# The Bökstedt spectral sequence and the Hopf structure on THH:

$R$ a $S$-algebra, using the skeleton filtration

$$E^2_{s,*}(R) = HH_s(H_*(R,\mathbb{F}_p)) \Longrightarrow H_{s+*}(THH(R),\mathbb{F}_p)$$

strongly convergent, of $A_*$-comodules $(A_* = H_*(H\mathbb{F}_p,\mathbb{F}_p))$
(apply mod $p$ homology to a filtered spectrum)

**Proposition:** $R$ commutative $S$-algebra. Then:
- $H_*(THH(R),\mathbb{F}_p)$ augmented commutative $A_*$-comodule $H_*(R,\mathbb{F}_p)$-algebra
- the s.s. is " " " "

$$THH(R) = R \otimes S^1$$

$$\begin{array}{ccc} * \longrightarrow S^1 & & R \longrightarrow THH(R) \\ S^1 \longrightarrow * & & THH(R) \longrightarrow R \\ S^1 \vee S^1 \longrightarrow S^1 & \rightsquigarrow & THH(R) \wedge_R THH(R) \longrightarrow THH(R) \\ S^1 \longrightarrow S^1 \vee S^1 & & THH(R) \longrightarrow THH(R) \wedge_R THH(R) \end{array}$$

**Theorem:** $R$ commutative $S$-algebra. Then:
- $THH(R)$ augmented commutative $R$-algebra
- in the homotopy category: $THH(R)$ $R$-Hopf-algebra.

**Theorem:** $R$ commutative $S$ algebra. $\Lambda = H_*(R,\mathbb{F}_p)$
1) if $H_*(THH(R),\mathbb{F}_p)$ is flat over $\Lambda$, there is a coproduct on $H_*(THH(R),\mathbb{F}_p)$ and it is a $A_*$-comodule $\Lambda$-Hopf-algebra.
2) if each $E^r_{**}(R)$ is flat $(r \geq 2)$ over $\Lambda$ (or up to a $n$) then there is a coproduct on $E^r_{**}(R)$ and we have a $A_*$-comodule $\Lambda$-Hopf algebra spectral sequence.

(The dual Steenrod algebra) HH of ...

$p=2$

$$A_* = P(\bar{\xi}_k,\ k \geq 1)$$

$$\deg \bar{\xi}_k = 2^k - 1$$

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{2^i}$$

$p$ odd

$$A_* = P(\bar{\xi}_k,\ k \geq 1) \otimes E(\bar{\tau}_k,\ k \geq 0)$$

$$\deg \bar{\xi}_k = 2p^k - 2 \qquad \deg \bar{\tau}_k = 2p^k - 1$$

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i} \qquad \bar{\xi}_0 = 1$$

$$\psi(\bar{\tau}_k) = 1 \otimes \bar{\tau}_k + \sum_{i+j=k} \bar{\tau}_i \otimes \bar{\xi}_j^{p^i}$$

$$\beta(\bar{\tau}_k) = \bar{\xi}_k$$

$\deg x = d$

$$HH_*(P(x)) \cong \underset{(0,d)}{P(x)} \otimes \underset{(1,d)}{E(\sigma x)}$$

$\to$ $A$ smooth $\Omega^*_A = \bigwedge_A \Omega_A \cong HH_*(A)$ Kähler diff. — Hochschild-Kostant-Rosenberg

as bi-algebras.

$$HH_*(E(x)) \cong E(x) \otimes \Gamma(\sigma x)$$

$\to$ direct computation

**Künneth's theorem:** $A$, $A'$ flat over $k$

$$HH_*(A) \otimes_k HH_*(A') \cong HH_*(A \otimes_k A')$$

$$HH_*(A_*) \cong \bigotimes_{k\geq 0} HH_*(E(\bar{\tau}_k)) \otimes \bigotimes_{k\geq 1} HH_*(P(\bar{\xi}_k))$$

$$\cong \bigotimes_{k\geq 0} E(\bar{\tau}_k)\otimes\Gamma(\sigma\bar{\tau}_k)\otimes\bigotimes_{k\geq 1} P(\bar{\xi}_k)\otimes E(\sigma\bar{\xi}_k)$$

$$\cong A_* \otimes \Gamma(\sigma\bar{\tau}_k, k\geq 0) \otimes E(\sigma\bar{\xi}_k, k\geq 1)$$

$(0, -)$ $\quad(1, 2p^k-1)$ $\quad(1, 2p^k-2)$

since over $\mathbb{F}_p$, generated by $\gamma_{p^i}(\sigma\bar{\tau}_k)$ $i\geq 0$

$p^i(1, 2p^k-1)$

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if $x$ is an infinite cycle in $E^2_{**}(R)$, we let $[x]$ be its classe in $H_*(THH(R), \mathbb{F}_p)$

$\alpha : THH(R)\wedge S^1_+ \longrightarrow THH(R)$ the circle action

$\eta : R \longrightarrow THH(R)$ the unit

$\rightsquigarrow$ get a map $R\wedge S^1_+ \longrightarrow THH(R)$

Denote $\sigma : H_*(R, \mathbb{F}_p) \longrightarrow H_{*+1}(THH(R), \mathbb{F}_p)$

$x \longmapsto \alpha(\eta(x)\otimes s_1)$ generator of $H_1(S^1_+, \mathbb{F}_p)$

Proposition: " $\sigma(x) = [\sigma x]$ " (McClure - Staffeldt)

Lemma: for $i\geq 0$, $\sigma(\bar{\xi}_{i+1}) = 0$

Thus $\sigma\bar{\xi}_{i+1} \in E^2_{**}(H\mathbb{F}_p)$ is not a permanent cycle.

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Dyer-Lashof operation:

'$H_\infty$ ring spectra and their applications'
Bruner - May - McClure - Steinberger

$H_*(R, \mathbb{F}_p)$, $R$ $E_\infty$-ring spectrum, admit operation:

$p=2$ $\quad Q^k : H_*(R, \mathbb{F}_2) \longrightarrow H_{*+k}(R, \mathbb{F}_2)$ $\quad Q^k(x) = 0$, $k<|x|$; $Q^k(x) = x^2$, $k=|x|$

$p$ odd $\quad Q^k : H_*(R, \mathbb{F}_p) \longrightarrow H_{*+2k(p-1)}(R, \mathbb{F}_p)$ $\quad Q^k(x) = 0$, $k<2|x|$; $Q^k(x) = x^p$, $k=2|x|$

+ Cartan formula

For $R = H\mathbb{F}_p$: $\quad Q^{p^k}(\bar{\xi}_k) = \bar{\xi}_{k+1}$ (all $p$) $\quad Q^{p^k}(\bar{\tau}_k) = \bar{\tau}_{k+1}$ ($p$ odd)

Proposition: (Bökstedt) $Q^k(\sigma(x)) = \sigma(Q^k(x))$

Proposition: (Angeltveit - Rognes) $Q^k(\alpha(x\otimes s_1)) = \alpha(Q^k(x)\otimes s_1)$

$x \in H_*(THH(R), \mathbb{F}_p)$

Proof: $\tilde{\alpha} : THH(R) \to F(S^1_+, THH(R))$

Let $DS^1_+ = F(S^1_+, S)$. The Dyer-Lashof operation $Q^k$ on $H_*(DS^1_+, \mathbb{F}_p)$ are trivial for $k \neq 0$.

$\nu : THH(R) \wedge DS^1_+ \xrightarrow{\text{composition}} F(S^1_+, THH(R))$ is an equivalence ($S^1_+$ finite CW-complex, Spanier-Whitehead duality)

$\nu, \tilde{\alpha}$ are maps of algebras $H_*(THH(R), \mathbb{F}_p) \xrightarrow{\tilde{\alpha}} H_*(F(S^1_+, THH(R)), \mathbb{F}_p)$

$\nu \cong \quad H_*(THH(R), \mathbb{F}_p) \otimes H_*(DS^1_+, \mathbb{F}_p)$

since $\tilde{\alpha}(x) = \alpha(x \otimes -)$, $\nu^{-1}\tilde{\alpha}(x) = x \otimes 1 + \alpha(x \otimes s_1) \otimes i_1$, $i_1 \in H^1(S^1_+, \mathbb{F}_p)$ dual of $s_1$.

maps of algebras: $Q^k(\nu^{-1}\tilde{\alpha}(x)) = \nu^{-1}\tilde{\alpha}(Q^k(x))$

External Cartan: $Q^k(x \otimes 1 + \alpha(x \otimes s_1) \otimes i_1) = Q^k(x) \otimes 1 + Q^k(\alpha(x \otimes s_1)) \otimes i_1$
$= Q^k(x) \otimes 1 + \alpha(Q^k(x) \otimes s_1) \otimes i_1$

Now $\sigma(x) = \alpha(\eta(x) \otimes s_1)$ and $Q^k \eta = \eta Q^k$.

Proof: (lemma) $\sigma(\overline{\xi}_{i+1}) = \sigma(\beta(\overline{\tau}_{i+1})) = \beta(\sigma(\overline{\tau}_{i+1}))$ naturality
$= \beta(\sigma(Q^{p^i}(\overline{\tau}_i))) = \beta(Q^{p^i}(\sigma(\overline{\tau}_i)))$ prop
$= \beta(\sigma(\overline{\tau}_i)^p)$ $\quad |\sigma(\overline{\tau}_i)| = p^i$
$= 0$ since the Bockstein is a derivation.

Proposition: In the spectral sequence, $d^r = 0$ for $2 \leq r \leq p-2$

There is $d^{p-1}(\gamma_{p+k}\sigma\overline{\tau}_i) \doteq \sigma\overline{\xi}_{i+1} \cdot \gamma_k \sigma\overline{\tau}_i$ (up to a unit in $\mathbb{F}_p$) $i \geq 0$, $k \geq 0$

$E^p_{**}(H\mathbb{F}_p) = A_* \otimes P_p(\sigma\overline{\tau}_1, \sigma\overline{\tau}_2, \ldots)$ and the s.s. collapse here

Lemma: $R$ commutative $S$-algebra, $\Lambda = H_*(R, \mathbb{F}_p)$ connected, $HH_*(\Lambda)$ flat over $\Lambda$. Then $E^2_{**}(R) = HH_*(\Lambda)$ is an $A_*$-comodule, $\Lambda$-Hopf-algebra and a shortest non-zero diff. $d^r_{s,t}$ in lowest total degree $s+t$ must map an algebra indecomposable to a coalgebra primitive and $A_*$-comod. prim.

Proof: • if $d^r(xy) \neq 0$, $d^r(xy) = d^r(x)y \pm x d^r(y)$, so $d^r(x) \neq 0$ or $d^r(y) \neq 0$ so that $xy$ is not in lowest total degree.

• if $d^r(z)$ is not coalgebra primitive: let $\psi(z) = z \otimes 1 + 1 \otimes z + \sum_i z_i' \otimes z_i''$
$\psi \circ d^r = (d^r \otimes 1 \pm 1 \otimes d^r) \circ \psi$ so some $d^r(z_i') \neq 0$ or $d^r(z_i'') \neq 0$ and $z$ cannot be in lowest possible degree.

• if $d^r(z)$ not $A_*$-comodule primitive: let $\nu(z) = 1 \otimes z + \sum_i a_i \otimes z_i$
$\nu \circ d^r = (1 \otimes d^r)\nu$ so some $d^r(z_i) \neq 0$
and " " " " " "

Proof: (Prop) shortest non zero $d^r$, $d^r(z) \neq 0$ with $z$ in lowest total degree

$z$ indecomposable $\Rightarrow$ the filtration degree of $z$ is $\geq p$

$d^r(z)$ coalgebra primitive $\Rightarrow$ the filtration degree of $d^r(z)$ is $0$ or $1$

so $r \geq p-1$, and then $d^r = 0$ for $2 \leq r \leq p-2$

Let $i \geq 0$. Suppose for all $0 \leq j \leq i-1$, there is a

$$d^{p-1}(\gamma_{p+k}\,\sigma\overline{\tau_j}) \doteq \sigma\overline{\xi_{j+1}} \cdot \gamma_k\,\sigma\overline{\tau_j} \qquad \text{for all } k \geq 0$$

$E^p_{**}(H\mathbb{F}_p)$ is a subquotient $B_{**}$ of

$$A_* \otimes E(\sigma\overline{\xi_{i+1}}, \sigma\overline{\xi_{i+2}}, \dots) \otimes \underbrace{P_p(\sigma\overline{\tau_0}, \dots \sigma\overline{\tau_{i-1}})}_{\Gamma(\sigma\overline{\tau_0}, \dots, \sigma\overline{\tau_{i-1}})} \otimes \Gamma(\sigma\overline{\tau_i}, \sigma\overline{\tau_{i+1}}, \dots)$$

- The spectral sequence is unital and augmented: the filtration degree $0$ elem. are permanent cycle (the edge homomorphism is a split injection)

And so if $r \geq p$, and $d^r(x) \neq 0$ then the filtration degree of $x$ is $\geq p+1$

- algebra generators in $B_{**}$ of filtration degree $\geq p+1$ have total degree $\geq 2p^{i+2}$ (these are the $\gamma_{p^k}(\sigma\overline{\tau_i})$ of bidegree $(p^k, 2p^{i+k}-p^k)$, $k \geq 2$)

Therefore: either classes in total degree $\leq 2p^{i+2}-2$ are not killed by $d^{p-1}$, nor by $d^r$ $r \geq p$ or not, that is: a $d^{p-1}$ diff. kills a total degree $\leq 2p^{i+2}-2$ class.

Since $\sigma\overline{\xi_{i+1}}$ of total degree $2p^{i+1}-1$ is killed, it must be by a $d^{p-1}$ diff. $(1, 2p^{i+1}-2)$

from a class in bidegree $(p, 2p^{i+1}-p) = (1+p-1, 2p^{i+1}-2-p+2)$: $\gamma_p\,\sigma\overline{\tau_i}$

Fact: the $A_*$-coaction $\nu(\gamma_p\,\sigma\overline{\tau_i}) = \overline{\tau_0} \otimes \sigma\overline{\xi_i}\,\gamma_{p-1}\,\sigma\overline{\tau_i} + 1 \otimes \gamma_p\,\sigma\overline{\tau_i}$

By hypothesis $\sigma\overline{\xi_i} = d^{p-1}(\gamma_p\,\sigma\overline{\tau_{i-1}})$, so $d^r(\sigma\overline{\xi_i}) = 0$

and $d^{p-1}(\gamma_{p-1}\,\sigma\overline{\tau_i}) = 0$ because in filtration $0$, permanent cycle.

So $d^{p-1}(\sigma\overline{\xi_i} \cdot \gamma_{p-1}\,\sigma\overline{\tau_i}) = 0$

and $d^{p-1}(\gamma_p\,\sigma\overline{\tau_i})$ is an $A_*$-comodule primitive, that is $\doteq \sigma\overline{\xi_{i+1}}$

Now $\psi(\gamma_{p+k}\,\sigma\overline{\tau_i}) = \sum_{\alpha+\beta = p+k} \gamma_\alpha\,\sigma\overline{\tau_i} \otimes \gamma_\beta\,\sigma\overline{\tau_i}$

$d^{p-1}(\gamma_{p+k}\,\sigma\overline{\tau_i}) - \gamma_k\,\sigma\overline{\tau_i} \cdot \sigma\overline{\xi_{i+1}}$ is coalgebra primitive in filtration $k+1$, thus $0$

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Now: $(\sigma\overline{\tau_k})^p = \sigma\overline{\tau_{k+1}}$ so $H_*THH(\mathbb{F}_p) \cong A_* \otimes P(\sigma\overline{\tau_0})$

$THH(\mathbb{F}_p)$ is an $H\mathbb{F}_p$ module, so the Hurewicz homomorphism is an injection with image the $A_*$-primitive: $\pi_*THH(\mathbb{F}_p) = P(\mu)$ degree $2$ $|\mu| = 2$

The following* "should be true":

If $f: X \longrightarrow BG$ ($G$ monoid) is an $E_n$-map we shall have an $E_{n-1}$-equivalence $THH(Mf) \simeq Mf \wedge BX_+$

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<u>The Lewis-May Thom spectrum functor:</u>

$F(n)$ the topological monoid of base point preserving equivalence of $S^n$

($BF(n) = B(*, F(n), *)$ $\quad EF(n) = B(*, F(n), S^n)$ $\quad EF(n) \to BF(n)$ induced by $S^n \to *$)

($* \to S^n$ gives a section $BF(n) \to EF(n)$)

<u>Thom space</u> $\quad f: X \to BF(n) \quad T(f) = f^*EF(n)/X$

<u>Thom spectrum</u>: $BF = \operatorname{colim} BF(n) \quad f: X \to BF \quad X(n) = f^{-1}(BF(n))$

induces $f_n: X(n) \longrightarrow BF(n)$

Let $T(f)_n = T(f_n)$ with structure maps induced by

fibrewise

$$\begin{tikzcd} S^1 \wedge f_n^*(EF(n)) \arrow[r] \arrow[d] & f_{n+1}^*(EF(n+1)) \arrow[d] \\ X(n) \arrow[r] & X(n+1) \end{tikzcd}$$

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<u>$H\mathbb{F}_p$ as a Thom spectrum:</u>

$H\mathbb{F}_2$: $\quad \psi: S^1 \longrightarrow BO \quad$ the non trivial element of $\pi_1(BO)$

induces, (since $BO$ is an infinite loop space) $\gamma: \Omega^2 S^3 \longrightarrow BO$

$\to \quad M\gamma \simeq H\mathbb{F}_2$

$M\gamma$ is 2 local (since $M\psi$ is the moore spectrum of $\mathbb{F}_2$ and $M\gamma$ is $(-1)$-connected)

Steenrod algebra

Thus it suffices to show that $A \longrightarrow H^*(M\gamma, \mathbb{F}_2)$ is an isomorphism

(evaluation on the Thom class; map of modules over $A$)

We have the dual $H_*(M\gamma, \mathbb{F}_2) \longrightarrow A_*$ map of $A_*$-comodules

come from $H_1(S^1)$

$H_*(M\gamma, \mathbb{F}_2) \cong H_*(\Omega^2 S^3, \mathbb{F}_2) \cong P(x_n, n \geq 0)$ generated by $x_0$ over the Dyer Lashof Algebra

Thus $H_*(M\gamma, \mathbb{F}_2)$ and $A_*$ have the same dimension in each rank, we show that the map is surjective

Thom isomorphism and $\gamma_* : H_*(\Omega^2 S^3, \mathbb{F}_2) \to H_*(BO, \mathbb{F}_2)$ commutes with the Dyer Lashof operations. $H_*(BO, \mathbb{F}_2)$ is generated under the Dyer Lashof by the class of degree 1, which is $\gamma_*(x_0)$

$$\begin{tikzcd} H_*(M\gamma, \mathbb{F}_2) \arrow[r] & H_*(MO, \mathbb{F}_2) \arrow[r, "\alpha_*"] & A_* \\ H_*(\Omega^2 S^3, \mathbb{F}_2) \arrow[u, "S"] \arrow[r, "\gamma_*"] & H_*(BO, \mathbb{F}_2) \arrow[u, "S"'] & \end{tikzcd}$$

one can show that $\alpha_*(\gamma_*(x_k)) = \xi_k$ + decomposable.

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$H\mathbb{F}_p$: $BF_{(p)}(n)$ classifies ~~the~~ fibration with fiber $S^n_{(p)}$ (p-local)

$\pi_1(BF_{(p)}) \cong \mathbb{Z}_p^*$ the units

$\phi : S^1 \to BF_{(p)}$ representing $p+1 \in \mathbb{Z}_p^\times$ : $M\phi$ the Moore spectrum of $\mathbb{F}_p$

induces $\gamma : \Omega^2 S^3 \to BF_{(p)}$. Similar argument give $M\gamma = H\mathbb{F}_p$.

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<u>Theorem</u>: (Blumberg) Let $f: X \to BG$ be an $E_2$-map. Assume $Mf$ is equivalent as a homotopy commutative $S$-algebra to some strictly commutative $S$-algebra $M'$.
Then there is an isomorphism in the derived category
$$THH(Mf) \simeq BX_+ \wedge Mf.$$

<u>Proof</u>: The Thom isomorphism implies that there is an equivalence
$$M' \wedge Mf^{op} \to M' \wedge X_+^{op}$$
So $B(Mf, Mf \wedge Mf^{op}, Mf) \to B(M', Mf \wedge Mf^{op}, M') \to B(M', M' \wedge X_+^{op}, M')$

is a w.e.

On the $k^{th}$ simplicial level: $M' \wedge X_+^{op}$ act on $M'$ by projecting
$M' \wedge X_+^{op} \to M'$ and multiplying.

There is an isomorphism $M' \wedge (M' \wedge X_+^{op})^{\wedge k} \wedge M' \to M' \wedge (M')^{\wedge k} \wedge M' \wedge (X_+^{op})^{\wedge k}$

that induces an equivalence $B(M', M' \wedge X_+^{op}, M') \simeq B(M', M', M') \wedge B(S, \Sigma^\infty X_+^{op}, S)$
is
$M' \wedge \Sigma^\infty BX_+$

An $E_\infty$-result:

$\mathcal{L}$ the linear isometries operad

Theorem: (Blumberg) Let $f: X \to BG$ be a good maps of $\mathcal{L}$-spaces such that ~~A~~ $X$ is a cofibrant and grouplike $\mathcal{L}$-space. Then there is a weak equivalence of commutative S-algebras

$$THH(Mf) = Mf \wedge BX_+$$

(Schlichtkrull ~~the~~ gives a similar result for symmetric spectra)

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Theorem: (Blumberg - Cohen - Schlichtkrull)

1) If $f: X \to BF$ is a $E_1$-map of spaces. Then there is a natural stable equivalence

$$THH(Mf) \simeq \cancel{M}\, M(L^\eta(Bf))$$

$$L^\eta(Bf): \underset{\text{free loop space}}{L(BX)} \xrightarrow{L(Bf)} L(B^2F) \simeq BF \times B^2F \xrightarrow{id \times \eta} \underset{\text{Hopf fibration}}{BF \times BF} \to \underset{\text{multiplication}}{BF}$$

2) If $f$ is $E_2$, then there is a stable equivalence

$$THH(Mf) \simeq Mf \wedge M(\eta \circ Bf)$$

3) If $f$ is $E_3$, then there is stable equivalence

$$THH(Mf) \simeq Mf \wedge BX_+.$$

Sketch "~~Proof~~": ~~Key~~ Construct a rigid Thom spectrum functor from a symmetric monoidal category of spaces to a " " " of spectra where BF can be seen as a commutative topological monoid. (it isn't a generalised E-M space)

$$B^{cy}_\bullet(f): B^{cy}_\bullet(X) \to B^{cy}_\bullet(BF) \to BF$$ ← constant simplicial object; ↖ multiplication

$$\cancel{M}\ B^{cy}(Mf) = M(\underbrace{B^{cy}(f)}_{\text{can be identified with } L^\eta(Bf)})$$

## References:

Most of the computation of $THH(\mathbb{F}_p)$ follows J. Rognes' notes, available as I write at:

folk.uio.no/rognes/papers/thh-fp-3.pdf

Structural results can be found in:

V. Angeltveit and J. Rognes, 'Hopf algebra structure on topological Hochschild homology'

The computation of $d^{p-1}$ is adapted from that for $\mathbb{Z}_p$ found in:

C. Ausoni, 'Topological Hochschild homology of connective complex K-theory'

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Results for $THH(Tf)$ are from:

A.J. Blumberg, 'Topological Hochschild homology of Thom spectra which are $E_\infty$-ring spectra'

A.J. Blumberg, R. Cohen and C. Schlichtkrull, 'Topological Hochschild homology of Thom spectra and the free loop space'

Beware: for that last paper, be sure to get the published version, as the conjectured statement for a multiplicative result is different in the preprint. x