The right adjoint of the parabolic induction

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Abstract

We extend the results of Emerton on the ordinary part functor to the category of the smooth representations over a general commutative ring R, of a general reductive p-adic group G (F-points of a reductive connected F-group over a local non archimedean field of residual characteristic p). In Emerton's work, $F = \mathbb{Q}_p$, R is a complete artinian local $\mathbb{Z}_p$-algebra having a finite residual field, and the representations are admissible. We show:

The (smooth) parabolic induction functor admits a right adjoint. The center-locally finite part of the (smooth) right adjoint is equal to the (admissible) right adjoint of the admissible parabolic induction functor when R is noetherian. The (smooth or admissible) parabolic induction functor is fully faithful when 0 is the only infinitely p-divisible element in R.

Contents

1 Introduction

Let R be a commutative ring, let F be a local non archimedean field of finite residual field of characteristic p, let $\mathbf{G}$ be a reductive connected F-group. Let $\mathbf{P}, \overline{\mathbf{P}}$ be two opposite parabolic F-subgroups of unipotent radical $\mathbf{N}, \overline{\mathbf{N}}$ and Levi subgroup $\mathbf{M} = \mathbf{P} \cap \overline{\mathbf{P}}$ of center $\mathbf{Z}(\mathbf{M})$. The groups of F-points are denoted by the same letter but not in bold. The parabolic induction functor $\mathrm{Ind}_P^G : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(G)$ between the categories of smooth R-representations of M and of G is the right adjoint of the N-coinvariant functor, and respects admissibility.

For any (R,F,G), we show that Ind_P^G admits a right adjoint R_P^G.

When R is noetherian, we show that the $Z(M)$-locally finite part of R_P^G respects admissibility, hence is the right adjoint of the functor Ind_P^G between admissible R-representations. If p is invertible in R we show that the N-coinvariant functor respects admissibility, hence is the left adjoint of the functor Ind_P^G between admissible R-representations.

When 0 is the only infinitely p-divisible element in R, we show that the counit of the adjoint pair $(-_N, \mathrm{Ind}_P^G)$, is an isomorphism, hence Ind_P^G is fully faithful and the unit of the adjoint pair $(\mathrm{Ind}_P^G, R_P^G)$ is an isomorphism.

The results of this paper have already be used [HV] in the comparison between parabolic and compact induction for smooth representations over an algebraically closed field of characteristic p for **any** $(F, \mathbf{G})$, following the arguments of Herzig when the characteristic of F is 0 and $\mathbf{G}$ is split. The comparison is a basic step in the classification of the non-supersingular admissible irreducible representations of G (work in progress with Abe, Henniart, and Herzig, see also Ly's work [Ly] for $GL(n,D)$ where D/F is finite dimensional division algebra).

When p is invertible in R, it was known that Ind_P^G has a right adjoint. When R is the field of complex numbers, Casselman for admissible representations and Bernstein in general proved that the right adjoint is equal to the $\overline{N}$-coinvariant functor multiplied by the modulus of P ("the Bernstein second adjointness"), and stressed in his unpublished notes the incredible power of this innocent statement. A proof was published by Bushnell. Both proofs rely on the property that the category $\mathrm{Mod}_{\mathbb{C}}(G)$ is noetherian. Conversely, Dat [Dat] proved that the Bernstein second adjointess implies the noeheriannity of $\mathrm{Mod}_R(G)$ and prove it assuming the existence of certain idempotents (that he constructed using the theory of types for linear groups, classical groups if $p \neq 2$, and groups of semi-simple rank 1). Under the same hypothesis on G, Dat proved also that N-coinvariant functor respects admissibility. See [Dat].

When $F = \mathbb{Q}_p$ and R is a complete artinian local $\mathbb{Z}_p$-algebra having finite residual field, Emerton [Emerton] showed that Ind_P^G restricted to admissible representations has a right adjoint equal to the ordinary part functor $\mathrm{Ord}_{\overline{P}}$. Studying its derived functors he showed that the N-coinvariant functor respects admissibility [Emerton2].

In section 2 we give precise definitions and references to the litterature on adjoint functors and on grothendieck abelian locally small categories.

The existence of a right adjoint of $\mathrm{Ind}_P^G : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(G)$ is proved using that $\mathrm{Mod}_R^\infty(G)$ is a grothendieck abelian locally small category and that Ind_P^G is an exact functor commuting with small direct sums, in sections 3 and 4. When the ring R is noetherian, this method cannot be applied to the functor $\mathrm{Ind}_P^G : \mathrm{Mod}_R^{adm}(M) \to \mathrm{Mod}_R^{adm}(G)$ because the category of smooth admissible R-representations is not grothendieck. It is not even known if it is an abelian category when the characteristic of F is p and p is not invertible in R.

In section 5, we assume that 0 is the only infinitely p-divisible element in R; we show the vanishing of the N-coinvariants of ind_P^{PgP} when $PgP \neq P$ and that the counit of the adjunction $(-_N, \mathrm{Ind}_P^G)$ is an isomorphism; the general arguments of section 2 imply that the unit of the adjunction $(\mathrm{Ind}_P^G, R_P^G)$ is an isomorphism and that Ind_P^G is fully faithful. When R is noetherian, $\mathrm{Ind}_P^G : \mathrm{Mod}_R^{adm}(M) \to \mathrm{Mod}_R^{adm}(G)$ is also obviously fully faithful.

The set $P\overline{P}$ is open dense in G. The partial compact induction functor $\mathrm{ind}_P^{P\overline{P}}$: $\mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(\overline{P})$ admits a right adjoint $R_P^{P\overline{P}}$ by the general method of section 2. Let z be an arbitrary element of the center $Z(M)$ of M strictly contracting N. We prove that the z-locally finite parts of $R_P^G(V)$ and of $R_P^{P\overline{P}}(V)$ for $V \in \mathrm{Mod}_R^\infty(G)$, are isomorphic in section 6.

The right adjoint $R_{\overline{P}}^{\overline{P}P} : \mathrm{Mod}_R^\infty(P) \to \mathrm{Mod}_R^\infty(M)$ of the functor $\mathrm{ind}_{\overline{P}}^{\overline{P}P}$ is explicit. It is the smooth part of the functor $\mathrm{Hom}_{R[N]}(C_c^\infty(N,R),-)$. In section 7, following Casselman and Emerton, we give another description of $R_{\overline{P}}^{\overline{P}P}$. Choosing an arbitrary open compact subgroup N_0 of N, the submonoid M^+ of elements of M contracting N_0 acts on V^{N_0} by the Hecke action. We have the smooth induction functor $\mathrm{Ind}_{M^+}^M : \mathrm{Mod}_R^\infty(M^+) \to \mathrm{Mod}_R^\infty(M)$. We show that $R_{\overline{P}}^{\overline{P}P} : \mathrm{Mod}_R^\infty P \to \mathrm{Mod}_R^\infty(M)$ is equal to the functor $V \mapsto \mathrm{Ind}_{M^+}^M(V^{N_0})$. The $Z(M)$-locally finite part of this functor is the Emerton's ordinary part functor $\mathrm{Ord}_P : \mathrm{Mod}_R^\infty P \to \mathrm{Mod}_R^\infty(M)$.

In section 8 we assume that R is noetherian and we show that $\mathrm{Ord}_P(V)$ is admissible when V is an admissible R-representation of G. Therefore the parabolic induction functor $\mathrm{Ind}_P^G : \mathrm{Mod}_R^{adm} M \to \mathrm{Mod}_R^{adm} G$ admits a right adjoint equal to the functor $\mathrm{Ord}_{\overline{P}} : \mathrm{Mod}_R^{adm} G \to \mathrm{Mod}_R^{adm} M$. If p is invertible in R, we prove that V_N is admissible when V is an admissible R-representation of G. Therefore the admissible parabolic induction functor Ind_P^G admits a left adjoint equal to the admissible N-coinvariant functor.

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2 Review on adjunction between grothendieck abelian categories

In a small category, the collection of objects and the morphisms $\mathrm{Hom}(A,B)$ between two objects A and B form sets. In a locally small category only the morphisms $\mathrm{Hom}(A,B)$ between two objects A and B must form a set. The category $\mathbb{S}et$ is locally small but not small. A category which is not locally small is big.

Let I be a small category, $\mathcal{C},\mathcal{D}$ locally small categories, $\mathcal{C}^{op}$ the locally small opposite category of $\mathcal{C}$ and $\mathcal{D}^{\mathcal{C}}$ the category of functors $\mathcal{C} \to \mathcal{D}$. The categories $\mathbb{S}et^{\mathcal{C}^{op}}, \mathbb{S}et^{\mathcal{C}}$ are big ([KS] Def. 1.4.2). A contravariant functor $\mathcal{C} \to \mathcal{D}$ is a functor $\mathcal{C}^{op} \to \mathcal{D}$.

Proposition 2.1. ([KS] Def. 1.2.11, Cor. 1.4.4)
The covariant Yoneda functor : $C \mapsto \mathrm{Hom}(-,C) : \mathcal{C} \to \mathbb{S}et^{\mathcal{C}^{op}}$ and the contravariant Yoneda functor : $C \mapsto \mathrm{Hom}(C,-) : \mathcal{C} \to \mathbb{S}et^{\mathcal{C}}$ are fully faithful.

A functor $F \in \mathbb{S}et^{\mathcal{C}}$ is called representable when there exists $C \in \mathcal{C}$ such that $F \simeq \mathrm{Hom}(C,-)$. A functor $F \in \mathbb{S}et^{\mathcal{C}^{op}}$ is called representable when there exists $C \in \mathcal{C}$ such that $F \simeq \mathrm{Hom}(-,C)$. In both cases, the object C which is unique modulo unique isomorphism is called a representative of the functor F.

A functor $F : \mathcal{I} \to \mathcal{C}$ defines functors

$$\varinjlim F : \mathcal{C} \to \mathbb{S}et \quad C \mapsto \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, ct_C), \quad \varprojlim F : \mathcal{C}^{op} \to \mathbb{S}et \quad C \mapsto \mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(ct_C, F),$$

where $ct_C : \mathcal{I} \to \mathcal{C}$ is the constant functor defined by $C \in \mathcal{C}$.

When the functor $\varinjlim F \in \mathbb{S}et^{\mathcal{C}}$ is representable, a representative is called the injective limit (or colimit or direct limit) of F and denoted also by $\varinjlim F$. We have natural isomorphisms ([ML] III.4 (2), (3)), for $C \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, ct_C) \simeq \mathrm{Hom}_{\mathcal{C}}(\varinjlim F, C).$$

When the functor $\varprojlim F \in \mathbb{S}et^{\mathcal{C}^{op}}$ is representable, a representative is called the projective limit (or inverse limit or limit) of F and denoted also by $\varprojlim F$. We have natural isomorphisms, for $C \in \mathcal{C}$,

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(ct_C, F) \simeq \mathrm{Hom}_{\mathcal{C}}(C, \varprojlim F).$$

One says that $(F(i))_{i\in\mathcal{I}}$ is an inductive (resp. projective) system in $\mathcal{C}$ indexed by $\mathcal{I}$ (resp. I^{op})) and one writes $\varinjlim(F(i))_{i\in\mathcal{I}}$ or $\varprojlim(F(i))_{i\in\mathcal{I}^{op}}$ for the object $\varinjlim F$ or $\varprojlim F$.

Example 2.2. 1) A set of objects $(C_i)_{i\in\mathcal{I}}$ of $\mathcal{C}$ indexed by a set $\mathcal{I}$ can be viewed as a functor $F : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is identified with a discrete category (the only morphisms are the identities). When they exist, $\varinjlim F = \oplus_{i\in\mathcal{I}} C_i$ is the direct sum, or coproduct, or disjoint union $\sqcup_{i\in\mathcal{I}} C_i$, and $\varprojlim F = \prod_{i\in\mathcal{I}} C_i$ is the direct product.

2) When $\mathcal{I}$ has two objects with two parallel morphisms other than the identities, F is nothing but two parallels arrows $C_1 \overset{g}{\underset{f}{\rightrightarrows}} C_2$ in $\mathcal{C}$. When they exist, $\varinjlim F$ is the cokernel of (f, g) and $\varprojlim F$ is the kernel of (f, g) ([KS] Déf. 2.2.2).

3) When they exist, it is possible to construct the inductive (resp. projective) limit of a functor $F : I \mapsto \mathcal{C}$, using only coproduct and cokernels (resp. products and kernels) ([KS] Prop. 2.2.9). If $\mathrm{Ob}(I)$ and $\mathrm{Hom}(I)$ denote the set of objects and of morphisms of the small category I, and if $s : \sigma(s) \to \tau(s)$ with $\sigma(s), \tau(s) \in \mathrm{Ob}(I)$ for $s \in \mathrm{Hom}(I)$, then

$$(1) \qquad \varinjlim F \ \text{ is the cokernel of } f, g : \oplus_{s\in\mathrm{Hom}(I)} F(\sigma(s)) \overset{g}{\underset{f}{\rightrightarrows}} \oplus_{i\in\mathrm{Ob}(I)} F(i),$$

where f, g correspond respectively to the two morphisms $\mathrm{id}_{F(\sigma(s)}, F(s)$, for $s \in \mathrm{Hom}(I)$,

$$\varprojlim F \ \text{ is the kernel of } \prod_{i\in\mathrm{Ob}(I)} F(i) \overset{g}{\underset{f}{\rightrightarrows}} \prod_{s\in\mathrm{Hom}(I)} F(\sigma(s)),$$

where f, g are deduced from the morphisms $\mathrm{id}_{F(\tau(s)}, F(s) : F(\tau(s)) \times F(\sigma(s)) \overset{g}{\underset{f}{\rightrightarrows}} F(\tau(s)$ for $s \in \mathrm{Hom}(I)$.

Let $F : \mathcal{C} \mapsto \mathcal{D}$ be a functor. For $U \in \mathcal{D}$, we have the category $\mathcal{C}_U$ whose objects are the pairs (X, u) with $X \in \mathcal{C}, u : F(X) \to U$. We say that F is right exact if the category $\mathcal{C}_U$ is filtrant for any $U \in \mathcal{D}$, and that F is left exact if the functor $F^{op} : \mathcal{D}^{op} \to \mathcal{C}^{op}$ is right exact ([KS] 3.1.1, 3.3.1).

Proposition 2.3. *Let a functor $F : \mathcal{C} \mapsto \mathcal{D}$.*

1) When $\mathcal{C}$ admits finite projective limits, F is left exact if and only it commutes with finite projective limits. In this case, F commutes with the kernel of parallel arrows.

2) When $\mathcal{C}$ admits small projective limits, F is left exact and commutes with small direct products, if and only if F commutes with small projective limits.

3) The similar statements hold true for right exact functors, inductive limits, small direct sums, and cokernels.

Proof. 1) ([KS] Prop. 3.3.3, Cor. 3.3.4).

2) If F preserves small projective limits, F is left exact and preserves small direct products (Example 2.2 1)). Conversely, from (**??**), a left exact functor which commutes wit small direct products preserves small projective limits because it commutes with the kernel of the parallel arrows by 1).

3) Replace $\mathcal{C}$ by $\mathcal{C}^{op}$. □

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be the functors. Then (F, G) is a pair of adjoint functors, or F is the left adjoint of G, or G is the right adjoint if F, if their exists an isomorphism of bifunctors from $\mathcal{C}^{op} \times \mathcal{C}$ to $\mathbb{S}et$

$$\mathrm{Hom}_{\mathcal{D}}(F(.), .) \simeq \mathrm{Hom}_{\mathcal{C}}(., G(.)),$$

called the adjunction isomorphism ([KS] Def. 1.5.2). The functor F determines the functor G up to unique isomorphism and G determines F up to unique isomorphism ([KS] Thm. 1.5.3). For $X \in \mathcal{C}$, the image of the identity $\mathrm{id}_{F(X)}$ in $\mathrm{Hom}_{\mathcal{D}}(F(X), F(X))$ by the adjunction isomorphism is a morphism $X \mapsto G \circ F(X)$. Similarly, for $Y \in \mathcal{D}$, the image of $\mathrm{id}_{G(Y)}$ is a morphism $F \circ G(Y) \to Y$. The morphisms are functorial in X and Y. We have constructed morphisms of functors, called the unit and the counit :

$$\epsilon : 1_{\mathcal{C}} \to G \circ F, \quad \eta : F \circ G \to 1_{\mathcal{D}}.$$

Proposition 2.4. *Let (F, G) be a pair of adjoint functors.*

F is fully faithful iff the unit $\epsilon : 1 \to G \circ F$ is an isomorphism.

G is fully faithful iff the counit $\eta : F \circ G \to 1$ is an isomorphism.

F and G are fully faithful iff F is an equivalence (fully faithful and essentially surjective ([KS] Def. 1.2.11,1.3.13)*) iff G is an equivalence. In this case F and G are quasi-inverse one to each other.*

Proof. ([KS] Prop. 1.5.6). □

Proposition 2.5. *Let (F, G) be a pair of adjoint functors. Then F is right exact and G is left exact.*

Proof. ([KS] Prop. 3.3.6) □

Let $\mathcal{A}$ be a locally small abelian category. A generator of $\mathcal{A}$ is an object E of $\mathcal{A}$ such that the functor $\mathrm{Hom}(E, -) : \mathcal{A} \to \mathbb{S}et$ is faithful (i.e. any object of $\mathcal{A}$ is a quotient of a small direct sum $\oplus_i E$).

If $\mathcal{A}$ admits small inductive limits, the functor between abelian categories

$$F \mapsto \varinjlim F : \mathcal{A}^{\mathcal{I}} \to \mathcal{A}$$

is additive and right exact.

The category $\mathcal{I}$ is filtrant when it is not empty, for any objects i, j of $\mathcal{I}$ there exist morphisms $i \to k, j \to k$, and for any parallel morphisms $f, g : i \to j$ there exists a morphism $h \to k$ such that $h \circ f = h \circ g$ ([KS] Def. 3.1.1).

Definition 2.6. ([KS] Déf. 8.3.24) *The locally small abelian category $\mathcal{A}$ is called grothendieck if it admits a generator, small inductive limits, and the small filtered inductive limits are exact.*

Example 2.7. *The locally small abelian category of left modules over a ring is grothendieck.*

Proof. ([KS] Ex. 8.3.25). □

Proposition 2.8. *A grothendieck abelian locally small category admits small projective limits.*

Proof. ([KS] Prop. 8.3.27). □

Proposition 2.9. *Let a functor $F : \mathcal{A} \to \mathcal{C}$ where $\mathcal{A}$ is a grothendieck abelian locally small category. The following properties are equivalent:*

1) F admits a right adjoint,

2) F commutes with small inductive limits,

3) F is right exact and commutes with small direct sums.

Proof. This is a combination of Prop. 2.3 with ([KS] Thm. 5.2.6, Prop. 5.2.8, Prop. 5.2.9). □

3 The category $\mathrm{Mod}_R^\infty(G)$

Let R be a commutative ring, let G be a secound countable locally profinite group (for instance, for a parabolic subgroup of a reductive group), and let $(K_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence $(K_n)_{n\in\mathbb{N}}$ of pro-p-open subgroups of G, with trivial intersection. We suppose R_n normal in R_0 for all n.

The category $\mathrm{Mod}_R(G)$ of R-representations of G is the grothendieck abelian locally small category of left $R[G]$-modules (Ex. 3).

Let $V \in \mathrm{Mod}_R(G)$. A vector in V is called smooth when it is fixed by an open subgroup of G. The set of smooth vectors of V is a $R[G]$-submodule of V, equal to $V^\infty = \cup_{n\in\mathbb{N}} V^{K_n}$ where V^{K_n} is the submodule of the vectors $v \in V$ fixed by R_n. When $V = V^\infty$, V is called smooth.

The same definition applies when G is a locally profinite monoid (its maximal subgroup is open and locally profinite).

3.1 $\mathrm{Mod}_R^\infty(G)$ is grothendieck

The module $C_c(G,R)$ of functions $f : G \to k$ with compact support is a $R[G\times G]$-module for the left and right translations. For $n \in \mathbb{N}$, the submodule $C_c(K_n\backslash G, R)$ of functions left invariant by K_n, is a smooth representation of G for the right translation. They form a strictly increasing sequence of union $C_c^\infty(G,R)$, equal to the smooth part of the $R[G\times G]$-module $C_c(G,R)$.

The subcategory $\mathrm{Mod}_R^\infty(G) \subset \mathrm{Mod}_R(G)$ of smooth R-representations of G, is an abelian locally small full subcategory.

Lemma 3.1. *$\mathrm{Mod}_R^\infty(G)$ is a grothendieck abelian locally small category.*

Proof. A direct sum of smooth representations is smooth, the cokernel of two parallel arrows in $\mathrm{Mod}_R^\infty(G)$ is smooth hence $\mathrm{Mod}_R^\infty(G)$ admits small inductive limits (1). Small filtered inductive limits are exact because they are already exact in $\mathrm{Mod}_R(G)$. A generator is $\oplus_{n\in\mathbb{N}} C_c(K_n\backslash G, R)$. ☐

For $W \in \mathrm{Mod}_R^\infty(G), V \in \mathrm{Mod}_R(G)$ we have $\mathrm{Hom}_{R[G]}(W,V) = \mathrm{Hom}_{R[G]}(W, V^\infty)$. The smoothification

$$V \mapsto V^\infty : \mathrm{Mod}_R(G) \to \mathrm{Mod}_R^\infty(G)$$

has a left adjoint: the inclusion $\mathrm{Mod}_R^\infty(G) \to \mathrm{Mod}_R(G)$ hence is left exact (Prop. 2.5). The smoothification is never right exact ([Viglivre] I.4.3) hence does not have a right adjoint (Prop. 2.5).

3.2 Admissibility and z-finiteness

A smooth R-representation V of G is called admissible when for any compact open subgroup H of G, the R-module V^H of H-fixed elements of V is finitely generated. The category $\mathrm{Mod}_R^{adm}(G)$ does not have a generator or small inductive limits. Worse, it is not known if the quotient of an admissible representation remains admissible.

Let z be an element in the center $Z(G)$ of G.

Definition 3.2. *Let $V \in \mathrm{Mod}_R(G)$. An element $v \in V$ is called z-finite if the R-module $R[z]v$ is contained in a finitely generated R-submodule of V.*

The subset V^{z-lf} of z-finite elements is a R-subrepresentation of V, called the z-locally finite part of V. When $V = V^{z-lf}$, the representation V is called z-locally finite.

The category $\mathrm{Mod}_R^{z-lf}(G)$ of z-locally finite smooth R-representations of G is a full abelian subcategory of $\mathrm{Mod}_R^\infty(G)$. The z-locally finite functor

$$V \mapsto V^{z-lf} : \mathrm{Mod}_R^\infty(G) \to \mathrm{Mod}_R^{z-lf}(G) \tag{2}$$

is the right adjoint of the inclusion $\mathrm{Mod}_R^{z-lf}(G) \to \mathrm{Mod}_R^\infty(G)$.

The z-locally finite part is contained in the $z^{\pm 1}$-locally finite part $V^{z^{\pm 1}-lf} = V^{z-lf} \cap V^{z^{-1}-lf}$, and in the $Z(G)$-locally finite part $V^{Z(G)-lf} = \cap_{z\in Z(G)} V^{z-lf}$ (this is a finite intersection when $Z(G)$ is a finitely generated monoid).

Lemma 3.3. *An admissible R-representation of G is $Z(G)$-locally finite.*

Proof. If $v \in V$ there exists $n \in \mathbb{N}$ such that $v \in V^{K_n}$, and V^{K_n} is a $Z(G)$-stable finitely generated R-module. □

4 The right adjoint R_P^G of $\mathrm{Ind}_P^G : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(G)$

let F be a local non archimedean field of residual characteristic p, let $\mathbf{G}$ be a reductive connected F-group. We fix a maximal F-split subtorus $\mathbf{S}$ of $\mathbf{G}$, and a minimal parabolic F-subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{S}$. Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. The $\mathbf{G}$-centralizer $\mathbf{Z}$ of $\mathbf{S}$ is a Levi subgroup of $\mathbf{B}$. We choose a parabolic F-subgroup $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{B}$ of unipotent radical $\mathbf{N}$ and Levi $\mathbf{M}$ containing $\mathbf{Z}$. We denote by X the group of F-rational points of an algebraic group $\underline{X}$ over F, with the exception that we write $N_G(S)$ for the group of F-rational points of the $\mathbf{G}$-normalizer $N_{\mathbf{G}}(\mathbf{S})$ of $\mathbf{S}$. Let $\mathbf{Z}$ be the $\mathbf{G}$- centralizer of $\mathbf{S}$ and $W_0 = N_{\mathbf{G}}(\mathbf{S})/Z$ the finite Weyl group.

Let $(K_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence of pro-p-open subgroups of G with trivial intersection, with R_n normal in R_0 for all n. We write $M_n = K_n \cap M, N_n = K_n \cap N, \overline{N}_n = K_n \cap \overline{N}$. We suppose, as we may, that $R_n = \overline{N}_n M_n N_n = N_n M_n \overline{N}_n$ has an Iwahori decomposition.

Lemma 4.1. *The quotient space $P\backslash G$ is compact and the quotient map $G \to P\backslash G$ admits a continuous section $\varphi : P\backslash G \to G$.*

Proof. This is well known: G is a finite disjoint union $\sqcup_{h\in X_n} P\overline{N}_n h$ for a finite subset $X_n \subset G$ for n large enough ([SVZ] Prop. 5.3), and $\varphi(Pxh) = xh, x \in \overline{N}_n, h \in X_n$ is a section. □

The smooth parabolic induction

$$\mathrm{Ind}_P^G : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(G)$$

sends $W \in \mathrm{Mod}_R^\infty(M)$ to the smooth representation $\mathrm{Ind}_P^G(W)$ of G acting by right translations on the module of functions $g : G \to W$ such that $f(mnxgx) = mf(x)$ for $m \in M, n \in N, x \in G, x \in K_n$ where $n \in \mathbb{N}$ depends on f. We write $C^\infty(P\backslash G, W)$ for the locally constant functions on the compact set $P\backslash G$ with values in W. The smooth parabolic induction

$$\mathrm{Ind}_P^G : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(G)$$

is the right adjoint of the N-coinvariant functor

$$V \mapsto V_N \ : \ \mathrm{Mod}_R^\infty(G) \to \mathrm{Mod}_R^\infty(M)$$

([Viglivre] I.5.7 (i), I.A.3 Proposition). The N-coinvariant functor $\mathrm{Mod}_R(P) \to \mathrm{Mod}_R(N)$ is the right adjoint of the inflation functor $\mathrm{Infl}_M^P : \mathrm{Mod}_R(M) \to \mathrm{Mod}_R(P)$ sending a representation of $M = P/N$ to the natural representation of P trivial on N.

Remark 4.2. *The N-coinvariants of the inflation functor Infl_M^P is the identity functor of $\mathrm{Mod}_R M$ (the unit $1_{\mathrm{Mod}_R M} \mapsto -_N \circ \mathrm{Infl}_M^P$ of the adjunction $(\mathrm{Infl}_M^P, -_N)$ is an isomorphism).*

Proposition 4.3. *The smooth parabolic induction functor $\mathrm{Ind}_P^G : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(G)$ Ind_P^G is exact, commutes with small direct sums, and admits a right adjoint*

$$R_P^G : \mathrm{Mod}_R^\infty(G) \to \mathrm{Mod}_R^\infty(M).$$

Proof. By Lemma 4.1, the R-linear map

$$f \mapsto f \circ \varphi : C^\infty(P\backslash G, W) \to \mathrm{Ind}_P^G(W) \tag{3}$$

is an isomorphism. We have a natural isomorphism

$$C^\infty(P\backslash G, W) \simeq C^\infty(P\backslash G, k) \otimes_R W \simeq C^\infty(P\backslash G, \mathbb{Z}) \otimes_{\mathbb{Z}} W. \tag{4}$$

The $\mathbb{Z}$-module $C^\infty(P\backslash G, \mathbb{Z})$ is free, because it is the union of the increasing sequence of the $\mathbb{Z}$-modules $(C^\infty(P\backslash G/K_n, \mathbb{Z})_{n\in\mathbb{N}}$ which are free of finite rank. Hence the tensor product by $C^\infty(P\backslash G, \mathbb{Z})$ is exact, and the smooth parabolic induction functor is exact.

The smooth parabolic induction commutes with small direct sums $\oplus_{i\in\mathcal{I}} W_i$ because a function $f \in C^\infty(P\backslash G, W)$ takes only finitely many values.

Applying Prop. 2.9 and Lemma 3.1, the parabolic induction admits a right adjoint. $\square$

Remark 4.4. When p is invertible in R, Dat ([Dat] between Cor. 3.7 and Prop. 3.8) showed that

$$R_P^G(V) = ([\mathrm{Hom}_{R[G]}(C_c^\infty(G, R), V)]^N)^\infty \quad (V \in \mathrm{Mod}_R^\infty(G)).$$

In this case, the modulus δ_P of P is well defined, and

$$R_P^G(V) \simeq \delta_P V_N$$

when R is the field of complex numbers (Bernstein) or when G is a linear group, a classical group when $p \neq 2$, or of semi-simple rank 1 [Dat].

Let $g \in G$ and $Q = P$ or $Q = \overline{P}$ the parabolic subgroup of G opposite to P, $\overline{P} \cap P = M$. The partial compact parabolic induction functor

$$\mathrm{ind}_P^{PgQ} : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(Q)$$

associates to $W \in \mathcal{M}_k^\infty(M)$ the smooth representation $\mathrm{ind}_P^{PgQ}(W)$ of Q by right translation on the submodule of functions with support contained in PgQ,, with compact support modulo left multiplication by P (PgQ is generally not closed in the compact set $P\backslash G$).

Proposition 4.5. *The partial compact parabolic induction functor ind_P^{PgQ} is exact, commutes with small direct sums, and admits a right adjoint*

$$R_P^{PgQ} : \mathrm{Mod}_R^\infty(Q) \to \mathrm{Mod}_R^\infty(M).$$

Proof. Same proof as for the functor Ind_P^G (Prop. 4.3). $\square$

Remark 4.6. *When $PgP = P$, the partial compact parabolic induction functor ind_P^P is the inflation functor Infl_M^P.*

Lemma 4.7. *$W \in \mathrm{Mod}_R^\infty(M)$ is admissible if and only if $\mathrm{Ind}_P^G(W) \in \mathrm{Mod}_R^\infty(G)$ is admissible.*

Proof. This is well known and follows from the decomposition ([Viglivre] I.5.6, II.2.1):

$$(\mathrm{Ind}_P^G W)^{K_n} \simeq \oplus_{PgK_n} (\mathrm{Ind}_P^{PgK_n} W)^{K_n} \simeq \oplus_{PgK_n} W^{M\cap gK_n g^{-1}} \quad (n \in \mathbb{N}, g \in G),$$

where the sum is finite and $\mathrm{Ind}_P^{PgK_n} W \subset \mathrm{Ind}_P^G W$ is the R-submodule of functions with support contained in PgK_n. □

Corollary 4.8. *The smooth parabolic induction restricts to a functor*

$$\mathrm{Ind}_P^G : \mathrm{Mod}_R^{adm}(M) \to \mathrm{Mod}_R^{adm}(G).$$

We will later show that this functor admits also a right adjoint when the ring is noetherian.

5 Ind_P^G is fully faithful if $R_{p-ord} = \{0\}$.

Definition 5.1. *The p-ordinary part R_{p-ord} of R is the subset of $x \in k$ which are infinitely p-divisible.*

When R is a field, $R_{p-ord} = \{0\}$ if and only if the characteristic of R is p.

Let Φ_G be the set of roots of S in G. Let $\alpha \in \Phi_G$. We write U_α for the root subgroup of G associated to α. We have $R_{p-ord} \neq \{0\}$ if and only if there exists a Haar measure on U_α with values in R if and only if

$$C_c^\infty(U_\alpha, R)_{U_\alpha} \neq 0 \tag{5}$$

([Viglivre] I (2.3.1)).

Proposition 5.2. *We suppose $R_{p-ord} = \{0\}$. The N-coinvariants of the partial compact induction functor ind_P^{PgP} is 0 if $PgP \neq P$.*

Proof. Let $W \in \mathrm{Mod}_R M$. The action of N on

$$\mathrm{ind}_P^{PgP}(W) = C_c^\infty(P\backslash PgP, R) \otimes_R W$$

is trivial on W and the right translation on $C_c^\infty(P\backslash PgP, R)$. By the Bruhat decomposition, $PgP = PnP$ for an element $n \in N_G(S)$ of non trivial image $w \in W_0$. There exists a root $\alpha \in \Phi_G$ such that $U_\alpha \subset N$ and $U_{w(\alpha)}$ is not contained in N, and the multiplication map $PgMN_\alpha \times U_\alpha \to PgP$ is an homeomorphism, where N_α is a subgroup of N normalized by U_α. We deduce

$$C_c^\infty(P\backslash PgMN_\alpha U_\alpha, R)_{U_\alpha N_\alpha} = (C_c^\infty(P\backslash PgMN_\alpha) \otimes_R C_c^\infty(U_\alpha, R)_{U_\alpha})_{N_\alpha}.$$

Applying (5), we obtain $C_c^\infty(P\backslash PgP, R)_N = 0$. □

Theorem 5.3. *We suppose $R_{p-ord} = \{0\}$. Then*

1. *The parabolic induction $\mathrm{Ind}_P^G : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(G)$ is fully faithful,*
2. *The unit $\mathrm{id}_{\mathrm{Mod}_R^\infty(M)} \to R_P^G \circ \mathrm{Ind}_P^G$ of the adjoint pair $(\mathrm{Ind}_P^G, R_P^G)$ is an isomorphism.*
3. *The counit $\eta : -_N \circ \mathrm{Ind}_P^G \to \mathrm{id}_{\mathrm{Mod}_R^\infty(M)}$ of the adjoint pair $(-_N, \mathrm{Ind}_P^G)$ is an isomorphism.*

Proof. By Prop. 2.4 the 3 properties are equivalent. The counit η of the adjoint pair $(-_N, \mathrm{Ind}_P^G)$ is an isomorphism, because the N-coinvariant of Ind_P^G is isomorphic to the N-coinvariants of Infl_P^G. This follows from Prop. 5.2 and Remarks 4.2, 4.6.

a) Beeing a right adjoint, the N-coinvariant functor $\mathrm{Mod}_R^\infty(P) \to \mathrm{Mod}_R^\infty(M)$ is right exact.

b) The representation W inflated to M is a quotient of the restriction to P of $\mathrm{Ind}_P^G(W)$. The Bruhat decomposition implies that the kernel of the quotient map $\mathrm{Ind}_P^G(W) \to W$ admits a finite filtration $F_1 \subset F_2 \subset \ldots \subset F_r$ of quotients ind_P^{PgP} for all double cosets $PgP \neq P$ of G. □

6 The z-locally finite parts of R_P^G and of $R_P^{P\overline{P}}$ are equal

We compare the right adjoint $R_P^G : \mathrm{Mod}_R^\infty(G) \to \mathrm{Mod}_R^\infty(M)$ of the smoooth parabolic induction Ind_P^G and the right adjoint $R_P^{P\overline{P}} : \mathrm{Mod}_R^\infty \overline{P} \to \mathrm{Mod}_R^\infty M$ of the partial compact parabolic induction $\mathrm{ind}_P^{P\overline{P}}$.

Let $\mathbf{Z}(\mathbf{M}) \subset \mathbf{S}$ be the split center of M and let $z \in Z(M)$ be an element strictly contracting N : for any open compact subgroup $N_0 \subset N$, the sequence $(z^n N_0 z^{-n})_{n\in\mathbb{N}}$ is strictly decreasing of trivial intersection. We denote by

$$R_P^{G,z-lf} : \mathrm{Mod}_R^\infty G \to \mathrm{Mod}_R^{z-lf} M, \quad \text{resp. } R_P^{P\overline{P},z-lf} : \mathrm{Mod}_R^\infty \overline{P} \to \mathrm{Mod}_R^{z-lf} M,$$

the z-locally finite part of R_P^G, resp. $R_P^{P\overline{P}}$. The restriction to $\overline{P}$ is a faithful functor

$$\mathrm{Res}_{\overline{P}}^G : \mathrm{Mod}_R^\infty(G) \to \mathrm{Mod}_R^\infty \overline{P}.$$

Theorem 6.1. *The functors $R_P^{G,z-lf}$ and $R_P^{P\overline{P},z-lf} \circ \mathrm{Res}_{\overline{P}}^G$ are isomorphic.*

Proof. We want to prove that there exists a functorial isomorphism

$$\mathrm{Hom}_{R[M]}(W, R_P^{G,z-lf}(V)) \to \mathrm{Hom}_{R[M]}(W, R_P^{P\overline{P},z-lf}(V)) \tag{6}$$

in $(W,V) \in \mathrm{Mod}_R^{z-lf}(M) \times \mathrm{Mod}_R^\infty(G)$. The image of W by a $R[M]$-homomorphism in $R_P^G(V)$, resp. $R_P^{P\overline{P}}(V)$ is contained in $R_P^{G,z-lf}(V)$, resp. $R_P^{P\overline{P},z-lf}(V)$. Also, we can replace in (6) $R_P^{G,z-lf}$ by R_P^G and $R_P^{P\overline{P},z-lf}$ by $R_P^{P\overline{P}}$. Then we use the adjunctions $(\mathrm{Ind}_P^G, R_P^G)$ and $(\mathrm{ind}_P^{P\overline{P}}, R_P^{P\overline{P}})$. We are reduced to find a functorial isomorphism

$$J : \mathrm{Hom}_{R[G]}(\mathrm{Ind}_P^G W, V) \to \mathrm{Hom}_{R[\overline{P}]}(\mathrm{ind}_P^{P\overline{P}} W, V) \tag{7}$$

in $(W,V) \in \mathrm{Mod}_R^{z-lf}(M) \times \mathrm{Mod}_R^\infty(G)$. There is an obvious candidate: the restriction of a $R[G]$-homomorphism $\varphi : \mathrm{Ind}_P^G W \to V$ to the submodule $\mathrm{ind}_P^{P\overline{P}} W \subset \mathrm{Ind}_P^G W$ is a $R[\overline{P}]$-homomorphism $J(\varphi) : \mathrm{ind}_P^{P\overline{P}} W \to V$. This is functorial. Two $R[G]$-homomorphisms $\mathrm{Ind}_P^G(W) \to V$ equal on $\mathrm{Ind}_P^{P\overline{P}}(W)$ are equal because an arbitrary open subset of $P\backslash G$ is a finite disjoint union of G-translates of compact open subsets of $P\backslash P\overline{P}$ ([SVZ] Prop. 5.3).

To study the image of J we introduce more notations. For $r \in \mathbb{N}, \overline{n} \in \overline{N}$ and a non-zero element $w \in W^{M_r}$, let $f_{r,\overline{n},w} \in \mathrm{ind}_P^{P\overline{P}}(W)$ be the function of support $P\overline{N}_r\overline{n}$ and equal to w on $\overline{N}_r\overline{n}$. For $g \in G$, the support of the function $gf_{r,\overline{n},w} \in \mathrm{Ind}_P^G(W)$ is $P\overline{N}_r\overline{n}g^{-1}$. We say that $(g, r, \overline{n}, w)$ is admissible when

$$P\overline{N}_r g = P\overline{N}_r\overline{n} \quad (r \in \mathbb{N}, g \in G, \overline{n} \in \overline{N}, w \in W^{M_r}).$$

Let $\Phi \in \mathrm{Hom}_{R[\overline{P}]}(\mathrm{ind}_P^{P\overline{P}} W, V)$. We will prove:

1) Φ belongs to the image of J when $\Phi(gf_{r,\overline{n},w}) = g\Phi(f_{r,\overline{n},w})$ for all admissible $(g, r, \overline{n}, w)$.

2) $\Phi(gf_{r,\overline{n},w}) = g\Phi(f_{r,\overline{n},w})$ for all admissible $(g, r, \overline{n}, w)$ (it is only here that we use that W is z-locally finite).

The proof of 1), resp. 2), follows ([Emerton] 4.4.6, resp. 4.4.3).

Proof of 1). Φ belongs to the image of J if and only if $\sum_i g_i\Phi(f_i) = 0$ for all $g_1, \ldots, g_n$ in G and $f_1, \ldots, f_n$ in $\mathrm{ind}_P^{P\overline{P}}(W)$ such that $\sum_i g_i f_i = 0$.

Let $g_1, \ldots, g_n, f_1, \ldots, f_n$ satisfying this property. We choose $r \in \mathbb{N}$ large enough, such that the f_i seen as elements of $C_c^\infty(\overline{N}, W)$ are left $\overline{N}_r$-invariant with values in W^{M_r}. Let $Y_i \subset X_r \cap \overline{N}$ such that the support of f_i is $\sqcup_{\overline{n}\in Y_i} P\overline{N}_r\overline{n}$. We have $f_i|_{P\overline{N}_r\overline{n}} = f_{r,\overline{n},f_i(\overline{n})}$.

Let $h \in X_r$. We have $\sum_i (g_i f_i)|_{P\overline{N}_r h} = 0$. In the sum, the non-zero elements

$$(g_i f_i)|_{P\overline{N}_r h} = g_i(f_i|_{P\overline{N}_r h g_i}),$$

are those such that $P\overline{N}_r h g_i = P\overline{N}_r\overline{n}$ for some $\overline{n} \in Y_i$. We have also $\sum_i hg_i(f_i|_{P\overline{N}_r h g_i}) = 0$. This implies

$$\sum_i hg_i\Phi(f_i|_{P\overline{N}_r h g_i}) = 0 \ ,$$

under the hypothesis of the proposition. We have also $\sum_i g_i\Phi(f_i|_{P\overline{N}_r h g_i}) = 0$.

For all i, $P\backslash G$ is the finite disjoint union $\cup_{h\in X_r} P\overline{N}_r h g_i$ hence $f_i = \sum_{h\in X_r} f_i|_{P\overline{N}_r h g_i}$. We deduce

$$\sum_i g_i\Phi(f_i) = \sum_{h\in X_r}\sum_i g_i\Phi(f_i|_{P\overline{N}_r h g_i}) = 0 \ .$$

Proof of 2). We have $\overline{n}^{-1} f_{r,\overline{n},w} = f_{r,1,w}$, and $f_{r,1,w}$ is fixed by R_r.

First we reduce to $\overline{n} = 1$ by replacing $(g, \overline{n})$ by $(g\overline{n}^{-1}, 1)$.

We choose three integers $r' \in \mathbb{Z}, r'' \geq r, a \in \mathbb{N}$ as follows. The subset $\overline{N}_r g^{-1} \subset P\overline{N}_r$ beeing compact, its projection onto the first factor of the isomorphism $N \times M \times \overline{N} \to P\overline{N}$ is compact and contained in $N_{r'}$, hence $\overline{N}_r g^{-1} \subset N_{r'}\overline{P}$. The R-submodule generated by $\Phi(1_{r,1,w'})$ for $w' \in R[z]w$ is contained in a finitely generated R-submodule of V, contained in $V^{K_{r''}}$. We have $z^a N_{r'} z^{-a} \subset N_{r''} \subset N_r$.

The function $f_{r,1,w}$ is the sum of its restrictions to $Pz^{-a}\overline{N}_r z^a\overline{v} = P\overline{N}_r z^a\overline{v}$ for the cosets $z^{-a}\overline{N}_r z^a\overline{v}$ contained in $\overline{N}_r$,

$$f_{r,1,w} = \sum_{\overline{v}\in z^{-a}\overline{N}_r z^a\backslash\overline{N}_r} (z^a\overline{v})^{-1} f_{r,1,z^a(w)}.$$

By $R[\overline{P}]$-linearity of Φ, we are reduced to prove, for $\overline{v} \in \overline{N}_r$,

$$\Phi(g\overline{v}^{-1}z^{-a}f_{r,1,z^a(w)}) = g\Phi(\overline{v}^{-1}z^{-a}f_{r,1,z^a(w)}) = g\overline{v}^{-1}z^{-a}\Phi(f_{r,1,z^a(w)}).$$

We may write $g\overline{v}^{-1} = \overline{p}n_{r'}$ with $n_{r'} \in N_{r'}, \overline{p} \in \overline{P}$. We have to prove that

$$\Phi(n_{r'}z^{-a}f_{r,1,z^a(w)}) = n_{r'}z^{-a}\Phi(f_{r,1,z^a(w)}).$$

Applying z^a, it is equivalent to prove

$$\Phi(z^a n_{r'}z^{-a}f_{r,1,z^a(w)}) = z^a n_{r'}z^{-a}\Phi(f_{r,1,z^a(w)}).$$

As $f_{r,1,z^a(w)}$ and $\Phi(f_{r,1,z^a(w)})$ are fixed by $z^a n_{r'} z^{-a}$, the equality is obvious. □

7 The Hecke description of $R_{\overline{P}}^{\overline{P}P} : \mathrm{Mod}_R^\infty(P) \to \mathrm{Mod}_R^\infty(M)$

We choose a compact open subgroup N_0 *of* N. The submonoid $M^+ \subset M$ contracting N_0 is the set of $m \in M$ such that $mN_0m^{-1} \subset N_0$. The inverse monoid $M^- \subset M$ is the submonoid of M dilating N_0. The group $M^+ \cap M^-$ is open in M. We recall the element $z \in Z(M)$ strictly contracting N. We have the induction functor

$$I_{M^+}^M : \mathrm{Mod}_R(M^+) \to \mathrm{Mod}_R(M)$$

sending $W \in \mathrm{Mod}_R(M^+)$ to the representation of M by right translations on the module of R-linear maps $\psi : M \to W$ such that $\psi(mx) = m\psi(x)$ for all $m \in M^+, x \in M$. The smooth induction functor

$$\mathrm{Ind}_{M^+}^M : \mathrm{Mod}_R^\infty(M^+) \to \mathrm{Mod}_R^\infty(M)$$

is the smoothification of the induction functor $I_{M^+}^M$.

Definition 7.1. *Let* $V \in \mathrm{Mod}_R^\infty(P)$. *The submonoid* $M^+ \subset M$ *contracting* N_0 *acts on* V^{N_0} *by the Hecke action* $(m,v) \mapsto h_{N_0,m}(v)$,

$$h_{N_0,m}(v) = \sum_{n \in N_0/mN_0m^{-1}} nmv \quad (m \in M^+, v \in V^{N_0}). \tag{8}$$

The Hecke action of M^+ on V^{N_0} is smooth because it extends the natural action of the open subgroup $M^+ \cap M^-$.

Theorem 7.2. *The functor*

$$V \mapsto \mathrm{Ind}_{M^+}^M(V^{N_0}) : \mathrm{Mod}_R^\infty(P) \to \mathrm{Mod}_R^\infty(M) \tag{9}$$

is isomorphic to $R_{\overline{P}}^{\overline{P}P}$.

The theorem implies that the functor

$$R_{\overline{P}}^{\overline{P}P, z-lf} : \mathrm{Mod}_R^\infty(P) \to \mathrm{Mod}_R^{z-lf}(M)$$

is isomorphic to the z-locally finite part of the functor (9). The Emerton's ordinary functor is the $Z(M)$-locally finite part of the functor (9):

$$\mathrm{Ord}_P = R_{\overline{P}}^{\overline{P}P, Z(M)-lf} : \mathrm{Mod}_R^\infty(P) \to \mathrm{Mod}_R^{Z(M)-lf}(M)$$

is isomorphic to Ord_P. Applying Thm. 6.1, we obtain:

Corollary 7.3. *The functor*

$$V \mapsto (\mathrm{Ind}_{M^+}^M(V^{N_0}))^{z-lf} : \mathrm{Mod}_R^\infty(G) \to \mathrm{Mod}_R^{z-lf}(M)$$

is isomorphic to $R_P^{G,z-lf}$. *The functor*

$$V \mapsto \mathrm{Ord}_P(V) = (\mathrm{Ind}_{M^+}^M(V^{N_0}))^{Z(M)-lf} : \mathrm{Mod}_R^\infty(G) \to \mathrm{Mod}_R^{z-lf}(M)$$

is isomorphic to $R_P^{G,Z(M)-lf}$.

The functor $R_{\overline{P}}^{\overline{P}P}$ is the right adjoint of

$$\mathrm{ind}_{\overline{P}}^{\overline{P}P} = \mathrm{ind}_{\overline{P}}^{\overline{P}N} \simeq C_c^\infty(N,R) \otimes_R - : \mathrm{Mod}_R^\infty(M) \to \mathrm{Mod}_R^\infty(P)$$

Lemma 7.4. *The functor* $R_{\overline{P}}^{\overline{P}P} : \mathrm{Mod}_R^\infty(P) \to \mathrm{Mod}_R^\infty(M)$ *is the smoothification of the functor* $\mathrm{Hom}_{R[N]}(C_c^\infty(N,R),-)$.

Thm. 7.2 follows from:

Proposition 7.5. *Let* $V \in \mathrm{Mod}_R^\infty(P)$. *The map:*

$$\Phi : \mathrm{Hom}_{R[N]}(C_c^\infty(N,R),V) \to I_{M^+}^M(V^{N_0}) \quad f \mapsto \Phi(f)(m) = (mf)(1_{N_0}) \ (m \in M), \tag{10}$$

is an isomorphism of $R[M]$*-modules.*

Proof. For $V \in \mathrm{Mod}_R^\infty(P)$, $m \in M$ and $n \in N$ act on $f \in \mathrm{Hom}_k(C_c^\infty(N,R),V)$ by

$$mf = m \circ f \circ m^{-1} : x \mapsto mf(m^{-1}xm), \ nf : x \mapsto f(xn).$$

The map (10) is $R[M]$-linear by adjunction because the map

$$\varphi : \mathrm{Hom}_{R[N]}(C_c^\infty(N,R),V) \to V^{N_0} \quad , \quad f \mapsto f(1_{N_0}).$$

is $R[M^+]$-linear: For $m \in M^+, f \in \mathrm{Hom}_k(C_c^\infty(N,R),V)$, we compute

$$\begin{aligned}(mf)(1_{N_0}) = m \circ f \circ m^{-1}(1_{N_0}) = m \circ f(1_{m^{-1}N_0 m}) &= m \sum_{n \in m^{-1}N_0 m/N_0} f(1_{N_0 n^{-1}}) \\ = m \sum_{n \in m^{-1}N_0 m/N_0} nf(1_{N_0}) = \sum_{n \in N_0/mN_0m^{-1}} &nm\varphi(1_{N_0}) = h_{N_0,m}(f(1_{N_0})) \ .\end{aligned}$$

The map Φ (10) is injective because the $R[P]$-module $C_c^\infty(N,R)$ is generated by 1_{N_0}, and we compute:

$$\Phi(f)(m) = (mf)(1_{N_0}) = m \circ f \circ m^{-1}(1_{N_0}) = m \circ f(1_{N_0}(m - m^{-1})) = m \circ f(1_{m^{-1}N_0 m}).$$

If $\Phi(f) = 0$ then $f(1_{m^{-1}N_0 m}) = 0$ for all $m \in M$. As f is N-equivariant, we have also $0 = f(1_{m^{-1}N_0 mn}) = f(mn1_{N_0})$ for all $m \in M, n \in N$, hence $f = 0$.

We show the surjectivity of Φ. An arbitrary element of $C_c^\infty(N,R)$ is a sum

$$\sum_i x_i n_i z^a 1_{N_0} = \sum_i x_i 1_{z^a N_0 z^{-a} n_i^{-1}} = \sum_i x_i n_i 1_{z^a N_0 z^{-a}} = \sum_i x_i n_i z^a 1_{N_0}$$

where $x_i \in k, n_i \in N, a \in \mathbb{N}$ and the open sets $z^a N_0 z^{-a} n_i^{-1}$ are disjoint. Let $\psi \in \mathrm{Ind}_{M^+}^M(V^{N_0})$. There is a $R[N]$-linear map

$$f_\psi : C_c^\infty(N,R) \to V \quad \text{such that } f_\psi(z^a 1_{N_0}) = h_{N_0,z^a}(\psi(z^{-a})) \quad (a \in \mathbb{N}).$$

We have $\Phi(f_\psi) = \psi$. □

Remark 7.6. $(\mathrm{Ind}_{M^+}^M(V^{N_0}))^{z^{-1}-lf} = (I_{M^+}^M(V^{N_0}))^{z^{-1}-lf}$, *for* $V \in \mathrm{Mod}_R^\infty(P)$.

Proof. Let $\varphi \in (I_{M^+}^M(V^{N_0}))^{z^{-1}-lf}$. We have to prove that φ is smooth. Let $W_\varphi \subset I_{M^+}^M(V^{N_0})$ be a finitely generated R-module containing $R[z^{-1}]\varphi$. The image of W_φ by the map $f \mapsto f(1)$ is a a finitely generated R-submodule of V^{N_0} containing $\varphi(z^{-a})$ for all $a \in \mathbb{N}$. The action of M^+ on V^{N_0} is smooth. This implies that there exists a large integer $n \in \mathbb{N}$ such that $M_n = K_n \cap M \subset M^+ \cap M^-$ fixes $\varphi(z^{-a})$ for all $a \in \mathbb{N}$. The element φ is fixed by M_n because $M = \cup_{a \in \mathbb{N}} M^+ z^{-a}$, hence two elements of $I_{M^+}^M(V^{N_0})$ equal on z^{-a} for all $a \in \mathbb{N}$ are equal. □

Remark 7.7. *Let $W \in \mathrm{Mod}_R^\infty(M^+)$ and suppose $M_r \subset M^+ \cap M$. The map*

$$f \mapsto f|_{z^\mathbb{Z}} : (I_{M^+}^M W)^{M_r} \to I_{z^\mathbb{N}}^{z^\mathbb{Z}}(W^{M_r})$$

is a $R[z^\mathbb{Z}]$-isomorphism.

Proof. The map $f \mapsto f|_{z^\mathbb{Z}} : I_{M^+}^M W \to I_{z^\mathbb{N}}^{z^\mathbb{Z}} W$ is a $R[z^\mathbb{Z}]$-isomorphism because $M = \cup_{a\in\mathbb{N}} M^+ z^{-a}$. Let $f \in I_{M^+}^M W$. For $m^+ \in M^+, m_r \in M_r, a \in \mathbb{Z}$, we have $f(m^+ z^a m_r) = m^+ f(m_r z^a) = m^+ m_r f(z^a)$ because $M_r \subset M^+$ and $z \in Z(M)$. If f is fixed by M_r, then $f(z^a)$ is fixed by M_r for all $a \in \mathbb{Z}$ because $f(z^a) = (z^a m_r) = m_r f(z^a)$. Conversely, if $f(z^a)$ is fixed by M_r for all $a \in \mathbb{Z}$, then f is fixed by M_r because $f(m^+ z^a m_r) = m^+ f(z^a) = f(m^+ z^a)$ and $M = \cup_{a\in\mathbb{Z}} M^+ z^{-a}$. □

8 The right adjoint $\mathrm{Ord}_{\overline{P}}$ of $\mathrm{Ind}_P^G : \mathrm{Mod}_R^{adm}(M) \to \mathrm{Mod}_R^{adm}(G)$

Theorem 8.1. *We suppose that R is noetherian. For $V \in \mathrm{Mod}_R^{adm}(G)$, the representation $(I_{M^+}^M(V^{N_0}))^{z^{-1}-lf}$ of M is admissible.*

Proof. Let r large enough so that M_r is an open compact subgroup of $M^+ \cap M$. Note that $M_r N_0$ is a group. The R-module of elements fixed by M_r in $(I_{M^+}^M(V^{N_0}))^{z^{-1}-lf}$ is isomorphic to

$$X = (I_{z^\mathbb{N}}^{z^\mathbb{N}}(V^{N_0 M_r}))^{z^{-1}-lf}$$

by the map $f \mapsto f|_{z^\mathbb{Z}}$ (Remark 7.7). We want to show that X is finitely generated.

The image Y of X by $f \mapsto f(1)$ is a submodule of $V^{N_0 M_r}$ containing $f(z^a)$ for all $a \in \mathbb{Z}$ and $f \in X$. We have $X \subset I_{z^\mathbb{N}}^{z^\mathbb{Z}}(Y)$. We have the compact open subgroup $\overline{N}_r M_r N_0$. We will prove (Prop. 8.2) that

$$Y \subset V^{\overline{N}_r M_r N_0}.$$

Admitting this, Y is a finitely generated R-module because V is admissible and R is noetherian. The action $h_{N_0,z}$ of z on Y is surjective because, for $f \in X$ we have $f(1) = f(zz^{-1}) = h_{N_0,z} f(z^{-1})$. A surjective endomorphism of a finitely generated R-module is bijective (this is an application of Nakayama lemma [Matsumura] Thm. 2.4). Hence the action of z on Y is bijective. Hence $Y \simeq I_{z^\mathbb{N}}^{z^\mathbb{Z}}(Y)$ is a finitely generated R-module. As R is noetherian, X is a finitely generated R-module. □

Proposition 8.2. *We suppose R noetherian. If $f \in (I_{z^\mathbb{N}}^{z^\mathbb{Z}}(V^{M_r N_0}))^{z^{-1}-lf}$, then $f(1) \in V^{\overline{N}_r M_r N_0}$.*

Proof. We have

$$V^{M_r N_0} = \cup_{t\geq r} V^{\overline{N}_t M_r N_0}, \tag{11}$$

where $\overline{N}_t M_r N_0 = K_t M_r N_0 \subset G$ is a compact open subgroup as $M_r N_0 \subset K_0$ normalizes R_t, and the sequence $(\overline{N}_t M_r N_0)_{t\geq r}$ is strictly decreasing of intersection $M_r N_0$.

We suppose $t \geq r$. We write $n(r,t) \in \mathbb{N}$ for the smallest integer such that $z^{-n}\overline{N}_r z^n \subset \overline{N}_t \subset \overline{N}_r$ for $n \geq n(r,t)$. We show that

1) $h_{N_0,z^n}(V^{\overline{N}_t M_r N_0})$ *is fixed by* $\overline{N}_r M_r N_0$ *when* $n \geq n(r,t)$. .

Let $v \in V^{\overline{N}_t M_r N_0}$. Let $\overline{n}_r \in \overline{N}_r$ and $(n_i)_{i\in I}$ a system of representatives of $N_0/z^n N_0 z^{-n}$. We compute:

$$\overline{n}_r h_{N_0,z^n}(v) = \sum_{i\in I} \overline{n}_r n_i z^n v = \sum_{i\in I} n_i' \overline{b}_i z^n v = \sum_{i\in I} n_i' z^n v, \tag{12}$$

where $\overline{n}_r n_i = n'_i \overline{b}_i$ with $n'_i \in N_0, \overline{b}_i \in \overline{N}_r M_r$ using the Iwahori decomposition of the group $\overline{N}_r M_r N_0 = N_0 \overline{N}_r M_r$. For the last equality, we use that $z^n v$ is fixed by $z^n M_r \overline{N}_t z^{-n} = M_r z^n \overline{N}_t z^{-n}$ which contains $M_r \overline{N}_r$. We show that $(n'_i)_{i\in I}$ is a system of representatives of $N_0/z^n N_0 z^{-n}$, hence that $\overline{n}_r$ fixes $h_{N_0,z^n}(v)$. We have to prove that ${n'_i}^{-1} n'_j \in z^n N_0 z^{-n}$ implies $i = j$. If $\overline{b}_i n_i^{-1} n_j \overline{b}_j^{-1} \in z^n N_0 z^{-n}$, then $n_i^{-1} n_j$ belongs to the group generated by $\overline{N}_r M_r$ and $z^n N_0 z^{-n}$. This group is $\overline{N}_r M_r z^n N_0 z^{-n}$ as the subgroup $z^n N_0 z^{-n} \subset N_0$ normalizes the group $\overline{N}_r M_r$. Hence $n_i^{-1} n_j \in z^n N_0 z^{-n}$. This implies $i = j$.

2) $V^{\overline{N}_t M_r N_0}$ is a $R[z]$-submodule of $V^{M_r N_0}$.

When $t = r$, $n(t,t) = 0$. Applying 1) $V^{\overline{N}_t M_t N_0}$ is a $R[z]$-submodule of V^{N_0}. Hence $V^{M_r N_0} \cap V^{\overline{N}_t M_t N_0}$ is a $R[z]$-submodule of $V^{M_r N_0}$. The group generated by the groups $M_r N_0$ and $\overline{N}_t M_t N_0$ is $\overline{N}_t M_r N_0$ because M_r contains M_t and normalizes $N_0, \overline{N}_t$. Hence $V^{\overline{N}_t M_r N_0}$ is a $R[z]$-submodule of $V^{M_r N_0}$.

3) Let $f \in (I_{z^{\mathbb{N}}}^{z^{\mathbb{Z}}}(V^{M_r N_0}))^{z^{-1}-lf}$. The R-module generated by $f(z^{-a})$ for $a \in \mathbb{Z}$ is finitely generated because f is z^{-1}-finite and there exists $t \geq r$ such that $f(z^{-a}) \subset V^{\overline{N}_t M_r N_0}$. By 2), $f \in I_{z^{\mathbb{N}}}^{z^{\mathbb{Z}}}(V^{\overline{N}_t M_r N_0})$. We have $f(1) \in \cap_{n\geq 1} h_{N_0,z^n} V^{\overline{N}_t M_r N_0}$. By 1), $h_{N_0,z^n} V^{\overline{N}_t M_r N_0} \subset V^{\overline{N}_r M_r N_0}$ when $n \geq n(r,t)$. Hence $f(1) \in V^{\overline{N}_r M_r N_0}$. □

This ends the proof of Thm. 8.1. An admissible representation of M is $Z(M)$-locally finite (Lemma 3.3). By Thm. 8.1, Remark 7.6, and Corollary 7.3, we deduce :

Corollary 8.3. *The functor*

$$(R_P^G)^{Z(M)-lf} \simeq \mathrm{Ord}_{\overline{P}} : \mathrm{Mod}_R^{adm}(G) \to \mathrm{Mod}_R^{adm}(M)$$

is the right adjoint of the parabolic induction $\mathrm{Ind}_P^G : \mathrm{Mod}_R^{adm}(M) \to \mathrm{Mod}_R^{adm}(G)$.

Corollary 8.4. *When* $R_{p-ord} = \{0\}$, *the parabolic induction* $\mathrm{Ind}_P^G : \mathrm{Mod}_R^{adm}(M) \to \mathrm{Mod}_R^{adm}(G)$ *is fully faithful, and the unit* $\mathrm{id} \mapsto \mathrm{Ord}_{\overline{P}} \circ \mathrm{Ind}_P^G$ *of the adjunction* $(\mathrm{Ind}_P^G, \mathrm{Ord}_{\overline{P}})$ *is an isomorphism.*

Proof. Lemma 4.7, Cor. 5.3. □

Emerton [Emerton2] showed that the N-coinvariant functor preserves admissibility when the characteristic of F is 0 and R is a complete artinian local $\mathbb{Z}_p$-algebra having finite residual field, using the derived functors of the ordinary part functor Ord_P. This method does not apply when the characteristic of F is p.

It is not difficult to extend the classical proof for complex representations showing that the N-coinvariant functor preserves admissibility to **any** $(F, \mathbf{G}, R)$ with a commutative noetherian ring R where p is invertible, but this is not in the litterature. We end this article by a proof of this result.

Proposition 8.5. *We suppose that* R *is a commutative noetherian ring where* p *is invertible. Then the* N*-coinvariant functor* $V \mapsto V_N : \mathrm{Mod}_R^{\infty}(G) \to \mathrm{Mod}_R^{\infty}(M)$ *preserves admissibility.*

Proof. Let $V \in \mathrm{Mod}_R^{\infty}(G)$, $\kappa : V \to V_N$ the natural map, $a \in \mathbb{N}$, and let $v \in V^{N_0}$. Let $r \in \mathbb{N}$. For $m \in M, v \in V$, we have $m\kappa(v) = \kappa(mv)$ hence $\kappa(V^{M_r N_0}) \subset (V_N)^{M_r}$. Conversely let $w \in (V_N)^{M_r}$ and $v \in V$ with $\kappa(v) = w$. The fixator H_r of v in the pro-p-group $M_r N_0$ is open. As p is invertible in R, the element $[M_r N_0 : H_r]^{-1} \sum_{b \in M_r N_0/H_r} bv$ is well defined and fixed by $M_r N_0$ and its image in V_N is equal to w. Hence

$$\kappa(V^{M_r N_0}) = (V_N)^{M_r}. \tag{13}$$

As $V^{N_0} = \cup_{r\in\mathbb{N}} V^{M_r N_0}$ and V_N is a smooth representation of M, we get $\kappa(V^{N_0}) = V_N$. From (11), (13), we have

$$(V_N)^{M_r} = \cup_{t\geq r}\kappa(V_0^{\overline{N}_t M_r N_0}). \tag{14}$$

Applying κ to

$$h_{N_0,z^a}(v) = \sum_{u\in N_0/z^a N_0 z^{-a}} u z^a v = z^a \sum_{u\in z^{-a}N_0 z^a/N_0} uv$$

we obtain

$$\kappa(h_{N_0,z^a}(v)) = [N_0 : z^a N_0 z^{-a}] z^a \kappa(v).$$

The index $[N_0 : z^a N_0 z^{-a}]$ is a power of p which goes to infinity with a. As p is invertible in R, the $R[z,z^{-1}]$-module generated by $\kappa(v)$ is equal to the $R[z,z^{-1}]$-module generated by $\kappa(h_{N_0,z^a}v)$ for $a\in\mathbb{N}$. We apply this to $a\geq n(r,t)$ to obtain

$$\kappa(V^{\overline{N}_t M_r N_0}) = \kappa(h_{N_0,z^a}(V^{\overline{N}_t M_r N_0})) \subset \kappa(V^{\overline{N}_r M_r N_0}).$$

because $h_{N_0,z^a}(V^{\overline{N}_t M_r N_0}) \subset V^{\overline{N}_r M_r N_0}$ (Part 1) in the proof of Prop. 8.2). This is valid for all $t\geq r$. With (14), we get

$$(V_N)^{M_r} = \kappa(V^{\overline{N}_r M_r N_0}). \tag{15}$$

If $V\in \mathrm{Mod}_R^{adm}(G)$, the R-module $V^{\overline{N}_r M_r N_0}$ is finitely generated, and the same is true for $(V_N)^{M_r}$ by (15). □

Remark 8.6. When a power of p vanishes in R, there is a large integer $a\in\mathbb{N}$ such that $h_{N_0,z^a}(V^{N_0})$ is contained in the kernel of κ.

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