M. Morrow
TC Study group
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Cyclotomic Spectra

§1. Cyclotomic Spectra

<u>Def^n of objects</u> (i) A cyclotomic spectrum is a spectrum X equipped with an S^1-action (i.e. $X \in Sp^{BG}$) and an $S^1 \cong S^1/C_p$-equivariant map

$$\varphi_p : X \longrightarrow X^{tC_p}$$

for each prime number p.

(ii) Sometimes we work with a fixed prime p: a p-cyclotomic spectrum is a spectrum X equipped with a C_{p^∞} ($= \mathbb{Q}_p/\mathbb{Z}_p = \mathbb{Z}/p^\infty$ = p-power torsion in S^1)-action and a $C_{p^\infty} \cong C_{p^\infty}/C_p$-equiv^t map

$$\varphi_p : X \longrightarrow X^{tC_p}$$

(Rmk: If X is p-complete then can replace C_{p^∞} by S^1)

In both cases φ_p is called the p^{th} cyclotomic Frobenius.

<u>Def^n</u> (topol. cyclic hom.)
(i) If $(X, (\varphi_p)_p)$ is a cyclotomic spectrum, its topological cyclic homology is the spectrum

$$TC(X) := \mathrm{Eq}\left(X^{hS^1} \underset{\prod_p \mathrm{can}}{\overset{\prod_p \varphi_p^{hS^1}}{\rightrightarrows}} \prod_p (X^{tC_p})^{hS^1} \right)$$

where the maps are

- $\varphi_p^{hS^1} : X^{hS^1} \longrightarrow (X^{tC_p})^{hS^1}$
- can : $X^{hS^1} \simeq (X^{hC_p})^{hS^1/C_p} \underset{\substack{\text{via}\\ S^1/C_p \cong S^1}}{\simeq} (X^{hC_p})^{hS^1} \longrightarrow (X^{tC_p})^{hS^1}$

(iii) If (X, φ_p) is a p-Cyclotomic Spectrum then its top. cyclic homol. is

$$TC(X) := Eq\left(X^{hC_{p^\infty}} \underset{can}{\overset{\varphi_p^{hC_{p^\infty}}}{\rightrightarrows}} (X^{tC_p})^{hC_{p^\infty}} \right)$$

Also define the negative cyclic homology and periodic as
$TC^-(X) := X^{hS^1}$, $TP(X) := X^{tS^1}$ ($S^1 \leadsto C_{p^\infty}$ in p case)

<u>Important special case</u>: If X is a bdd-below, p-complete spectrum, then we will see in Ausoni's talk that

- S^1-action on X $\equiv$ C_{p^∞}-action on X
- $X^{tC_q} \simeq 0$ for any prime $q \neq p$ ($\Rightarrow \varphi_q = 0$)
- $(X^{tC_p})^{hS^1} \simeq X^{tS^1}$ by clever applications of Tate orbit lemma
  is $(X^{tC_p})^{hC_{p^\infty}} \simeq X^{tC_{p^\infty}}$ is

So in this case a cycl. structure on X is the same as a p-cycl. structure, and

$$TC(X) = Eq\left(TC^-(X) \underset{can}{\overset{\varphi_p^{hS^1}}{\rightrightarrows}} TP(X) \right)$$

where
- $\varphi_p : X^{hS^1} \to (X^{tC_p})^{hS^1} \simeq X^{tS^1}$
- $can : X^{hS^1} \to X^{tS^1}$ usual map.

Examples (in another talk) $\widehat{\text{in } p \text{ case}}$

(i) For any $\mathbb{E}_1$-alg R we will see that $THH(R)$ is a cyclotomic spectrum. φ_p will be induced by the ~~maps~~ Tate diagonals

$$R^{\otimes n} \xrightarrow[\Delta_p]{} (R^{\otimes np})^{tC_p}$$

for all $n \geq 0$. Write $TC(R) = TC(THH(R))$ similarly TC^-, TP.

ii) If X is any spectrum we may give it the trivial $\mathbb{S}^1$-action and let φ_p be "inclusions"

$$\varphi_p : X \xrightarrow[?]{} \underset{\simeq F(BC_{p+}, X)}{X^{hC_p}} \xrightarrow{\text{can.}} X^{tC_p}$$

$\rightsquigarrow$ cyclotomic spectrum X^{triv}

iii) In particular, get $\mathbb{S}^{triv}$ ($\simeq THH(\mathbb{S})$ as cycl. spectra)

<u>Proposition</u> (Properties of LEq). Let $\mathcal{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$ be functors between ∞-cats.

(i) The square LEq is homotopy cartesian in Cat_∞

(ii) Let $X = (C_X, f_X)$, $Y = (C_Y, f_Y) \in \mathrm{LEq}(F,G)$. Then

$$\mathrm{Map}_{\mathrm{LEq}(F,G)}(X,Y) \simeq \mathrm{Eq}\left(\mathrm{Map}_{\mathcal{C}}(C_X, C_Y) \overset{f_X^* G}{\underset{f_Y^* F}{\rightrightarrows}} \mathrm{Map}_{\mathcal{D}}(F(C_X), G(C_Y))\right)$$

($\Rightarrow \pi_0 \mathrm{Map}(X,Y)$ = maps $h : C_X \to C_Y$ in $\mathcal{C}$ s.t the foll. diag. comm:

$$\begin{array}{ccc} F(C_X) & \xrightarrow{f_X} & G(C_X) \\ {\scriptstyle F(h)}\downarrow & & \downarrow{\scriptstyle G(h)} \\ F(C_Y) & \longrightarrow & G(C_Y) \end{array}$$
)

Also, $h \in \mathrm{Map}(X,Y)$ is an equivalence iff its image in $\mathrm{Map}_{\mathcal{C}}(C_X, C_Y)$ is an equivalence.

(iii) $\mathcal{C}, \mathcal{D}$ stable and F, G exact $\Rightarrow$ $\mathrm{LEq}(F,G)$ is stable and $\mathrm{LEq}(F,G) \to \mathcal{C}$ is exact

(iv) $\mathcal{C}$ presentable, $\mathcal{D}$ accessible, F colimit-preserving, G access.

$\Rightarrow$ $\mathrm{LEq}(F,G)$ is presentable and $\mathrm{LEq}(F,G) \to \mathcal{C}$ is colimit-preser.

(<u>Def$^{\text{ns}}$</u>: • $\mathcal{D}$ access means $\exists$ a regular cardinal s.t (a) $\mathcal{D}$ is locally small and has small κ-filtered colimits (b) full subcat $\mathcal{D}^\kappa \subseteq \mathcal{D}$ of κ-compact objts is ess. small (c) $\mathcal{D}^\kappa$ gens $\mathcal{D}$ under small κ-filtered colimits

• $\mathcal{C}$ presentable means accessible + has small colimits • G access means $\exists$ κ st. preserves small κ-filtered colimits)

(v) Suppose $p : K \to \mathrm{LEq}(F,G)$ is a diagram s.t $K \to \mathrm{LEq} \to \mathcal{C}$ has a limit preserved by G. Then p has a limit preserved by $\mathrm{LEq} \to \mathcal{C}$.

§2 The category of cyclotonic spectra

<u>Def^n</u> $\mathcal{C}, \mathcal{D}$ ∞-cats ; $F, G : \mathcal{C} \to \mathcal{D}$ functors. The <u>lax equaliser</u> of F and G is the ∞-cat defined as the pullback of simplicial sets

$$\begin{tikzcd} \mathrm{LEq}(F,G) \arrow[r] \arrow[d] & \mathcal{D}^{\Delta^1} \arrow[d, "(ev_0, ev_1)"] \\ \mathcal{C} \arrow[r, "(F,G)"'] & \mathcal{D} \times \mathcal{D} \end{tikzcd} \qquad (\mathrm{LEq})$$

<u>Rmk</u>: Objects of $\mathrm{LEq}(F,G)$ are pairs (c,f) where
- $c \in \mathrm{Ob}\,\mathcal{C}$
- $f : F(c) \to G(c)$ is a morphism in $\mathcal{D}$

<u>Def^n</u> (i) The ∞-cat CycSp of cycl. spectra is lax equaliser of

$$Sp^{BS^1} \underset{\prod_p -^{tC_p}}{\overset{\prod_p \mathrm{id}}{\rightrightarrows}} \prod_p Sp^{BS^1}$$

(ii) Similarly, CycSp_p is lax equaliser of

$$Sp^{BC_{p^\infty}} \underset{-^{tC_p}}{\overset{\mathrm{id}}{\rightrightarrows}} Sp^{BC_{p^\infty}}$$

<u>NB</u> By Rmk above, objects of CycSp and CycSp_p are as in 1st def^n.

Idea of Proof

∞-Cat Yoga (all refs to Lurie omitted in following.)

(i) the functor (ev_0, ev_1) • is an inner fibration
• given an equivalence in its codomain $D \times D$ and a lift of the source to D^{Δ^1}, the equiv. lifts

$\Downarrow$ Thm of Joyal

(ev_0, ev_1) is a categorical fibration (i.e. fibration for Joyal model str. on sSets)

(ii) Have pullback of sSets:

$$\begin{tikzcd} \mathrm{Hom}^R(X,Y) \arrow[r] \arrow[d] & \mathrm{Hom}^R(fX,fY) \arrow[d] \\ \mathrm{Hom}^R_{\mathcal{C}}(C_X,C_Y) \arrow[r] & \mathrm{Hom}^R(F(C_X),F(C_Y)) \times \mathrm{Hom}^R(G(C_X),G(C_Y)) \end{tikzcd}$$

Kan fibr. since Hom^R takes inner fibs to Kan fibs.

$\Rightarrow$ square is hom. cart. in Quillen model str. on sSets

$\Rightarrow$ pullback in Kan after replacing Hom^R by Map.

$\Rightarrow$ Get claims.

(iii) Hypotheses $\Rightarrow$ $D \times D$, D^{Δ^1} are stable $\Rightarrow$ the hom. pullback LEq is also stable.

(iv) Hypotheses $\Rightarrow$ $D \times D$, D^{Δ^1} are acces. $\Rightarrow$ hom. pullback LEq is also access.
Must show LEq has all small colimits — omitted.

(v) Dual argument to (iv).

Corol: We have the following properties about CycSp and the forgetful functor

$$\mathrm{CycSp} \longrightarrow \mathrm{Sp}$$
$$(X, (\varphi_p)_p) \longmapsto X$$

- CycSp is presentable stable ∞-cat (also sym. monoidal) greatly ⊗ in Sp^{BS^1}
- forgetful functor is exact, preserves all colimits, reflects equivalences.
- A map $(X, \varphi_p) \to (Y, \varphi_p)$ of cycl. sp. is an S^1-equiv map $X \to Y$ and a 2-cell

$$\begin{tikzcd} X \arrow[r, "\varphi_p"] \arrow[d] & X^{tC_p} \arrow[d] \\ Y \arrow[r, "\varphi_p"'] & Y^{tC_p} \end{tikzcd}$$

for each p.

Proof
Special case of previous prop^n since
- Sp^{BS^1} is stable ~~exact~~ ~~$t_{C_p}: \mathrm{Sp}^{BS^1} \to \mathrm{Sp}^{BS^1}$ is exact~~
- forget: $\mathrm{Sp}^{BS^1} \to \mathrm{Sp}$ preserves all colimits and reflects equivalences ($\Rightarrow$ exact)
- $t_{C_p}: \mathrm{Sp}^{BS^1} \to \mathrm{Sp}^{BS^1}$ is exact and $\underbrace{\text{accessible}}$

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$-_{hC_p}$, $-^{hC_p}$ are access.
(since they have adjoints and Sp^{BS^1} is access.)

$\square$

Next we show that TC is corepresentable.
Not^n: $\mathcal{C}$ ∞-cat, $X, Y \in \mathcal{C}$ $\rightsquigarrow$ Mapping spectrum $\mathrm{map}_{\mathcal{C}}(X,Y)$
s.t. • $\Omega^\infty \mathrm{map}_{\mathcal{C}}(X,Y) = \mathrm{Map}_{\mathcal{C}}(X,Y)$
• $\mathrm{map}_{\mathcal{C}}(X,-)$ exact.

Thm: $TC(X) \simeq \mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{triv}, X)$

Proof

For $X, Y \in \mathrm{CycSp}$ we have an fibre sequ.

$$\mathrm{map}_{\mathrm{CycSp}}(Y,X) \longrightarrow \mathrm{map}_{\mathrm{Sp}^{BS^1}}(Y,X) \underset{\prod_p \mathrm{can}}{\overset{\prod_p \varphi_p}{\rightrightarrows}} \prod_p \mathrm{map}_{\mathrm{Sp}^{BS^1}}(Y, X^{tC_p})$$

Why? - true for Map instead of map by Prop (ii).
- hence true after Ω^∞
- $\mathrm{Fun}^{\mathrm{Ex}}(\mathrm{CycSp}, \mathrm{Sp}) \underset{\Omega^\infty}{\simeq} \mathrm{Fun}^{\mathrm{Lex}}(\mathrm{CycSp}, \mathrm{Kan})$

Take $Y = \mathbb{S}^{\mathrm{triv}}$ to get

$$\mathrm{map}_{\mathrm{CycSp}}(\mathbb{S}^{\mathrm{triv}}, X) = \mathrm{Eq}\Big(\underbrace{\mathrm{map}_{\mathrm{Sp}^{BS^1}}(\mathbb{S}, X)}_{= X^{hS^1}} \rightrightarrows \underbrace{\prod_p \mathrm{map}_{\mathrm{Sp}^{BS^1}}(\mathbb{S}, X^{tC_p})}_{= (X^{tC_p})^{hS^1}} \Big)$$