An elementary introduction to the local trace formula of J.Arthur. The case of finite groups.

Marie-France Vigneras

Introduction. This paper starts with an elementary introduction to the theory of representations of a finite group G on an algebraically closed field L of good characteristic ℓ not dividing $|G|$, starting from the trace formula of the regular representation of $G \times G$ on $L[G]$. We found that this could be useful, because such a trace formula has been recently proved by James Arthur for a reductive group $G(F)$ over a non-archimedean local field F and called the *local trace formula*. This can be used to help to understand better the local trace formula, the strength of it, the problems that remain open and the nature of the applications that we can expect. This is the subject of the rest of this paper which is aimed at a non-specialist audience. The global trace formula is a powerful tool to prove correspondences between automorphic representations for different groups, called *Langlands correspondences.* The multiplicative groups of two division algebras of the same degree over F have the same character table. It would be very nice to find a proof of this fundamental example using the local trace formula. This is another reason to look at the finite group case, which shows quickly that the local trace formula is not enough. But which other ingredient should we add ?

It is a pleasure to thank the University of Geneva and the Deutsche Mathematiker-Vereinigung, for inviting the author to give talks on this material.

Table of contents,

3. APPLICATIONS

3.1 Density of the tempered characters. 3.2 Relation between character and coefficient. 3.3 Rank. 3.4 Spectral form of $h(C_x)$.

§1. Finite group.

1.1 *Trace formula.* Let G be a finite group, let L be an infinite field of characteristic ℓ. Consider the group algebra $L[G]$ of G over L, and the natural representation R of $G \times G$ on $L[G]$

$$(R(g_1, g_2)f)(g) = f(g_1^{-1}\, g\, g_2).$$

The trace of R is very easy to compute. A basis of $L[G]$ consists of the characteristic functions 1_x of $x \in G$ and

$$\operatorname{tr} R(g_1, g_2) = |\{x \in G, 1_x(g_1^{-1}\, x\, g_2) \neq 0\}|.$$

Denote by C_g the conjugacy class and by Z_g the centralizer of an element $g \in G$. We have

$$\operatorname{tr} R(g_1, g_2) = \begin{cases} 0 & C_{g_1} \neq C_{g_2} \\ |Z_g| & \text{if } g = g_1 = g_2. \end{cases}$$

For a representation (ρ, V) of G over L, denote by

$$\rho(f) = \sum_{g \in G} f(g)\rho(g)$$

the natural action of $f \in L[G]$ on V, and denote

$$f(C_g) = \sum_{x \in C_g} f(x).$$

Let $f(g_1, g_2) = f_1(g_1)\, f_2(g_2)$ in $L[G \times G]$. *The geometrical side* of the trace formula for $R(f)$ is

$$\operatorname{tr} R(f) = \sum_{C_g} |Z_g|\, f_1(C_g)\, f_2(C_g).$$

Let S_L be the set of isomorphism classes of the irreducible representations of G over L. For $\pi, \sigma \in S_L$, let $m(\pi \otimes \sigma)$ be the multiplicity of $\pi \otimes \sigma$ in a Jordan-Hölder sequence of R (R could be not semi-simple). The *spectral side* of the trace formula for $R(f)$ is

$$\operatorname{tr} R(f) = \sum_{\pi, \sigma \in S_L} m(\pi \otimes \sigma) \operatorname{tr} \pi(f_1)\, \operatorname{tr} \sigma(f_2).$$

The trace formula is the equality between the geometric side and the spectral side:

Trace formula For $f \in L[G]$, we have

$$\sum_{C_g} |Z_g| f_1(C_g)\, f_2(C_g) \;=\; \sum_{\pi, \sigma \in S_L} m(\pi \otimes \sigma) \operatorname{tr} \pi(f_1)\, \operatorname{tr} \sigma(f_2).$$

When $f_2 = 1_x$ the trace formula becomes

$$|Z_x| f(C_x) \; = \; \sum_{\pi \in S_L} a_x(\pi) \; \mathrm{tr}\, \pi(f).$$

where $a_x(\pi) = \sum_{\sigma \in S_L} m(\pi \otimes \sigma) \; \mathrm{tr}\, \sigma(x)$. This equality for all $x \in G$ is equivalent to the trace formula.

1.2 *Schur lemma. Any G-endomorphism of an irreducible representation π of G over L is a multiple scalar of the identity.*

Denote by (π^*, V^*) the contragredient representation of a representation (π, V), where V^* is the space of linear forms on V, and

$$< \pi^*(g)v^*, \pi(g)v >=< v^*, v >, \;\; v \in V, \;\; v^* \in V^*, \;\; g \in G.$$

Let us consider the $G \times G$-homomorphisms: for $\pi, \; \sigma \in S_L$,

$$\pi^* \otimes \pi \overset{\phi_\pi}{\to} L[G] \overset{\sigma}{\to} \sigma \otimes \sigma^* \overset{j_\sigma}{\sim} \mathrm{End}\, \sigma$$

(Be careful with the position of the contragredient !)

1. $\phi_\pi(v^* \otimes v)(g) =< v^* \, , \pi(g) \, v >$
2. $\sigma(f)$ as in [1.1]
3. $j_\sigma(v \otimes v^*)(w) =< v^* \, w > \; v, \;\; (v, w \in \sigma, \;\; v^* \in \sigma^*)$.

Schur lemma implies: there exists $\lambda(\pi) \in L$ such that

$$\sigma \, \phi_\pi = \left\{ \begin{array}{ll} 0 & \text{if } \pi \neq \sigma^* \\ \lambda(\pi)\, \mathrm{id} & \text{if } \pi = \sigma^*. \end{array} \right.$$

In particular, a *matrix coefficient* $f_\pi(g) =< \pi(g)v, v^* >\in L[G]$ of π satisfies

$$\sigma(f_\pi)w = \left\{ \begin{array}{ll} 0 & \text{if } \pi \neq \sigma^* \\ \lambda(\pi) < v, w^* > v^* & \text{if } \pi = \sigma^*. \end{array} \right.$$

hence

$$\mathrm{tr}\, \sigma(f_\pi) = \left\{ \begin{array}{ll} 0 & \text{if } \pi \neq \sigma^* \\ \lambda(\pi) f_\pi(1) & \text{if } \pi = \sigma^*. \end{array} \right.$$

The character χ_π of π is the sum of the diagonal matrix coefficients and we deduce from above

$$\sigma(\chi_\pi) = \left\{ \begin{array}{ll} 0 & \text{if } \pi \neq \sigma^* \\ \lambda(\pi) \; \mathrm{id}_\pi & \text{if } \pi = \sigma^*. \end{array} \right.$$

Good characteristic: we suppose that ℓ doesn't divide $|G|$.

1.3 *Completeness of characters.* $\operatorname{tr}\pi(f) = 0$ *for all* $\pi \in S_L$ *is equivalent to* $f(C_x) = 0$ *for all conjugacy classes* C_x.

Proof: clear from [1.1] and the equality

$$\operatorname{tr}\pi(f) = \sum_{C_x} \chi_\pi(x)\, f(C_x).$$

where $\chi_\pi(x) = \operatorname{tr}_\pi(x)$ is the character of π.

1.4 *Completeness of representations. If* $f \in L[G]$ *satisfies* $\pi(f) = 0$ *for all* $\pi \in S_L$*, then* $f = 0$.

As $L[G]$ is finite dimensional over L, this means by a general theorem in algebra that *the group algebra is semi-simple.* The trace formula [1.1] implies that $f(1) = 0$, from which we deduce that $f = 0$ because for all $x \in G$, $f'(g) = f(xg)$ has the same property than f, hence $f'(1) = f(x) = 0$.

1.5 $\lambda(\pi) \neq 0$. The trace formula [1.1] for $x = 1$ applied to a coefficient f_π such that $f_\pi(1) = 1$,

$$|G| = \lambda(\pi)\, a_1(\pi^*)$$

shows that $\lambda(\pi) \neq 0$, because $|G| \neq 0$.

1.6 *Linear independence of characters.* Suppose that

$$\sum_{\pi \in S_L} a(\pi)\chi_\pi = 0,\ \ a(\pi) \in L.$$

Apply π. From the last formula of [1.2], $\lambda(\pi)\ a(\pi^*) = 0$, hence by [1.5], $a(\pi) = 0$ for all $\pi \in S_L$.

1.7 *Spectral decomposition of the regular representation.* The trace formula [1.1] applied to $f = \chi_\pi$ gives

$$|G|\,\chi_\pi \ = \ \lambda(\pi)\ \dim\pi \sum_{\sigma \in S_L} m(\pi^* \otimes \sigma)\, \chi_\sigma.$$

We deduce from [1.6]

- $m(\pi^* \otimes \sigma) \neq 0 \Longleftrightarrow \sigma = \pi$
- $|G| = \lambda(\pi) m(\pi^* \otimes \pi)\ \dim\pi$.

The multiplicity of π in $R|_{G\times\{1\}}$ is $m(\pi^*\otimes\pi)\,\dim\pi$. From *Frobenius reciprocity*

$$\dim \mathrm{Hom}_G(\pi, \mathrm{ind}_1^G(1)) = \dim\pi.$$

We deduce that $m(\pi\otimes\pi^*) = 1$. Therefore, we get:

1. *Spectral decomposition of R*
$$R = \sum_{\pi\in S_L} \pi\otimes\pi^*.$$

2. *Value of $\lambda(\pi)$*
$$|G| = \lambda(\pi)\,\dim\pi.$$

3. *Trace formula*
$$|Z_x|f(C_x) \;=\; \sum_{\pi\in S_L} \chi_\pi(x^{-1})\ \mathrm{tr}\,\pi(f).$$

1.8 *Relation between character and coefficient.* For a coefficient f_π of π such that $f_\pi(1) = 1$ the trace formula [1.7.3] gives

$$\frac{f_\pi(C_x)}{|C_x|} = \frac{\chi_\pi(x)}{\dim\pi}.$$

1.9 *Orthogonality relations.*

1. *first*
$$\sum_{g\in G} \chi_\pi(g)\chi_\sigma(g^{-1}) = \begin{cases} 0 & \text{if } \pi\neq\sigma \\ |G| & \text{if } \pi=\sigma. \end{cases}$$
Another way of writing (see [1.2]) $\mathrm{tr}\,\sigma(\chi_\pi) = |G|\,\delta_{\chi_{\pi^*},\sigma}$.

2. *second*
$$\sum_{\pi\in S_L} \chi_\pi(x)\chi_\pi(y^{-1}) = \begin{cases} 0 & \text{if } C_x\neq C_y \\ |Z_x| & \text{if } x=y. \end{cases}$$

Another way of writing the trace of $R(x,y)$ [1.1], [1.7.1].

1.10 *Primitive central idempotents.* The map $f\to\pi(f)_{\pi\in S_L}$ induces an algebra isomorphism
$$i_L : L[G] \mapsto \oplus_{\pi\in S_L}\mathrm{End}\,\pi$$
(It is an injection by the completeness result [1.3] and a surjection by the theorem of Burnside, or by a comparaison of dimensions). As a vector space, the center $Z_L[G]$ of $L[G]$ admits different canonical basis [1.7.3]:

1. the characteristic functions of the conjugacy classes of G,
2. the characters of the irreducible representations of G over L,
3. *the primitive central idempotents of* $L[G]$,

$$e_\pi = \frac{\dim \pi}{|G|}\,\chi_{\pi^*}$$

(e_π projects $L[G]$ on $\operatorname{End}\pi$.)

We identify C_g with the characteristic function of C_g. The product in $Z_L[G]$ is evident on the idempotents, or on the characters

$$e_{_\pi}^2 = e_{_\pi},\ e_{_\pi}\, e_{_\pi} = 0, \quad \text{if } \sigma \neq \pi,$$

but not on the conjugacy classes. Let $\{C_1, \dots C_n\}$ be the conjugacy classes of G. There are integers $m_{i,j,k},\ i,j,k = 1 \dots n$ such that

$$C_i\, C_j = \sum m_{i,j,k} C_k$$

.

1.11 *Characters of* $Z_L[G]$ **[V]**Goldschmidt 2-15. They are indexed by S_L so that

$$\omega_\pi e_\sigma = \delta_{\pi,\sigma}, \quad \sigma, \pi \in S_L.$$

Lemma. The value of ω_π *on a conjugacy class is given by*

$$\omega_\pi(C_g) = \frac{\chi_\pi(C_g)}{\dim \pi}.$$

(in our notation, $\chi_\pi(C_g) = |C_g|\,\chi_\pi(g)$.)

Proof: Multiply $e_{_\pi}$ by $\chi_\pi(g)\,|G|/\dim \pi$, and sum over $\pi \in S_L$. Use the trace formula [1.7.3], and we get

$$\sum_{\pi \in S_L} \frac{\chi_\pi(g)\,|G|}{\dim \pi} e_{_\pi} = |Z_g|\, C_g.$$

Apply ω_π.

We suppose for the next two lemma that the characteristic of L is zero.

1.12 *The values of* ω_π *are algebraic integers.*

Proof: Apply ω_π to the product formula of conjugacy classes. On deduces that $\omega_\pi(C_i)$ is an eigenvalue of the matrix whose (j,k)-entry is $m_{i,j,k}$.

1.13 $\dim \pi$ *divides the order of* $|G|$.

The first orthogonality relations imply

$$\frac{|G|}{\dim \pi} = \sum_{i=1,\dots n} \omega_\pi(C_i)\,\chi_{\pi^*}(C_i).$$

The values of χ_π are algebraic integers. Therefore, $|G|/\dim \pi$ is a rational algebraic integer. We present another proof in [1.18].

1.14 *Groups with the same character table. Let G and H be two groups, and let $\phi : C_i \to C_i'$ be a bijection from the conjugacy classes of G to those of H. ϕ defines by linearity an isomorphism from $Z_L[G]$ to $Z_L[H]$:*

$$C_i' \, C_j' = \sum m_{i,j,k}\, C_k',$$

if and only if ϕ induces a bijection $\chi' \to \chi'\phi$ from the irreducible characters of H to those of G.

This is another instance where a geometrical result is equivalent to a spectral one.

The proof goes as follows. If $\phi : Z_L[G] \to Z_L[H]$ is a bijection, the image of a primitive central idempotent of G is a primitive central idempotent of H. For any $\pi \in S_L(G)$, there is $\pi' \in S_L(H)$ such that [1.10.3]

$$\frac{\dim \pi}{|G|}\chi_\pi = \frac{\dim \pi'}{|H|}\chi_{\pi'}\phi.$$

Apply this to $g = 1$. The image of $1 \in G$ which is the unit of $Z_L[G]$ is the unit $1 \subset H$ of $Z_L[H]$. We get

$$\frac{(\dim \pi)^2}{|G|} = \frac{(\dim \pi')^2}{|H|}.$$

By duality, $\omega_{\pi'}\,\phi = \omega_\pi$. By [1.11]

$$\frac{|C_g|\,\chi_\pi(g)}{\dim \pi} = \frac{|\phi(C_g)|\,\chi_{\pi'}(g')}{\dim \pi'},\ (g' \in \phi(C_g)).$$

We deduce

$$|C_g| = |\phi(C_g)|,$$

then $|G| = |H|$, and $(\dim \pi)^2 = (\dim \pi')^2$. But the dimension is a positive integer, hence for all $\pi \in S_L(G),\ g \in G$ we have:

$$\chi_{\pi'}\phi = \chi_\pi.$$

Conversely, if for all $\pi' \in S_L(H)$ we have $\chi_{\pi'}\phi = \chi_\pi \in S_L(G)$, then $|G| = |H|$ by [1.9], and ϕ induces by linearity a bijection from the primitive central idempotents of $L[G]$ to those of $L[H]$ [1.10.3]. This implies that ϕ induces by linearity a bijection from $Z_L[G]$ to $Z_L[H]$.

It is easy to generalize 1.14 to profinite groups: this is the aim of the paragraph 1.15 and 1.16. Then we will apply this to show in 1.17 that the Langlands correspondence for the irreducible representations of the multiplicative groups of division algebras over a local non-archimedean field, is equivalent to a statement concerning their conjugacy classes.

1.15 *Quotient groups.* Let $t : G \to G/G'$ a quotient. Then t defines an homomorphism

$$Z_L[G] \to Z_L[G/G']$$

which is compatible with the canonical basis

1. $t(C_g) = C_{t(g)}$,
2. $t(\chi_\pi) = \begin{cases} 0 & \text{if } \pi \text{ non trivial on } G', \\ \chi_{\overline{\pi}} & \text{if } \pi \text{ trivial on } G' \text{ and } \overline{\pi}\, t = \pi. \end{cases}$
3. $t(e_\pi) = \begin{cases} 0 & \text{if } \pi \text{ non trivial on } G', \\ e_{\overline{\pi}} & \text{if } \pi \text{ trivial on } G'. \end{cases}$

The representation π is trivial on G' if and only if the restriction of the character χ_π on G' is constant. Let $t' : H \to H/H'$ be another quotient. We have

Lemma. Let $\phi : C_i \to C'_i$ be a bijection between the conjugacy classes of G and H, and $\pi_i \to \pi'_i$ a bijection between $S_L(G)$ and $S_L(H)$ such that for all i, j,

$$\chi_{\pi_i}(C_j) = \chi'_{\pi'_i}(C'_j).$$

If $C_i \subset G' \iff C'_i \subset H'$, then ϕ induces an algebra isomorphism between $Z_L[G/G']$ and $Z_L[H/H']$.

[1.16] *Profinite groups.* A profinite group G is a projective limit of finite groups, with the projective topology

$$G = \varprojlim_{G'} G/G'.$$

The normal groups $G' \subset G$ are open and compact. G is a compact totally disconnected group. By definition, a representation of G factorizes through a finite quotient. The theory of representations of a profinite group is almost the same than for a finite group.

Let $G = \varprojlim_{G_n} G/G_n$ and $H = \varprojlim_{H_n} H/H_n$ be two profinite groups. Let $t_n : G \to G/G_n$, and let $t'_n : H \to H/H_n$. Suppose that there is a bijection

$\phi : C_i \to C'_i$ from the conjugacy classes of G to those of H. From [1.15], we deduce

Lemma. The properties 1 and 2 are equivalent

1. *There is a bijection $\pi_i \to \pi'_i$ between $S_L(G)$ and $S_L(H)$ such that*

$$\chi_{\pi_i}(C_j) = \chi_{\pi'_i}(C'_j).$$

   For all n, $C_i \subset G_n \Longleftrightarrow C'_i \subset H_n$.

2. *For all n, $\phi_n : t_n(C_i) \to t'_n(C'_i)$ is well defined, and induces an isomorphism between $Z_L[G/G_n]$ and $Z_L[H/H_n]$.*

[1.17] *Multiplicative group of a local division algebra.* Let F be a local non-archimedean field of residual characteristic p, of uniformizing parameter ω_F. As well known, if D^* and D'^* are the multiplicative groups of two division algebras of the same degree n^2 over F, then the Langlands correspondence tells us that for all irreducible complex representation π' of D'^*, there exists an irreducible complex representation π of D^* such that for all $g \in D^*$, $g' \in D'^*$ with the same characteristic polynomial we have $\chi_\pi(g) = \chi_{\pi'}(g')$. This is equivalent to a geometric property of the conjugacy classes, that we are going to describe.

Let us denote by $P_g(X) \in F[X]$ the characteristic polynomial of $g \in D^*$, and by C_g its conjugacy class in D^*. The conjugacy classes of D^* are parametrized by the characteristic polynomial:

$$P_g(X) = P_{g'}(X) \Longleftrightarrow C_g = C_{g'}.$$

The center of D^* is F^* and π has a central character. We can suppose after torsion by a character of D^* that the action of the group $\omega_F^{\mathbf{Z}}$ is trivial.

The quotient group $G = D^*/\omega_F^{\mathbf{Z}}$ is profinite. Let D_m be the m-th congruence subgroup of D^*. If P is the maximal ideal of the integers of D, then $D_m = 1 + P^m$. The subgroups $G_m = \omega_F^{\mathbf{Z}} D_m/\omega_F^{\mathbf{Z}} \subset G$ are open and compact and

$$G = \varprojlim G/G_m.$$

The characteristic polynomial gives a bjection between the conjugacy classes of D^* and D'^*, such that $C_g \subset \omega_F^{\mathbf{Z}} D_m \Rightarrow C_{g'} \subset \omega_F^{\mathbf{Z}} D'_m$. We deduce easily from [1.16] that the Langlands correspondence between D and D' is equivalent to:

Geometric property of conjugacy classes. The characteristic polynomial induces for all m, a bijection between the conjugacy classes of the finite groups G/G_m and G'/G'_m, that respects the product.

Can we prove directly the geometric statement ? There are two proofs of the Langlands conjecture. One uses the global correspondence with the automorphic cuspidal representations of $GL(n)$, the other one uses the complete description of the representations of D^*.

Remark. A natural question is to wonder if the multiplicative groups D^* and D'^* are isomorphic, as they have the same character table ! One has the following result due to J.-P. Serre.

Proposition. Let F and F' be two local non archimedean fields, let D be a division algebra over F and let D' be a division algebra over F'. Then D^* and D'^* are locally isomorphic if and only if the fields F and F' are isomorphic, and if the Brauer invariants of D and D' are equal or opposite.

In the case where $F = F'$ and D' is the opposite algebra of D, the map $x \to x^{-1}$ gives an isomorphism between D^* and D'^*. The groups are also isomorphic as F-algebraic groups.

Question: What is the first value of m such that the two finite quotients G/G_m and G'/G'_m are different ?

One can see easily that $m \geq 2$. J.-P. Serre showed that $m \geq 3$.

We show now that the trace formula can be used to prove that in good characteristic, the theory of representations of a finite group is the same than in characteristic zero. The same idea can be used for reductive p-adic groups.

1.18 *Existence of stable lattices.*
Suppose that the characteristic of L is zero. There exists a number field E such that the primitive central idempotents of $L[G]$ belong to $E[G]$. Let A_E be the ring of algebraic integers in E. Let $V \subset e_\pi\, E[G]$ a simple $E[G]$-module, isomorphic to π. For any $v \in e_\pi\, E[G]$ there is $x \in E$ such that $xv \in A_E[G]$, and

$$\Lambda(\pi) = V \cap A_E[G].$$

is a $A_E[G]$-lattice of π (a projective and finitely generated A-module which is G-stable such that

$$V = \Lambda(\pi) \otimes_A E.$$

This implies that the values of the character χ_π belong to A_E. The values of the idempotent e_{χ_π} do not belong to A_E, because of the factor $\dim \pi/|G|$. Choose $v \neq 0$ in the lattice Λ_π, and $v^* \neq 0$ in the dual lattice Λ_π^* such that $< v, v^* >= 1$, and apply the last formula of [1.2]:

$$\mathrm{tr} < v, \pi(g)v^* >= \lambda(\pi) = |G|/\dim \pi$$

is equal to an algebraic rational integer. Hence, $\dim \pi$ divides $|G|$ (see[1.13]).

1.19 We suppose that $(K, A, \mathcal{M}, k)$ is a *modular data* where K is a finite extension of $\mathbf{Q}_\ell$, A is the algebraic closure of $\mathbf{Z}_\ell$ in K, $\mathcal{M}$ the maximal ideal of A, and $k = A/\mathcal{M}$. We suppose that every absolutely irreducible representation of G is realizable over K. We denote by $p_{\mathcal{M}}$ the reduction modulo $\mathcal{M}$.

Proposition. The reduction modulo $\mathcal{M}$ defines a bijection from S_K to S_k.

Proof: Every primitive central idempotent of $K[G]$ belongs to $A[G]$ (the integer $\lambda(\pi)$ divides $|G|$, hence is a unit of A). Therefore the decomposition of $A[G]$ is (see [1.10])

$$A[G] \simeq \oplus_{\pi \in S_K} e_\pi \, A[G],$$

where $e_\pi \, A[G]$ is an order of $\mathrm{End}_K \, \pi \sim e_\pi \, K[G]$ contained in $\mathrm{End}_A \, \Lambda_\pi$ for any $A[G]$-lattice Λ_π of π. We have

$$e_\pi \, A[G]/\mathcal{M} e_\pi = \mathrm{End}_k \, \Lambda_\pi/\mathcal{M}\Lambda_\pi$$

because the first one is a k-subspace of the second one, of the same dimension. By reduction modulo $\mathcal{M}$ of $A[G]$,

$$k[G] \simeq \oplus_{\pi \in S_K} e_\pi \, A[G]/\mathcal{M} e_\pi \, A[G] \simeq \oplus_{\sigma \in S_k} \mathrm{End}_k \, \sigma.$$

This implies that $\{p_{\mathcal{M}} e_\pi\}$ are the primitive central idempotents of $k[G]$, and that the S_k is the set of isomorphism classes of the irreducible representations $\Lambda_\pi/\mathcal{M}\Lambda_\pi$ of G over k. This is true for any modular data as above. We deduce that $\Lambda_\pi/\mathcal{M}\Lambda_\pi$ is an absolutely irreducible representation of k.

§2. Reductive p-adic groups.

Although J. Arthur considers connected reductive groups over any local field of characteristic zero, we will consider only the case of a connected algebraic reductive group G defined over a local non-archimedean field of characteristic zero F.

Open problem: Extend the the local trace formula to the case where the characteristic of F is not zero, and where the group is not connected (this last generalization should be useful for the *base change* which is the natural generalisation of the norm for a cyclic extension E/F:

$$N : E^* \to F^*.$$

By duality, N induces a map from the characters of $GL_1(F)$ to the characters of $GL_1(E)$ which are invariant by $\mathrm{Gal}(E/F)$. The base change problem is to understand what happens when GL_1 is replaced by a reductive group $G(F)$. One is naturally lead to the study of the representations of the semi-direct product of $G(E)$ by the natural action of $\mathrm{Gal}(E/F)$, hence of a reductive not connected group).

2.1 *Elliptic elements.* The basic conjugacy classes of $G(F)$ are called *elliptic*. They behave in many aspects like the conjugacy classes in a finite group.

Definition. An element $x \in G(F)$ is elliptic, if it is semi-simple, and if its centralizer $Z_{G(F)}(x)$ is compact modulo the center of $G(F)$.

Suppose that $Z(F)$ is a compact group. Let us denote C_x the conjugacy class of an elliptic element x in $G(F)$. We can define

$$h(C_x) = \int_{G(F)} f(g^{-1}xg)\, dg$$

for a Haar measure dg on $G(F)$, $f \in \mathcal{C}$, where $\mathcal{C}$ is the set of locally constant compactly supported functions $f : G \to \mathbf{C}$, and $h = f\, dg \in H = \mathcal{C}\, dg$. We define a measure on the set C_{ell} of elliptic conjugacy classes of $G(F)$ by the formula

$$\int_{C_{ell}} h(C_x) = \int_{G(F)} h(g)$$

for all $h = f\, dg \in H$ with support in C_{ell}.

But we can't avoid a center, because the Levi subgroups have a non compact center (see the following lemma). In this case, we replace the integral defining $h(C_x)$ by an integral on $G(F)/Z(F)$ instead of $G(F)$.

Elements x such that the connected centralizer $Z_G(x)$ is a torus are called *regular*.

Lemma. A semi-simple regular element $m \in G(F)$ is an elliptic element in a Levi $M(F)$.

Example: $G = GL(n)$. Semi-simple conjugacy classes C_g are classified by their characteristic polynomial $P_g(X) \in F[X]$. Elliptic elements are those such that the quotient $F[X]/P_g(X)F[X]$ is a field of degree n over F. Regular means that the roots of $P_g(X)$ are simple.

The maximal split F-tori are conjuguate. We fix one A, and we consider only the F-Levi of F-parabolic subgroups containing A, called simply "Levi". The group $G(F)$ diagonally embedded in $G(F)\times G(F)$ is not discrete in $G(F)\times G(F)$. For this reason, in the local trace formula, not all the conjugacy classes appear. It is enough to consider only the conjugacy classes of the dense subset of semi-simple regular elements of $G(F)$. This is a very important fact, that explains why the local trace formula is much more easier than the global trace formula.

Remark. Another equivalent definition for an elliptic element x is: $Z_G(x)$ *is a F-anisotropic torus modulo Z.*

2.2 *Weighted orbital integral.* Selberg in the classical case , and Arthur in the local case proved that what appears in the trace formula is a weighted version of $h(C_x)$ if x is not elliptic. They were led to the notion of a *weighted orbital integral* of a regular semi-simple element, that we will define in this paragraph. What is very surprising is that it is not invariant by conjugacy. There is a not very convincing trick to make it invariant. However it is invariant when h is a matrix coefficient of an irreducible representation ! see [3.2].

Let $x \in G(F)$ be an M-elliptic, G-regular element. There are finitely many parabolic P of Levi M. By the Iwasawa decomposition, there is a maximal compact subgroup $K \in G(F)$ such that $G(F) = M(F)\,N(F)\,K$. We denote by $X(M)$ the group of F-rational characters of M,

$$M^o = \cap_{\chi \in X(M)} \operatorname{Ker}(\chi : M(F) \to \mathbf{C}^*),$$

and by $H_M : M(F) \to M(F)/M^o$ the projection. The quotient is a finitely generated free group, hence a lattice in a real vector space

$$\mathcal{A}_M = M(F)/M^o \otimes_{\mathbf{Z}} \mathbf{R}.$$

An element $g \in G(F)$ has a decomposition which depends on the choice of K

$$g = m_P(g)\,n_P(g)\,k_P(g).$$

and we set $H_P = H_M\,m_P$.

Again, it would be nice to suppose that Z is anisotropic. In this case, we consider the convex hull $\Pi_M(g)$ of the points $H_P(g)$ in the euclidean space $\mathcal{A}_M$. The volume of $\Pi(g)$ is denoted by $v_M(g)$. The *weight*

$$v_M : G(F) \to \mathbf{R}_{>0}$$

is left $M(F)$ and right K-invariant by construction. In the general case, one considers the volume of the projection of $\Pi_M(g)$ on $\mathcal{A}_M/\mathcal{A}_G$.

Example. Let $G = GL(2)$, $F = \mathbf{Q}_p$, P the group of upper triangular matrices, $\overline{P}$ the group of lower triangular matrices, M the group of diagonal matrices, $K = GL(2, \mathbf{Z}_p)$. The group $GL(2, \mathbf{Q}_p)$ acts on the tree of $PGL(2, \mathbf{Q}_p)$. The tree has a natural distance d. The group $GL(2, \mathbf{Z}_p)$ is the stabilizer of a vertice p_o of the tree, the $M(\mathbf{Q}_p)$-orbit of p_o is a line A, with elements

$$p_s = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & p^s \end{pmatrix} \right\} p_o.$$

Let

$$n_x = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}.$$

The point p_s is fixed by n_x if and only if $xp^s \in \mathbf{Z}_p$. The point $n_x\, p_o$ belongs to the line if $x \in \mathbf{Z}_p$. Otherwise, its distance to the line is equal to

$$d(n_x\, p_o, A) = d(n_x\, p_o, p_s) = d(n_x\, p_o, n_x\, p_s) = d(p_o, p_s) = s$$

where $x \in p^{-s}\mathbf{Z}_p^*$. If $t = m_{\overline{P}}(n_x)$ and $\nu = n_{\overline{P}}(n_x)$, then

$$d(n_x\, p_o, A) = d(\nu\, p_o, t^{-1}p_s) = d(p_o, t^{-1}p_s) = d(p_o, p_s).$$

This implies that $t^{-1}p_s = p_{-s}$, $t\, p_o = p_{2s}$ and therefore,

$$v_M(g) = 2\, d(gp_o, M(\mathbf{Q}_p)p_o).$$

Question : When G is F-split, Z anisotropic and M minimal, there is the Bruhat-Tits building that generalizes the tree. The group $G(F)$ acts on the building. The group K is the stabilizer of a point p_o, and $M(F)p_o$ is an appartment. How is the convex polytop $\Pi_M(g)$ related to the building ?

For a G-regular and M-elliptic element $x \in G(F)$, set

$$h(C_x, v_M) = \int_{G(F)/A_M(F)} f(g^{-1}xg)\, v_M(g)\, dg.$$

for $h \in H$. It is uniquely defined if one choose, once for all, Haar measures on the tori $A_M(F)$, and take the quotient measure. This is called a *weighted orbital integral* (this not the usual convention, we don't multiply by the jacobian appearing in the Weyl integration formula).

2.3 *Elliptic representations.* What is an elliptic complex representation π of $G(F)$ is not so clear, even for specialists. The usual definition is

Definition 1. The character of π doesn't vanish on the elliptic elements.

However, the local trace formula suggests a second definition. Let S be the set of isomorphism classes of irreducible representations of $G(F)$. A representation $\pi \in S$ is square-integrable, if it has a non zero matrix coefficient square-integrable modulo the center. Square-integrable representations are elliptic (with both definitions). A representation $\sigma \in S(M(F))$ lifted to $S(P(F))$ via the surjection $P(F) \to M(F)$, induces by unitary induction a finite length representation $I_P^G(\sigma)$ of $G(F)$. The Weyl group $W(A_M)$ acts on the euclidean space $\mathcal{A}_M/\mathcal{A}_G$. An element $t \in W(A_M)$ is *regular* if $\mathcal{A}_M/\mathcal{A}_G$ has no non-zero t-invariant vector. For instance, if M is minimal, the Coxeter element of the Weyl group is regular. The Weyl group $W(A_M)$ acts also on $S(M(F))$.

An elliptic representation is an irreducible quotient of an induced representation $I_P^G(\sigma)$ of a square-integrable representation σ of a Levi $M(F)$, fixed by a regular element $t \in W(A_M)$. But not every such quotient is elliptic. It is now that the definition of "elliptic" becomes very poorly understood.

For $h \in H$,

$$\pi(h) = \int_{G(F)} h(g)\,\pi(g)$$

is of finite rank and has a trace $\operatorname{tr}\pi(h)$. One defines a normalized intertwinning operator $R(\sigma,t)$ of $I_P^G(\sigma)$. The *weighted trace* is

$$\operatorname{tr} R(\sigma,t)\, I_P^G(\sigma)(h) = \sum_{\pi \in I_P^G(\sigma)} a(\pi,t)\ \operatorname{tr}\pi(h).$$

This defines numbers $a(\pi,t)$, because the traces of the elements of S are linearly independent. One proves that $a(\pi,t)$ is independent of P.

Open problem: understand the mysterious function $a(\pi,t)$.

Definition 2. We say that $\pi \in S_L$ is elliptic, if $\pi \in I_P^G(\sigma)$ where $a(\pi,t) \neq 0$ for some (σ,t) as above.

Open problem: Are the two definitions equivalent ?

Example : $G = GL(n)$. It is known that representations $\pi \in S$ induced by square-integrable representations $\sigma \in S(M(F))$ are irreducible. Their character doesn't vanish on the elliptic elements if and only if π is square-integrable. I think that $a(\pi,t) = 0$ when $I_P^G(\sigma)$ is irreducible, and $P \neq G$. Hence the answer should be yes for $GL(n)$.

2.4 *Local trace formula.*

$$\sum_M I_M^{geo}(h_1,h_2) = \sum_M I_M^{spec}(h_1,h_2), \quad h_1,h_2 \in H.$$

where M runs over the Levi. We will explicit only the "discrete" part given by the term $M = G$. We have

$$I_G^{geo}(h_1, h_2) = \int_{C_{ell}} h_1(C_x)\, h_2(C_x)$$

For $M \neq G$, the geometric term is much more complicated. It is an integral on the G-regular, M-elliptic elements of $G(F)$. The weighted orbital integral replaces $h(C_x)$.

We have when Z is anisotropic

$$I_G^{spec}(h_1, h_2) = \sum_{(\sigma,t)} \mathrm{tr}(R(\sigma, t)\, I_P^G(\sigma)(h_1))\, \mathrm{tr}(R(\sigma^*, t)\, I_P^G(\sigma^*)(h_2)).$$

The representation σ^* is the contragredient of σ.

The sum is finite, because there are only finitely many σ containing a non zero vector invariant by a given open compact subgroup. The proof of the local trace formula relies on the explicit Plancherel formula of Harish-Chandra. It is worth mentioning that no complete proof of this explicit Plancherel formula has been published by Harish-Chandra. He announced the result with a scheme for the proof in 2 articles published in 1973 and 1977 (see the volume IV of his collected papers (Springer-Verlag 1984)). Silberger published a complete proof for groups of semi-simple rank 1 (in his book Harmonic Analysis on p-adic reductive groups published by Princeton University Press).

2.5 *Tempered distributions.* There are two natural algebra attached to $G(F)$ that replace the group algebra of a finite group. We worked with the easiest one: H identified to an algebra of distributions (linear forms on $\mathcal{C}$) on $G(F)$. The other one $\mathcal{S}\, dg$ is bigger and more natural here

$$\mathcal{S} = \{f \in \cap_{d \in \mathbf{Z}} L^2(G(F), (1+r)^d) \text{ bi-invariant by some open subgroup}\},$$

where r is the natural length function on the compactly generated group $G(F)$. It contains H and is stable by holomorphic functional calculus in the reduced C_r^*-algebra of $G(F)$.

Distributions on $G(F)$ which extend to continuous functions on $\mathcal{S}$ are called *tempered*. Weighted orbital integrals $h(x, v_M)$ are tempered. It should follow from the "weighted Howe conjecture" that weighted orbital integrals are bounded on the elliptic elements on M. Representations with tempered traces are called tempered. The irreducible ones are identified the simple $\mathcal{S}$-modules V such that $\mathcal{S}V = V$ and are classified in terms of square integrable ones, like the elliptic ones but σ is supposed only square-integrable modulo the center. Let $S_t \subset S$ be the set of isomorphy classes of irreducible tempered representations.

The geometric side of the trace formula is an integral over conjugacy classes of semi-simple elements and the spectral side over S_t.

Open problem: Extend the local trace formula to $\mathcal{S}$.

§3. Applications. The applications that we give here, concern only the local theory and imitate what we developped in the paragraph 1. Up to now, only the compact trace formula (that is not considered in this article) and not the local trace formula has applications to the modular theory (using the ideas of 1.18 and 1.19). Let us hope that more applications of the local trace formula will be discovered in the future, in particular to the base change.

3.1 *Density of the tempered characters*:

$$\operatorname{tr}\pi(f) = 0 \text{ for all } \pi \in S_t \Rightarrow f(C_x) = 0 \text{ for all } C_x,\, x \in G(F).$$

page] This is an improvement on the usual proof which uses the global trace formula.

3.2 *Relation between character and coefficient.* To simplify, we suppose that the center Z is anisotropic. It has been shown by Labesse and Langlands of $SL(2)$ and later by Kazhdan [[**V**]Kazhdan] that there exists for a square-integrable representation $\pi \in S_t$, a function $h_\pi \in H$ such that for $\sigma \in S_t$ we have

$$\operatorname{tr}\sigma(h_\pi) = \delta_{\sigma,\pi^*}.$$

This is called a pseudo-coefficient because a matrix coefficient satifies the same relation, but is only square-integrable. By the density theorem, this defines the image of h_π in $H/[H,H]$, and the orbital integral of h_π. The orbital integral $h_\pi(C_x) = 0$ if the semi-simple element $x \in G(F)$ is not elliptic. As the characteristic of F is zero, one knows that there is a locally L^1, conjugation invariant function χ_π on the group $G(F)$ such that

$$\operatorname{tr}\pi(h) = \int_{G(F)} \chi_\pi(g)\, h(g)$$

called the character of π. Let $x \in G(F)$, M-elliptic, G-regular. Arthur proved

$$h_\pi(C_x, v_M) = (-1)^{\dim A_M}\chi_\pi(x)$$

3.3*Rank.* If $\pi \in S$, there is $h_\pi \in H$ such that

$$\int_{C_{ell}} \chi_\sigma(c)\chi_\pi(c) = \operatorname{tr}\sigma(h_\pi)$$

by the trace Paley-Wiener theorem.

The cohomological dimension of H is finite. The building gives a finite projective resolution of the trivial representation. There is a map from S to the Grothendieck group K_o generated by the finitely generated H-projective modules. By composition with the map from K_o to $H/[H,H]$, deduced from the trace maps $M(n,H) \to H/[H,H]$, we get a map called *the rank*

$$r : S \to H/[H,H]$$

For $\pi \in S$ cuspidal, it is easy to check that

$$r(\pi) = h_\pi \text{ modulo } [H,H]$$

Question. Is this equality true for $\pi \in S$?

Kazhdan announced that Bernstein proved that the answer to the question is yes. But no proof has been written (to my knowledge). This would imply for square-integrable π that $r(\pi)$ is the image of a pseudo-coefficient modulo $[H,H]$.

3.4 *Spectral form of $h(C_x)$.* For x elliptic, $h \in H$

$$h(C_x) = \sum a_x(\pi) \operatorname{tr} \pi(h)$$

where the sum is on the elliptic representations π. If $\pi \subset I_P^G(\sigma)$ where $\sigma \in S(M(F))$ is square-integrable,

$$a_x(\pi) = \sum a(\pi', t)\, a(\pi, t)\, \chi_{\pi^*}(x).$$

The sum is over all the regular $t \in W(A_M)$ such that $t\,\sigma = \sigma$ and on all the $\pi' \subset I_P^G(\sigma)$.

References

[1] Arthur, J. A local trace formula Preprint 1990.

[2] Arthur, J. Towards a local trace formula Preprint 1990.

[3] Arthur, Some problems in local harmonic analysis. Preprint 1990.

[4] Kazhdan, D. Cuspidal geometry of p-adic groups. Journal d'Analyse mathématique, Vol. 47 (1966)

[5] Goldschmidt, D.M. Lectures on Character Theory. Mathematics Lecture Series 8. Publish or Perish, Inc. (1980).