K-theoretic Gromov-Witten invariants and derived algebraic geometry

Marco Robalo (IMJ-PRG, UPMC)

Summary

Results in this talk: collaboration with E. Mann (Université d' Angers).

Introduction - GW-invariants

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Motivated by original ideas and suggestions of Manin and Toën

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(Kontsevich, Manin, Behrend, Fantechi, etc) - cohomological definition $\rightsquigarrow$ $I_d(X, \Gamma_1, ..., \Gamma_n) =$ obtained as intersection numbers for a good intersection product on the cohomology of a "nice" (ie. smooth and proper) moduli space of rational curves

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$\overline{\mathcal{M}}_{g.n}(X, d)$:= moduli space of stable maps $f : (C, x_1, ..., x_n) \to X$ with $f_*([C]) = d$ in $H_2(X, \mathbb{Z})$

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Remark In general, the stack $\overline{\mathcal{M}}_{g.n}(X, d)$ is not smooth $\rightsquigarrow$ cap product with fundamental class does not give the correct counting.

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(Givental-Lee) $\exists$ K-theoretic intersection product $\rightsquigarrow$ modify the structure sheaf $\rightsquigarrow$ virtual structure sheaf $\rightsquigarrow$

$$K(X)^{\otimes n} \to K(\overline{\mathcal{M}}_{g,n})$$

GW-invariants and Derived Categories

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Hypothesis (Manin-Toën) - GW-invariants are already present at the level of derived categories before passing to K-theory and cohomology.

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• derived structure sheaf $\mathcal{O}$ of $\mathbb{R}\overline{\mathcal{M}}_{g.n}(X,d) \rightsquigarrow$ virtual structure sheaf $(t_*)^{-1}(\mathcal{O}) = \Sigma(-1)^i \pi_i(\mathcal{O}) \in G(\overline{\mathcal{M}}_{g.n}(X,d))$.

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Theorem (Mann, R.)

X proj. algebraic variety $/\mathbb{C}$. $g=0$. Then, $D(X)$ admits categorical GW-intersection products

$$I_{0,n,d} : D(X)^{\otimes_n} \to D(\overline{\mathcal{M}}_{0,n})$$

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which endow $D(X)$ with the structure of a $D(\overline{\mathcal{M}})$-algebra, via

$$I_{0,n,d} := \mathbb{R}Stb_*(\mathbb{R}ev^*(-))$$

Virtual info $\subseteq \mathbb{R}ev^*(-)$

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Passing to K-theory we recover the formalism of Givental-Lee of K-theoretic GW-products

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- higher genus (brane actions for modular ∞-operads)

Technical Problem: How to construct categorical GW-products (easy) and how to show coherence under gluings of curves (hard)?

Remark I: Correspondences and pullback-pushforwards.

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- 1-morphisms in C^{corr}, $X \rightsquigarrow Y$ = diagrams

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- compositions of 1-morphisms= fiber products in C.
- 2-morphisms= 1-morphisms of diagrams.

Universal Property: C 1-category,

Brane Actions and Correspondances

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- For each 1-morphism $f : X \to Y$ in C, $F(f)$ has an adjoint $F(f)_*$ in S.

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$\exists$! 2-functor $\overline{F} : C^{corr} \to S$ given by pullback-pushforward along the correspondence

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Brane Actions and Correspondences

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Conclusion: We are reduced to show a theorem for correspondances in stacks

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Compose with $\overline{D}$: (derived Artin Stacks)$^{corr} \to S = dg - categories$ to get the categorical action.

Key idea

Key idea $\rightsquigarrow$ Brane actions for ∞-operads (discovered by Toën)

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Action de Membranes et Correspondances

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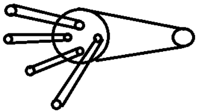

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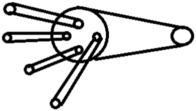

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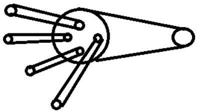

Brane Actions generalize this situation for general operads verifying a coherence condition $\rightsquigarrow$ cobordismes $\subseteq$ co-correspondances

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where the map $O(n+1) \to O(n)$ forgets the last entry. We say that O is coherent if for each pair of composable operations σ, τ, the natural square

$$\begin{tikzcd} Ext(Id) \arrow[r] \arrow[d] & Ext(\sigma) \arrow[d] \\ Ext(\tau) \arrow[r] & Ext(\sigma \circ \tau) \end{tikzcd}$$

is homotopy-cocartesian.

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Remark: In general if the operad is not coherent we still get a lax action.

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- The collection $O(n) := \mathfrak{M}_{0,n+1,\beta}$ forms a graded operad in derived stacks with $O(2)_0 = \mathfrak{M}_{0,3,0} = \overline{\mathcal{M}}_{0,3} = *$. Attention: Not coherent.

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$$O \text{ acts on } * \text{ via}$$

$$C \in O(n) = \mathfrak{M}_{0,n+1,\beta} \mapsto \coprod_{n \text{ first points}} * \to C \leftarrow * \text{ (last point)}$$

Brane Actions and Correspondences

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Corollary: Via composition with this map, $\overline{\mathcal{M}}_{0,n}$ acts on X via the correspondence of stable maps (only lax associative!).

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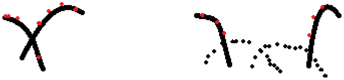

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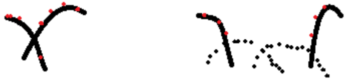

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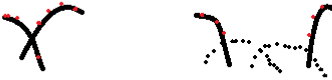

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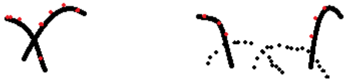

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$\rightsquigarrow$ Givental-Lee Metric in Quantum K-theory

Thank you for your attention.