})$ and the fact that the Kan extension is given by the underlying geometric realization (using the second-half of the Proposition 1.8).

Warning 1.10. The conclusion is that the complex $|(R, R)_\bullet|$ carries a circle action. We still have to explain why it also carries an algebra structure and the two are compatible. We will come back to this in two weeks.