The pro-p-Iwahori Hecke algebra of a reductive p-adic group I

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Abstract

Let R be a commutative ring, let F be a locally compact non-archimedean field of finite residual field k of characteristic p, and let $\mathbf{G}$ be a connected reductive F-group. We show that the pro-p-Iwahori Hecke R-algebra of $G = \mathbf{G}(F)$ admits a presentation similar to the Iwahori-Matsumoto presentation of the Iwahori Hecke algebra of a Chevalley group, and alcove walk bases satisfying Bernstein relations. This was known only for a F-split group $\mathbf{G}$.

Contents

1 Introduction

This paper extends to a general reductive p-adic group G the description of the pro-p-Iwahori Hecke algebra over any commutative ring R, that I gave 10 years ago for a split group G. This is a basic work which allows to describe the center and to prove finiteness results of the pro-p-Iwahori Hecke algebra for any (G, R). It is a fundamental tool for the theory of the representations of G over a field C of characteristic p : the inverse Satake isomorphism for spherical Hecke algebras, the classification of the supersingular simple modules of the pro-p-Iwahori Hecke C-algebra, and the classification of the irreducible admissible smooth C-representations of G in term of parabolic induction and irreducible supersingular representations of the Levi subgroups.

The study of congruences between classical modular forms naturally leads to representations over arbitrary commutative rings R, rather than to complex representations. In our local setting, that means studying R-modules with a smooth action of $G = \mathbf{G}(F)$ where F is a locally compact non-archimedean field of finite residue field k, and $\mathbf{G}$ is a connected reductive group over F.

When C is an algebraically closed field of characteristic equal to the characteristic p of k, very little is known about the theory of the smooth C-representations of G, besides the basic property that a non-zero representation has a non-zero vector invariant by a pro-p-Iwahori subgroup. The pro-p-Iwahori subgroups of G are the analogues of the p-Sylow subgroups of a finite group and the study of the smooth C-representations of G involves naturally the pro-p-Iwahori Hecke C-algebra $\mathcal{H}_C(1)$ of G. This is our motivation to study the pro-p-Iwahori Hecke algebra of G.

For **any** triple $(R, F, \mathbf{G})$, we show that the pro-p-Iwahori Hecke R-algebra $\mathcal{H}_R(1)$ of G admits a presentation, generalizing the Iwahori and Matsumoto presentation of the Iwahori Hecke R-algebra of a Chevalley group. The proof of the quadratic relations is done by reduction to the analog Hecke R-algebra of a finite reductive group. The Iwahori Hecke R-algebra $\mathcal{H}_R$ of G is a quotient of $\mathcal{H}_R(1)$ and all our results transfer to analogous and simpler results for the Iwahori Hecke R-algebra.

The Iwahori-Matsumoto presentation of the pro-p-Iwahori Hecke R-algebra of G leads naturally to the definition of R-algebras $\mathcal{H}_R(q_s, c_s)$ associated to a group $W(1)$ and parameters (q_s, c_s) satisfying simple conditions. The group $W(1)$ is an extension by a commutative group Z_k of an extended affine Weyl group W attached to a reduced root system Σ. The group W is more general than the group appearing in the Lusztig affine Hecke algebras $\mathcal{H}_R(q_s, q_s - 1)$. The R-algebra $\mathcal{H}_R(q_s, c_s)$ is a free R-module of basis indexed by the elements of $W(1)$ satisfying the braid relations and quadratic relations with coefficients (q_s, c_s).

We show that the algebra $\mathcal{H}_R(q_s, c_s)$ admits, for any Weyl chamber, an alcove walk basis indexed by the elements of $W(1)$, a product formula involving alcove walk bases associated to different Weyl chambers, and Bernstein relations. When the q_s are invertible in R we obtain a presentation of the algebra $\mathcal{H}_R(q_s, c_s)$ generalizing the Bernstein-Lusztig presentation for the Iwahori Hecke algebra of a split group. Our proofs proceed by reduction to the case $q_s = 1$.

We recall that we put no restriction on the triple $(R, F, \mathbf{G})$, the reductive group $\mathbf{G}$ may be not split, the local field may have characteristic p, the commutative ring R may be the ring of integers $\mathbb{Z}$ or a field of characteristic p.

When $\mathbf{G}$ is split, the complex Iwahori Hecke algebra $\mathcal{H}_{\mathbb{C}}$ of G was well understood. It is the affine Hecke algebra attached to a based root datum of G and to the cardinal q of k (the first proof is due to Iwahori and Matsumoto for a Chevalley group). Starting from the Iwahori-Matsumoto presentation of $\mathcal{H}_{\mathbb{C}}$, Bernstein and Lusztig gave another presentation of $\mathcal{H}_{\mathbb{C}}$, from which one can recover the center of $\mathcal{H}_{\mathbb{C}}$ and which is an essential step for the classification of its simple modules. The classification was done for $\mathbf{G} = GL(n)$ by

Zelevinski and Rogawski, and for $\mathbf{G}$ simple, simply connected with a connected center by Kazhdan-Lusztig, and Ginzburg, using equivariant K-theory of the variety of Steinberg triples. The condition "simply connected" was waived by Reeder. Görtz realized that the Bernstein basis could be understood using Ram's alcove walks, and gave a simpler proof of the Bernstein presentation of an affine Hecke complex algebra of a based root datum with unequal invertible parameters. When $\mathbf{G}$ is split but for any pair (R, F), I had shown that the pro-p-Iwahori Hecke algebra of G admits an Iwahori-Matsumoto presentation and an integral Bernstein basis, using Haines minimal expressions. A student Nicolas Schmidt of Grosse-Kloenne, in his unpublished diplomarbeit, defined the alcove walk basis, proved the product formula, and studied the Bernstein relations for algebras $\mathcal{H}_R(q_s, c_s)$ containing the algebras arising from a split $\mathbf{G}$, but not all those arising from a general $\mathbf{G}$.

For the field of complex numbers $\mathbb{C}$, the pro-p-Iwahori-invariant functor is an equivalence of categories from the $\mathbb{C}$-representations of G generated by their vectors invariant by a pro-p-Iwahori subgroup onto the category of right $\mathcal{H}_{\mathbb{C}}(1)$-modules. When $\mathbb{C}$ is replaced by an algebraically closed field C of characteristic p, this does not remain true : the functor does not always send irreducible representations onto simple modules. However the pro-p-Iwahori Hecke algebra $\mathcal{H}_C(1)$ appears constantly in the theory of smooth C-representations of G. The most striking example is the following:

For all integers $n > 2$, there exists a numerical Langlands correspondance for the pro-p-Iwahori-Hecke C-algebra $\mathcal{H}_{\mathbb{C}}(1)$ of $GL(n, F)$: a bijection between the simple supersingular $\mathcal{H}_{\mathbb{C}}(1)$-modules of dimension n and the dimension n irreducible continuous C-representations of the Galois group $\mathrm{Gal}(F^s/F)$ of the separable closure F^s of F.

In a forthcoming work with Abe, Henniart and Herzig [AHHV], we classify the irreducible admissible C-representations of G which are not supercuspidal, in term of the irreducible admissible supercuspidal (= supersingular) representations of the Levi subgroups and of parabolic induction. The Bernstein relations in the pro-p-Iwahori Hecke C-algebra $\mathcal{H}_C(1)$ of G, which is isomorphic to a C-algebra $\mathcal{H}_C(0, c_s)$ with parameters $q_s = 0$ is one of the main ingredients of the classification.

In a sequence to this paper, we describe the center of $\mathcal{H}_R(1)$, the inverse Satake isomorphisms and we give the classification of the simple supersingular $\mathcal{H}_C(1)$-modules, extending the work of myself and of Rachel Ollivier done only for split groups $\mathbf{G}$.

I had many conversations with Noriyuki Abe on the Bernstein relations and on their relations with the change of weight, during his stay at the "Institut de mathematiques de Jussieu" in 2013. The unpublished diplomarbeit of Nicolas Schmidt was very helpful for writing this article. I am very thankful to an excellent referee and Florian Herzig for their careful reading.

2 Main Results

2.1 Iwahori-Masumoto presentation.

Let $\mathbf{G}$ be a connected reductive group over a local non-archimedean field F of finite residue field k of characteristic p with q elements, and let R be a commutative ring. We fix an Iwahori subgroup I of G. Its pro-p-radical $I(1)$ is called a pro-p-Iwahori group. It is the unique pro-p-Sylow subgroup of I, and of every parahoric subgroup containing I. The pro-p-Iwahori subgroups of G are all conjugate. The Iwahori Hecke ring

$$\mathcal{H} = \mathbb{Z}[I \backslash G / I]$$

with the convolution product, is isomorphic to the ring of intertwiners $\mathrm{End}_{\mathbb{Z}[G]} \mathbb{Z}[I \backslash G]$ of the regular right representation $\mathbb{Z}[I \backslash G]$ of G associated to I. The Iwahori Hecke R-algebra obtained by base change

$$\mathcal{H}_R = R \otimes_{\mathbb{Z}} \mathcal{H} = R[I\backslash G/I]. \tag{1}$$

is isomorphic to the R-algebra of intertwiners $\mathrm{End}_{R[G]} R[I\backslash G]$. We replace I by $I(1)$ and define in the same way the pro-p-Iwahori Hecke ring $\mathcal{H}(1) = \mathbb{Z}[I(1)\backslash G/I(1)]$ and the pro-p-Iwahori Hecke R-algebra $\mathcal{H}_R(1)$.

The sets $I\backslash G/I$ and $I(1)\backslash G/I(1)$ have a natural group structure, isomorphic to the Iwahori Weyl group W and the pro-p-Iwahori Weyl group $W(1)$ defined as follows.

The Iwahori group I is the parahoric subgroup of G fixing an alcove $\mathfrak{C}$ in the building of the adjoint group of G. To define W and $W(1)$ we choose an apartment $\mathfrak{A}$ containing $\mathfrak{C}$. The apartment $\mathfrak{A}$ is associated to a maximal F-split subtorus $\mathbf{T}$ of $\mathbf{G}$. Let $\mathbf{Z}$ and $\mathbf{N}$ denote the centralizer and the normalizer of $\mathbf{T}$ in $\mathbf{G}$, and $Z := \mathbf{Z}(F), N := \mathbf{N}(F)$ their F-rational points. Then $W = N/N \cap I$ and $W(1) = N/N \cap I(1)$. We check that these maximal split tori are conjugate by I. The same is true for their normalizers and for the corresponding groups W and $W(1)$.

The apartment $\mathfrak{A}$ is a finite dimensional affine euclidean real space with a locally finite set $\mathfrak{H}$ of hyperplanes, such that the orthogonal reflections with respect to $H \in \mathfrak{H}$ generate an affine Weyl group $W(\mathfrak{H})$, and $\mathfrak{C}$ is a connected component of $\mathfrak{A} - \cup_{H\in\mathfrak{H}} H$. The group N acts on $\mathfrak{A}$ by affine automorphisms respecting $\mathfrak{H}$ and its subgroup Z acts by translations.

The parahoric subgroups of G generate a subgroup G^{aff}, which is also the kernel of the Kottwitz morphism κ_G, and G is generated by $Z \cup G^{aff}$. The maximal compact subgroup $\tilde{Z}_0$ of Z acts trivially on $\mathfrak{A}$ and contains the unique parahoric subgroup Z_0 of Z, of unique pro-p-Sylow subgroup $Z_0(1)$. The group Z_o is the kernel of the Kottwitz morphism κ_Z of Z and the quotient $Z_k = Z_0/Z_0(1)$ is the group of rational points of a k-torus. We have

$$Z \cap I = Z_0, \quad Z \cap I(1) = Z_0(1), \quad I = I(1)Z_0. \tag{2}$$

The action of $N^{aff} = N \cap G^{aff}$ on the apartment $\mathfrak{A}$ induces an isomorphism

$$W^{aff} = N^{aff}/Z_0 \to W(\mathfrak{H}). \tag{3}$$

The groups $Z_0(1) \subset Z_0$ are normalized by N and the maps $n \mapsto InI,\ n \mapsto I(1)nI(1)$ induce bijections

$$W = N/Z_0 \to I\backslash G/I, \quad W(1) = N/Z_0(1) \to I(1)\backslash G/I(1). \tag{4}$$

The pro-p-Iwahori-Weyl group $W(1)$ is an extension of the Iwahori-Weyl group W by $Z_0/Z_0(1) \simeq I/I(1)$

$$1 \to Z_k \to W(1) \to W \to 1. \tag{5}$$

The extension does not split in general (see [VigMA]). For a subset $X \subset W$, we denote by $X(1)$ the inverse image of X in $W(1)$. For an element $w \in W$, we denote by $\tilde{w} \in W(1)$ an element lifting w (hence $\tilde{w} \in w(1)$).

For $n \in N$, the double coset InI depends only on the image $w \in W$ of n and the corresponding intertwiner in the Iwahori Hecke ring $\mathcal{H}$ is denoted by T_w. Thus $(T_w)_{w\in W}$ is a natural basis of $\mathcal{H}$. We do the same for $\mathcal{H}(1)$ and $W(1)$. The relations satisfied by the products of the basis elements follow from the fact that W is a semi-direct product of the affine Weyl group W^{aff} by the image Ω in W of the N-normalizer of $\mathfrak{C}$,

$$W = W^{aff} \rtimes \Omega. \tag{6}$$

The group Ω identifies with the image of the Kottwitz morphism κ_G. Let $S^{aff} \subset W^{aff}$ be the set of orthogonal reflections with respect to the walls of $\mathfrak{C}$, using the isomorphism (3). The length ℓ of the Coxeter system (W^{aff}, S^{aff}) inflates to a length of W constant on the double cosets modulo Ω, and to a length of $W(1)$ constant on the double cosets modulo the inverse image $\Omega(1)$ of Ω. The Bruhat order of W^{aff} inflates to $W(1)$ and to W as in [Vig, Appendix].

For $n \in N$ of image w in W or in $W(1)$, the sets

$$InI/I \simeq I(1)nI(1)/I(1) \tag{7}$$

have the same number q_w of elements. The integer q_w is a power of the cardinal q of the residue field of F. When $s, s' \in S^{aff}$ are conjugate in W (denote $s \sim s'$), $q_s = q_{s'}$ and $q_w = q_{s_1} \dots q_{s_n}$ if $w = s_1 \dots s_n u$ with $s_i \in S^{aff}, u \in \Omega$ is a reduced decomposition.

Theorem 2.1. *The Iwahori Hecke ring $\mathcal{H}$ is the free $\mathbb{Z}$-module with basis $(T_w)_{w \in W}$ endowed with the unique ring structure satisfying*

- *The braid relations: $T_w T_{w'} = T_{ww'}$ if $w, w' \in W,\ \ell(w) + \ell(w') = \ell(ww')$.*
- *The quadratic relations: $T_s^2 = q_s + (q_s - 1)T_s$ if $s \in S^{aff}$.*

The Iwahori Hecke R-algebra $\mathcal{H}_R$ has the same presentation over R by base change (1).

The elements in the basis $(T_w)_{w \in W(1)}$ of $\mathcal{H}(1)$ verify the braid relations, and similar quadratic relations but the coefficient $q_s - 1$ is replaced by an element of $\mathbb{Z}[Z_k]$, that we define now.

Let $s \in S^{aff}$ and let $K_{\mathfrak{F}_s}$ be the parahoric subgroup of G fixing the face of $\mathfrak{C}$ supported on the wall fixed by s, of codimension 1. The quotient of $K_{\mathfrak{F}_s}$ by its pro-p-radical $K_{\mathfrak{F}_s}(1)$ is the group $K_{\mathfrak{F}_s,k}$ of k-points of a finite reductive connected group over k of semi-simple rank 1. Let T_0 be the maximal compact subgroup of the maximal split torus T of G, let $T_0(1)$ be the pro-p-Sylow subgroup of T_0 and let $T_k = T_0/T_0(1)$. The group T_k is a maximal split torus of $K_{\mathfrak{F}_s,k}$ of centralizer Z_k, and the root system $\Phi_{\mathfrak{F}_s}$ of $K_{\mathfrak{F}_s,k}$ with respect to T_k is contained in the root system Φ of G with respect to T. We denote by $N_{s,k}$ (the group of k-points of) the normalizer of T_k in $K_{\mathfrak{F}_s,k}$, by $U_{\alpha_s,k}$ the root subgroup associated to a reduced root $\alpha_s \in \Phi_{\mathfrak{F}_s}$ (we have $U_{2\alpha_s,k} \subset U_{\alpha_s,k}$ if $2\alpha_s \in \Phi_{\mathfrak{F}_s,k}$), by $K'_{\mathfrak{F}_s,k}$ the group generated by $U_{\alpha_s,k}$ and $U_{-\alpha_s,k}$, by $Z_{k,s}$ the intersection $Z_k \cap K'_{\mathfrak{F}_s,k}$, and by c_s the element defined by the formula:

$$c_s = (q_s - 1)|Z_{k,s}|^{-1} \sum_{t \in Z_{k,s}} T_t. \tag{8}$$

The group $U_{\alpha_s,k}$ is a p-Sylow subgroup of $K_{\mathfrak{F}_s,k}$, the integer q_s is the number of elements of $U_{\alpha_s,k}$ and $(q_s-1)|Z_{k,s}|^{-1}$ is an integer. For $u_k \in U^*_{\alpha_s,k} = U_{\alpha_s,k} - \{1\}$, the intersection

$$U_{-\alpha_s,k} u_k U_{-\alpha_s,k} \cap N_{s,k} = \{m_{\alpha_s}(u_k)\}$$

consists of a single element. The image of the map $m_{\alpha_s} : U^*_{\alpha_s,k} \to N_{s,k}$ is the coset $m_{\alpha_s}(u_k) Z_{k,s} = Z_{k,s} m_{\alpha_s}(u_k)$. The square $m_{\alpha_s}(u_k)^2$ is an element of $Z_{k,s}$. We choose an arbitrary element $u_k \in U^*_{\alpha_s,k}$ and we denote by $\tilde{s}$ the image of $m_{\alpha_s}(u_k)$ in $W(1)$ (as in section 4.2). Such a lift $\tilde{s}$ of s is called admissible. The quadratic relation of $T_{\tilde{s}}$ in $\mathcal{H}(1)$ is the same as the quadratic relation of $T_{\tilde{s}}$ in the finite Hecke algebra $\mathcal{H}(K_{\mathfrak{F}_s,k}, U_{\alpha_s,k})$. The quadratic relation in $\mathcal{H}_R(K_{\mathfrak{F}_s,k}, U_{\alpha_s,k})$ when R is a large field of characteristic p was computed by Cabanes and Enguehard [CE, Prop. 6.8].

Theorem 2.2. *The pro-p-Iwahori Hecke ring $\mathcal{H}(1)$ is the free $\mathbb{Z}$-module with basis $(T_w)_{w \in W(1)}$ endowed with the unique ring structure satisfying*

- *The braid relations:* $T_w T_{w'} = T_{ww'}$ *if* $w, w' \in W(1)$, $\ell(w) + \ell(w') = \ell(ww')$.
- *The quadratic relations:* $T_{\tilde{s}}^2 = q_s T_{\tilde{s}^2} + c_{\tilde{s}} T_{\tilde{s}}$ for $s \in S^{aff}$,

where $c_{\tilde{s}} = c_s$ *if the order of* $Z_{k,s}$ *is* $q_s - 1$ *(for example if* $\mathbf{G}$ *is* F*-split), and in general there are positive integers* $c_{\tilde{s}}(t) = c_{\tilde{s}}(t^{-1})$ *for* $t \in Z_{k,s}$*, constant on the coset* $t\{xs(x)^{-1} \mid x \in Z_k\}$*, of sum* $\sum_{t \in Z_{k,s}} c_{\tilde{s}}(t) = q_s - 1$ *such that* $c_{\tilde{s}} = \sum_{t \in Z_{k,s}} c_{\tilde{s}}(t) T_t$ *and*

$$c_{\tilde{s}} \equiv c_s \quad \text{modulo } p.$$

The $\mathbb{Z}$-module of basis $(T_w)_{w \in \Omega(1)}$ is a subalgebra of $\mathcal{H}(1)$ isomorphic to the group algebra $\mathbb{Z}[\Omega(1)]$ by the braid relations. The $\mathbb{Z}$-module of basis $(T_w)_{w \in W^{aff}(1)}$ for the inverse image $W^{aff}(1)$ of W^{aff} in $W(1)$, is a subalgebra $\mathcal{H}^{aff}(1)$ and $\mathcal{H}(1)$ is isomorphic to the twisted product

$$\mathcal{H}(1) \simeq \mathcal{H}^{aff}(1) \otimes_{\mathbb{Z}[Z_k]} \mathbb{Z}[\Omega(1)].$$

The pro-p-Iwahori Hecke R-algebra $\mathcal{H}_R(1)$ has the same presentation by base change. Theorems 2.1 and 2.2 imply (this can also proved directly):

Corollary 2.3. *The surjective* R*-linear map* $\mathcal{H}_R(1) \to \mathcal{H}_R$ *:*

$$T_{\tilde{w}} \mapsto T_w \quad \text{for } \tilde{w} \in W(1) \text{ of image } w \in W,$$

is an R*-algebra homomorphism.*

The properties of the pro-p-Iwahori Hecke R-algebra $\mathcal{H}_R(1)$ are transported to the Iwahori Hecke R-algebra $\mathcal{H}_R$ via this surjective R-algebra homomorphism.

We describe conditions on elements $(q_s, c_s) \in R \times R[Z_k]$ for $s \in S^{aff}(1)$, the inverse image of S^{aff} in $W(1)$, implying the existence of an R-algebra $\mathcal{H}_R(q_s, c_s)$ of basis $(T_w)_{w \in W(1)}$ satisfying braid and quadratic relations as in Thm. 2.2. We write $c_s = \sum_{t \in Z_k} c_s(t) t$ with $c_s(t) \in R$.

Theorem 2.4. *Let* $(q_s, c_s) \in R \times R[Z_k]$ *for* $s \in S^{aff}(1)$*. The following property A implies the property B :*

A *For all* $t \in Z_k, s \in S^{aff}(1), w \in W(1)$ *such that* $wsw^{-1} \in S^{aff}(1)$,

1) $q_s = q_{st} = q_{wsw^{-1}}$.

2) $c_{st} = c_s t$, *and* $c_{wsw^{-1}} = \sum_{t \in Z_k} c_s(t) wtw^{-1}$.

B *The free* R*-module* $\mathcal{H}_R(q_s, c_s)$ *of basis* $(T_w)_{w \in W(1)}$ *has a unique* R*-algebra structure satisfying*

- *The braid relations:* $T_w T_{w'} = T_{ww'}$ *if* $w, w' \in W(1)$, $\ell(w) + \ell(w') = \ell(ww')$.
- *The quadratic relations:* $T_s^2 = q_s s^2 + c_s T_s$ for $s \in S^{aff}(1)$.

In **B** the braid relations imply that the group algebra $R[Z_k]$ embeds in $\mathcal{H}_R(q_s, c_s)$ by the linear map sending $t \in Z_k$ to T_t.

If the q_s are not zero divisors in R, the properties A and B are equivalent. The maps $s \mapsto q_s$ from $S^{aff}(1)$ to R satisfying A1) are naturally in bijection with the maps from $S^{aff}/\sim$ to R.

For indeterminates $(\mathbf{q}_s)_{s \in S^{aff}/\sim}$, we have the generic $R[(\mathbf{q}_s)]$-algebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$. The R-algebra $\mathcal{H}_R(q_s, c_s)$ is a specialisation of the generic algebra.

Proposition 2.5. *Let* $(\mathfrak{q}_s)_{s \in S^{aff}/\sim}$ *be indeterminates,* $\mathbf{q}_s = \mathfrak{q}_s^2$*, and let* $(c_s)_{s \in S^{aff}(1)}$ *be elements of* R *satisfying A.2).*

The $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$*-algebra* $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1} c_s)$ *is isomorphic to* $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$.

The generic algebra is a $R[(\mathbf{q}_s)]$-subalgebra of $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$. Different properties of the R-algebra $\mathcal{H}_R(q_s, c_s)$, hence of the pro-p-Iwahori Hecke algebra, are proved by reduction to the simpler case $q_s = 1$ for all $s \in S^{aff}/\sim$, using this proposition.

Remark 2.6. The presentation of $\mathcal{H}$ (Thm. 2.1) generalizing the Iwahori-Matsumoto presentation for a Chevalley group [IM] cannot be found in the litterature for a general reductive group $\mathbf{G}$ but follows from different results of Bruhat and Tits [BT2, 5.2.12 Prop. (i) and (ii)] and exercises in Bourbaki [Bki, Ch. IV, §2, Ex. 8, 22, 23, 24, 25]. Borel [Borel] considered a semi-simple group $\mathbf{G}$ and a "non-connected" Iwahori subgroup $\tilde{I} = I\tilde{Z}_0$. When $\mathbf{G}$ is F-quasisplit and $Z_0 = \tilde{Z}_0$, Z is a torus, $I = \tilde{I}$, then $\mathcal{H}$ is a Lusztig affine Hecke algebra [Lusztig] attached to a based root datum of G and to a system of unequal parameters (q_s).

It is possible to compute the quadratic relations using the Bruhat-Tits theory without reduction to finite reductive groups (when $\mathbf{G}$ is F-split [VigMA].)

2.2 Alcove walk bases and Bernstein presentation

We chose an alcove $\mathfrak{C}$ to define I and an apartment $\mathfrak{A}$ containing $\mathfrak{C}$. To define a new basis of the algebra $\mathcal{H}_R(q_s, c_s)$, we choose first a special vertex x_0 of $\mathfrak{C}$. The special vertices of $\mathfrak{C}$ may be not conjugate by an element of G. The orthogonal reflections with respect of the walls containing x_0 generate a group isomorphic to W_0. The Iwahori-Weyl group W is the semi-direct product

$$W = \Lambda \rtimes W_0 \quad \text{where } \Lambda = Z/Z_0. \tag{9}$$

This implies

$$W(1) = \Lambda(1)W_0(1) \quad \text{where } \Lambda(1) = Z/Z_0(1),\ \Lambda(1) \cap W_0(1) = Z_k.$$

The new basis is related to this decomposition.

We identify the apartment $\mathfrak{A}$ with an euclidean real vector space V, the vertex x_0 becoming the null vector 0 of V. We recall the natural bijections between:

the (open) Weyl chambers of $\mathfrak{A}$ of vertex x_0,

the alcoves of $\mathfrak{A}$ of vertex x_0,

the bases of the root system Φ of $\mathbf{T}$ in $\mathbf{G}$,

the (spherical) orientations of $(\mathfrak{A}, \mathfrak{H})$ (see section 5.2).

A Weyl chamber $\mathfrak{D}$ of V contains a unique alcove $\mathfrak{C}_{\mathfrak{D}}$ of vertex 0, the basis $\Delta_{\mathfrak{D}}$ of Φ consists of the reduced roots α positive on $\mathfrak{D}$ such that $\operatorname{Ker}\alpha$ is a wall of $\mathfrak{C}_{\mathfrak{D}}$. The orientation $o_{\mathfrak{D}}$ is such that the $o_{\mathfrak{D}}$-positive side of an hyperplane $H \in \mathfrak{H}$ is the set of $x \in V$ with $\alpha(x) + r > 0$ where $H = \operatorname{Ker}(\alpha + r)$ for $\alpha \in \Phi$ positive on $\mathfrak{D}$.

The simply transitive action of W_0 on the Weyl chambers of V inflates to an action of W and an action of $W(1)$ on the spherical orientations o of $(V, \mathfrak{H})$, trivial on $\Lambda(1)$. We denote $o \bullet w = w^{-1}(o)$. If $\mathfrak{D}_o$ denotes the Weyl chamber defining the orientation o, then $w^{-1}(\mathfrak{D}_o) = \mathfrak{D}_{o\bullet w}$.

To define the new basis, we choose an orientation o of $(\mathfrak{A}, \mathfrak{H})$. For a pair $(w, s) \in W^{aff} \times S^{aff}$ such that $\ell(ws) = \ell(w) + 1$ we set

$$\epsilon_o(w, s) = 1 \quad \text{if } w(\mathfrak{C}) \text{ is contained in the } o\text{-negative side of } w(H_s),$$

where H_s is the affine hyperplane of V fixed by s. Otherwise we set $\epsilon_o(w, s) = -1$. When we walk from $w(\mathfrak{C})$ to $ws(\mathfrak{C})$ we cross the hyperplane H_s in the $\epsilon_o(w, s)$-direction: positive direction if $\epsilon_o(w, s) = 1$ and negative direction if $\epsilon_o(w, s) = -1$.

For $w \in W^{aff}$ of reduced decomposition $w = s_1 \ldots s_r$, $s_i \in S^{aff}, r = \ell(w)$, walking in a minimal gallery of alcoves $\mathfrak{C}, s_1(\mathfrak{C}), s_1s_2(\mathfrak{C}), \ldots, w(\mathfrak{C})$, we cross the hyperplanes $H_{s_1}, s_1(H_{s_2}), \ldots, s_1 \ldots s_{r-1}(H_{s_r})$ in the $\epsilon_o(1, s_1), \epsilon_o(s_1, s_2), \ldots, \epsilon_o(s_1 \ldots s_{r-1}, s_r)$-directions.

For (w,s) in $W(1)\times S^{aff}(1)$ with $\ell(ws)=\ell(w)+1$, lifting an element $(w^{aff}u,\overline{s})$ in $W\times S^{aff}$, with $w^{aff}\in W^{aff}, u\in\Omega, \overline{s}\in S^{aff}$, we write

$$\epsilon_o(w,s):=\epsilon_o(w^{aff},\overline{s})$$
$$T_s^{\epsilon_o(w,s)}:=T_s \text{ if } \epsilon_o(w,s)=1 \text{ and } T_s^{\epsilon_o(w,s)}:=T_s-c_s \text{ if } \epsilon_o(w,s)=-1.$$

We recall the quadratic relation $(T_s-c_s)T_s=q_s s^2$; it is easy to check that c_s and T_s commute.

Theorem 2.7. *Let o be a spherical orientation and let $w=s_1\dots s_r u$ with $u\in\Omega(1)$ and $s_i\in S^{aff}(1)$ for $1\leq i\leq r=\ell(w)$. The element of $\mathcal{H}_R(q_s,c_s)$ defined by*

(10) $$E_o(w)=T_{s_1}^{\epsilon_o(1,s_1)}T_{s_2}^{\epsilon_o(s_1,s_2)}\dots T_{s_r}^{\epsilon_o(s_1\dots s_{r-1},s_r)}T_u$$

does not depend on the choice of the reduced decomposition of w, satisfies

(11) $$E_o(w)-T_w\in\oplus_{w'<w}\mathbb{Z}T_{w'}.$$

and we have the product formula, for $w,w'\in W(1)$,

(12) $$E_o(w)E_{o\bullet w}(w')=q_{w,w'}E_o(ww'),\quad q_{w,w'}=(q_w q_{w'}q_{ww'}^{-1})^{1/2}.$$

The theorem, proved by reduction to $q_s=1$, implies that $(E_o(w))_{w\in W(1)}$ is a basis of $\mathcal{H}_R(q_s,c_s)$.

Corollary 2.8. *The R-module of basis $(E_o(\lambda))_{\lambda\in\Lambda(1)}$ is a subalgebra $\mathfrak{A}_o(1)$ of $\mathcal{H}_R(q_s,c_s)$ with product*

$$E_o(\lambda)E_o(\lambda')=q_{\lambda,\lambda'}E_o(\lambda\lambda')\quad\text{for } \lambda,\lambda'\in\Lambda(1).$$

The Bernstein relations that we are going to present allow us to give a presentation of $\mathcal{H}_R(q_s,c_s)$, starting from the basis $(E_o(w))_{w\in W(1)}$ when the $(q_s)_{s\in S^{aff}/\sim}$ are invertible in R. But the Bernstein relations exist without any conditions on $(q_s)_{s\in S^{aff}/\sim}$ and their many applications will be developed in the sequel of this article.

Let $\nu:\Lambda(1)\to V$ be the homomorphism such that $\lambda\in\Lambda(1)$ act on V by translation by $\nu(\lambda)$, and for $\alpha\in\Phi$ let e_α be the positive integer such that the set $\{e_\alpha\alpha\mid\alpha\in\Phi\}$ is the reduced root system Σ defining W^{aff}. A root $\beta\in\Sigma$ taking positive values on the Weyl chamber $\mathfrak{D}^+$ of vertex x_0 containing the alcove $\mathfrak{C}$ is called positive. For an arbitrary spherical orientation o of Weyl chamber $\mathfrak{D}_o$, let Δ_o be the corresponding basis of the reduced root system Σ (not of Φ), S_o and S be the sets of orthogonal reflections with respect to the walls of $\mathfrak{D}_o$ and of $\mathfrak{D}^+$.

For $s\in(S\cap S_o)(1)$ and $\lambda\in\Lambda(1)$, the Bernstein relations show that

(13) $$E_o(s)(E_{o\bullet s}(\lambda)-E_o(\lambda))=E_o(s\lambda s^{-1})E_o(s)-E_o(s)E_o(\lambda).$$

belongs to $\mathfrak{A}_o(1)$ and give its expansion on the basis $(E_o(\lambda))_{\lambda\in\Lambda(1)}$. The proof proceeds by reduction to $q_s=1$.

For a root β in Σ, $s_\beta\in W_0$ and $\nu(\lambda)$ is fixed by s_β if and only if $\beta\circ\nu(\lambda)=0$. As the translation by $\nu(\lambda)$ stabilizes $\mathfrak{H}$, we have $\beta\circ\nu(\lambda)\in\mathbb{Z}$. When $\beta\circ\nu(\lambda)\neq 0$, we denote its sign by $\epsilon_\beta(\lambda)$.

Theorem 2.9. (Bernstein relation in the generic algebra $\mathcal{H}_{R[\mathbf{q}_s)]}(\mathbf{q}_s,c_s)$)
Let o be a spherical orientation, $s\in(S\cap S_o)(1)$, $\beta\in\Delta$ such that $s\in s_\beta(1)$, and let $\lambda\in\Lambda(1)$.

When $\beta\circ\nu(\lambda)=0$, we have $E_{o\bullet s}(\lambda)=E_o(\lambda)$.

When $\beta\circ\nu(\lambda)\neq 0$, we have

$$E_o(s)(E_{o\bullet s}(\lambda)-E_o(\lambda))=\epsilon_\beta(\lambda)\epsilon_o(1,s)\sum_{k=0}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k,\lambda)c(k,\lambda)E_o(\mu(k,\lambda)),$$

where $c(k,\lambda)\in\mathbb{Z}[Z_k]$, $\mu(k,\lambda)\in\Lambda(1)$, $(\beta\circ\nu)(\mu(k,\lambda))=2k-|\beta\circ\nu(\lambda)|$,

$$\mathbf{q}(k,\lambda)=\prod_{s\in S^{aff}/\sim}\mathbf{q}_s^{m_{k,\lambda}(s)},\quad m_{k,\lambda}(s)\in\mathbb{N},\quad \sum_s m_{k,\lambda}(s)=\ell(\lambda)-\ell(\mu(k)).$$

The values of $\mathbf{q}(k,\lambda),c(k,\lambda)$ and $\mu(k,\lambda)$ are explicit (Cor. 5.43) and depend on s but not on o. They are simpler when the image of $\beta\circ\nu$ is $\mathbb{Z}$ (the other possibility is $2\mathbb{Z}$). When $\beta\circ\nu(\lambda)\neq 0$, moving the term indexed by $k=0$ from the right hand side to the left hand side, the Bernstein relation becomes:

$$E_o(s\lambda)-E_{o\bullet s}(s\lambda)=\epsilon_{o\bullet s}(1,s)\sum_{k=1}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k,\lambda)\mathbf{q}_s^{-1}c(k,\lambda)E_o(\mu(k,\lambda))\quad\text{if } \ell(s\lambda)<\ell(\lambda),$$

$$E_o(s\lambda)-E_o(s)E_o(\lambda)=\epsilon_o(1,s)\sum_{k=1}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k,\lambda)c(k,\lambda)E_o(\mu(k,\lambda))\quad\text{if } \ell(s\lambda)>\ell(\lambda).$$

The right hand side is 0 when $|\beta\circ\nu(\lambda)|=1$. Otherwise, we prove that
$\mathbf{q}(k,\lambda)\neq 1$ for the integers $0<k<\beta\circ\nu(\lambda)$.
When $\ell(s\lambda)<\ell(\lambda)$, the term $\mathbf{q}(k,\lambda)\mathbf{q}_s^{-1}$ is a product of $\mathbf{q}_s$ for $s\in S^{aff}/\sim$. We prove that $\mathbf{q}(k,\lambda)\mathbf{q}_s^{-1}\neq 1$ for $1<k<|\beta\circ\nu(\lambda)|-1$.

We obtain a presentation of the generic algebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s,c_s)$, and by specialisation a presentation of the R-algebra $\mathcal{H}_R(q_s,c_s)$ when the $(q_s)_{s\in S^{aff}/\sim}$ are invertible.

We choose the orientation o associated to the antidominant Weyl chamber $-\mathfrak{D}^+$. We have $S=S_o$ and $E_o(s)=T_s,\epsilon_o(1,s)=1$ for $s\in S(1)$. We denote by $\Lambda^s(1)$ the set of $\lambda\in\Lambda(1)$ such that $\nu(\lambda)$ is fixed by s. We set $E=E_o$.

Theorem 2.10. (Bernstein presentation of the generic algebra)
The $R[(\mathbf{q}_s)]$-algebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s,c_s)$ is isomorphic to the free $R[(\mathbf{q}_s)]$-module of basis $(E(w))_{w\in W(1)}$ endowed with the unique $R[(\mathbf{q}_s)]$-algebra structure satisfying:
- *Braid relations: $E(w)E(w')=E(ww')$ for $w,w'\in W_0(1)$, $\ell(w)+\ell(w')=\ell(ww')$.*
- *Quadratic relations: $E(s)^2=\mathbf{q}_s s^2+c_sE(s)$ for $s\in S(1)$.*
- *Product: $E(\lambda)E(w)=\mathbf{q}_{\lambda,w}E(\lambda w)$ for $\lambda\in\Lambda(1),w\in W(1)$.*
- *Bernstein relations : $E(s\lambda s^{-1})E(s)-E(s)E(\lambda)$*
$=0$ for (s,λ) in $S(1)\times\Lambda^s(1)$,
$=\epsilon_\beta(\lambda)\sum_{k=0}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k,\lambda)c(k,\lambda)E(\mu(k,\lambda))$ for (s,λ) in $S(1)\times(\Lambda(1)-\Lambda^s(1))$.

3 Review of Bruhat-Tits's theory

The aim of this chapter is to give precise references for the properties extracted from Bruhat-Tit's theory which will be used in the proofs of our results. The reader familiar with this theory should skip this chapter and proceed directly to chapter 4.

We keep the notations given in chapter 2.

For an algebraic group $\mathbf{H}$ defined over F, we denote $H=\mathbf{H}(F)$. Let $X^*(H)$ and $X_*(H)$ the group of F-characters and F-cocharacters of $\mathbf{H}$.

We insist on the fact that the characteristic of F may be 0 or p, and that the root system $\Phi \subset X^*(T)$ of $\mathbf{G}$ may be not reduced, Φ is the union of its irreducible components [Bki, VI. §1.2]

$$\Phi = \sqcup_{j=1}^{r} \Phi_j. \tag{14}$$

A basis Δ of Φ is the union of basis of Φ_j, $\Delta = \sqcup_{j=1}^{r} \Delta_j$. The set of coroots $\Phi^\vee \subset X_*(T)$ is the union of the sets of coroots of Φ_j, $\Phi^\vee = \sqcup_j \Phi_j^\vee$. The real vector space V generated by $\Phi^\vee$ is a product of the vector spaces V_j generated by $\Phi_j^\vee$,

$$V = V_1 \times \ldots \times V_r. \tag{15}$$

The Weyl group W_0 of Φ is the direct product of the Weyl groups of Φ_j, $W_0 = \prod_j W_{0,j}$. The actions of $W_{0,j}$ on V_j is irreducible, and the decomposition of V is orthogonal for a fixed positive definite bilinear form $(\ ,\ )$ on V invariant by the action of W_0 [Bki, VI §1.2,V §3.7]. For $\alpha \in \Phi$, we have the root group $\mathbf{U}_\alpha$ (containing $\mathbf{U}_{\mathbf{2}\alpha}$ if 2α is a root). We denote by ω the valuation of F normalized by $\omega(F - \{0\}) = \mathbb{Z}$.

The results contained in the 316 pages of [BT1] and [BT2] are valid for the group G, by the fundamental theorem [BT2, 5.1.20, 5.1.23] :

Theorem 3.1. *$(Z, U_\alpha)_{\alpha\in\Phi}$ is a root datum generating G and admitting a discrete valuation $\varphi = (\varphi_\alpha : U_\alpha - \{1\} \to \mathbb{R})_{\alpha\in\Phi}$ compatible with the valuation ω of F.*

The definition of "a root datum generating G" and of "a discrete valuation compatible with ω" is given in [BT1, 6.1.1 and 6.1.2 (8)] and in [BT1, 6.2.1, 6.2.21], [BT2, 5.1.23].

3.1 The element $m_\alpha(u)$ for $u \in U_\alpha^*$

For $\alpha \in \Phi$ and $u \in U_\alpha^* = U_\alpha - \{1\}$, there exists a unique triple $(v'_\alpha(u), m_\alpha(u), v''_\alpha(u))$ in $U_{-\alpha} \times N \times U_{-\alpha}$ such that [BT1, 6.1.2 (2)]:

$$u = v'_\alpha(u) m_\alpha(u) v''_\alpha(u). \tag{16}$$

Remark 3.2. *If $2\alpha \in \Phi$ and $u \in U_{2\alpha}^*$, we have $U_{2\alpha} \subset U_\alpha$ and $m_{2\alpha}(u) = m_\alpha(u)$ by unicity.*

If $\alpha \in \Phi, u \in U_\alpha^*$, then $m_\alpha(u^{-1}) = m_\alpha(u)^{-1}$.

The group $W_0 = N/Z$ identifies with the Weyl group of Φ [BoT, §5]. The image of $m_\alpha(u)$ in W_0 is the reflection s_α defined by α.

Lemma 3.3. *The group N is generated by Z and $\cup_{\alpha\in\Phi} m_\alpha(U_\alpha^*)$.*

Proof. [BT1, 6.1.2 (10), 6.1.3 c)] where can replace M_α by $m_\alpha(U_\alpha^*)$. □

Proposition 3.4. *Let Δ be a basis of Φ. We can choose $u_\alpha \in U_\alpha^*$ for all $\alpha \in \Delta$ such that, when $\alpha \neq \beta$,*

$$m_\alpha(u_\alpha) m_\beta(u_\beta) \ldots = m_\beta(u_\beta) m_\alpha(u_\alpha) \ldots, \tag{17}$$

where the number of factors is the order $n(\alpha, \beta)$ of $s_\alpha s_\beta \in W_0$.

Proof. a) When $\mathbf{G}$ is F-split, semisimple, and simply connected (we recall that $\mathbf{G}$ is connected), we choose a Chevalley system $x_\alpha : \mathbf{G_a} \to \mathbf{U}_\alpha$ for $\alpha \in \Phi$. The elements $u_\alpha = x_\alpha(1)$ for $\alpha \in \Delta$ satisfy the proposition [Steinberg, Lemma 56]. We reduce to this case in two steps.

b) From a) to the split case using a z-extension. We suppose that $\mathbf{G}$ is F-split. There exists a reductive connected F-group $\mathbf{H}$ with a simply-connected derived group $\mathbf{H}^{der}$ which is a central extension of $\mathbf{G}$ by a split F-torus [Milne-Shih, Prop. 3.1, Remark 3.3] when the characteristic of F is 0; their proof is valid in positive characteristic). There exists a maximal F-split subtorus T_H of H of image T in G, and $T_H \cap H^{der}$ is a maximal F-subtorus of H^{der}. The root groups of H^{der} are equal to the root groups in H and identify with the root groups in G by the map $H \to G$ [BorelLAG, Thm. 22.6]. The group H^{der} satisfies the condition of a). The image by the map $H \to G$ of a set of elements in H^{der} satisfying the proposition is a set of elements in G satisfying the proposition.

c) From the split case to the general case. By [BoT, Prop. 7.2 (11)] G contains a split connected subgroup G_{nm} with the same maximal split torus T and the following properties: the system of roots of T in G_{nm} is the subset $\Phi_{nm} \subset \Phi$ of non multipliable roots, the root group in G_{nm} of $\gamma \in \Phi_{nm}$ is $U_{\gamma/2} \cap G_{nm}$ if $\gamma/2 \in \Phi$ and $U_\gamma \cap G_{nm}$ otherwise. A basis Δ of Φ gives a basis $\Delta_{nm} = \{\alpha_{nm} \mid \alpha \in \Delta\}$ of Φ_{nm}, where $\alpha_{nm} = \alpha$ if α is not multipliable and $\alpha_{nm} = 2\alpha$ otherwise. The root subgroup in G_{nm} of α_{nm} is contained in U_α for $\pm\alpha \in \Delta$. The proposition is true for G_{nm} by b). A set of elements in G_{nm} satisfying the proposition is contained in G. Applying Remark 3.2 the proposition is true for G.

□

3.2 The group G'

Recall the decomposition (14) of the root system Φ into its irreducible components Φ_i, let G', resp. G'_i, be the subgroup of G generated by the root groups U_α for $\alpha \in \Phi$, resp. Φ_i, and let $Z' = Z \cap G', Z'_i = Z \cap G'_i$, for $1 \leq i \leq r$, be the intersections of Z with these groups. The subgroups G', G'_i are normal in G, each element of G'_i commutes with each element of G'_j if $i \neq j$, and $G'_i \cap G'_j$ is contained in the center of G'; the center of G' is contained in Z and $G = ZG'$. It is obvious that $(Z', U_\alpha)_{\alpha \in \Phi}$, resp. $(Z'_i, U_\alpha)_{\alpha \in \Phi_i}$, is a root datum generating G', resp. G'_i [BT1, 6.1.5].

Lemma 3.5. *The maps $(\varphi_\alpha)_{\alpha \in \Phi}$ defines on the root datum $(Z', U_\alpha)_{\alpha \in \Phi}$ generating G' a discrete valuation compatible with ω.*

The same statement is true for the maps $(\varphi_\alpha)_{\alpha \in \Phi_i}$ on the root datum $(Z'_i, U_\alpha)_{\alpha \in \Phi_i}$ generating G'_i.

Proof. The conditions (V 1), (V3), (V4), (V5) of the valuation [BT1, 6.2.1] remain obviously satisfied. The condition (V 2) is : for $\alpha \in \Phi$,

$$\text{the value } \varphi_{-\alpha}(u) - \varphi_\alpha(mum^{-1}) \text{ is constant for } u \in U_{-\alpha} - \{1\}$$

if $m \in M_\alpha := \{m \in ZG'_\alpha \mid mU_\alpha m^{-1} \subset U_{-\alpha},\ mU_{-\alpha}m^{-1} \subset U_\alpha\}$ where G'_α is the group generated by $U_\alpha \cup U_{-\alpha}$. The group $M'_\alpha = M_\alpha \cap G'_\alpha$ does not depend on Z. The group Z normalizes U_α and $U_{-\alpha}$ hence $M_\alpha = ZM'_\alpha$. As Z', Z'_i are contained in Z, the condition (V 2) remains satisfied. It is clear that the valuation remains discrete and compatible with ω [BT1, 6.2.21], [BT2, 5.1.23]. □

3.3 The apartment

The existence of the apartment is a consequence of the existence of the discrete valuation $\varphi = (\varphi_\alpha)_{\alpha \in \Phi}$ compatible with ω on the root datum $(Z, U_\alpha)_{\alpha \in \Phi}$ generating G.

Remark 3.6. A valuation is constructed for the classical groups [BT1, Chapter 10] , or using a Chevalley-Steinberg system when **G** is F-quasi-split [BT2, 4.1.3, 4.2.2, 4.2.3]. In general, **G** is quasi-split over an unramified finite Galois extension F'/F. A valuation for $\mathbf{G}(F')$ descends to G [BT1, 9.1.11, 9.2.10] but not necessarily the Chevalley-Steinberg valuation [BT2, 5.1.15].

We consider the unique homomorphism $v : Z \mapsto V$ such that

$$\alpha(v(z)) = (\omega \circ \alpha)(z) \quad (z \in S, \alpha \in \Phi). \tag{18}$$

The kernel of v contains the maximal compact subgroup $\tilde{Z}_0$ of Z and the center of G. The index of the subgroup $T\tilde{Z}_0 \subset Z$ is finite.

For $\alpha \in \Phi$, φ_α is a function from $U_\alpha^* = U_\alpha - \{1\}$ to $\mathbb{R}$, satisfying properties described in [BT1, 6.2.1 (V0) to (V5)], which is compatible with ω :

$$(19) \qquad \varphi_\alpha(zuz^{-1}) = \varphi_\alpha(u) + (\alpha \circ v)(z) \quad \text{for all } \alpha \in \Phi, u \in U_\alpha^*, z \in Z,$$

and discrete : $\Gamma_\alpha = \varphi_\alpha(U_\alpha^*)$ is a discrete subset in $\mathbb{R}$ [BT1, 6.2.21]. If Δ is a basis of Φ, φ is determined by $(\varphi_\alpha)_{\alpha\in\Delta}$ [BT1, 6.2.8]. We have [BT1, 6.2.2]:

$$(20) \qquad \Gamma_{-\alpha} = \Gamma_\alpha \quad \text{if } \alpha \in \Phi, \text{ and } \varphi_{2\alpha} = 2\varphi_\alpha|_{U_{2\alpha}^*},\ \Gamma_{2\alpha} \subset 2\Gamma_\alpha \quad \text{if } \alpha, 2\alpha \in \Phi.$$

For $u \in U_\alpha^*$, the elements $v'_\alpha(u), v''_\alpha(u) \in U_{-\alpha}^*$ defined in (16) satisfy [BT1, 6.2.1 (V5)]:

$$(21) \qquad \varphi_{-\alpha}(v'_\alpha(u)) = \varphi_{-\alpha}(v''_\alpha(u)) = -\varphi_\alpha(u).$$

For $x \in V$, the family $\varphi + x = ((\varphi + x)_\alpha)_{\alpha\in\Phi}$ defined by

$$(22) \qquad (\varphi + x)_\alpha(u) := \varphi_\alpha(u) + \alpha(x) \quad \text{for all } \alpha \in \Phi, u \in U_\alpha,$$

is also a discrete valuation compatible with ω [BT1, 6.2.5]. The set of discrete valuations compatible with ω on $(Z, U_\alpha)_{\alpha\in\Phi}$ is [BT2, 5.1.23]:

$$(23) \qquad \mathfrak{A} = \{\varphi + x, \quad \text{for} \quad x \in V\}.$$

This is an affine euclidean real space with an action ν of N by affine automorphisms [BT1, 6.2.5] such that for $n \in N$ of image $w \in W_0$,

$$(24) \quad \nu(n)(\varphi + x) = \varphi + \nu(n)(x), \quad \alpha(\nu(n)(x)) = w^{-1}(\alpha)(x) + \varphi_{w^{-1}(\alpha)}(n^{-1}un) - \varphi_\alpha(u),$$

for $\alpha \in \Phi, u \in U_\alpha^*$. Hence $z \in Z$ acts by translation by $\nu(z) := -v(z)$, and for $\beta \in \Phi, v \in U_\beta^*$, $m_\beta(v)$ (16) acts by orthogonal reflection $s_{\beta+\varphi_\beta(v)}$ with respect to the affine hyperplane $H_{\beta+\varphi_\beta(v)} = \varphi + \operatorname{Ker}(\beta + \varphi_\beta(v))$,

$$(25) \qquad \nu(m_\beta(v))(x) = x - (\beta(x) + \varphi_\beta(v))\beta^\vee = s_\beta(x) - \varphi_\beta(v)\beta^\vee,$$

where $\beta^\vee \in \Phi^\vee$ is the coroot of β. The action of N determines the valuation φ, and conversely. The set of hyperplanes

$$(26) \qquad \mathfrak{H} = \{H_{\alpha+r} = \varphi + \operatorname{Ker}(\alpha + r) \mid \alpha \in \Phi_{red}, r \in \Gamma_\alpha\}$$

is stable under the action ν of N. We have [BT1, 6.2.10]:

$$\nu(n)(H_{\alpha+\varphi_\alpha(u)}) = H_{w(\alpha)+\varphi_{w(\alpha)}(nun^{-1})}.$$

By (20), when $\alpha, 2\alpha \in \Phi, r \in \Gamma_{2\alpha}$, we have $r/2 \in \Gamma_\alpha$ and $H_{2\alpha+r} = H_{\alpha+r/2} \in \mathfrak{H}$.

The affine space $\mathfrak{A}$ contains a valuation ψ such that $0 \in \psi(U_\alpha^*)$ for all α in the set Φ_{nm} of non-multipliable roots [BT1, 6.2.15]. We suppose, as we may, that $0 \in \Gamma_\alpha$ for all $\alpha \in \Phi_{nm}$. By (20),

$$(27) \qquad 0 \in \Gamma_\alpha \quad \text{for all } \alpha \in \Phi.$$

In particular, φ is special [BT1, 6.2.13].

For $1 \leq j \leq r$, $\varphi_j = (\varphi_\alpha)_{\alpha\in\Phi_j}$ is a discrete valuation of the root datum $(Z'_j, (U_\alpha)_{\alpha\in\Phi_j})$ compatible with ω, $\mathfrak{A} = \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_r$ is a product of affine euclidean real spaces $\mathfrak{A}_j = \varphi_j + V_j$, the set $\mathfrak{H}$ is the union of the sets $\mathfrak{H}_j = \{\varphi_j + \operatorname{Ker}(\alpha + x) \mid \alpha \in \Phi_{j,red}, x \in \Gamma_\alpha\}$ of affine hyperplanes in $\mathfrak{A}_j$ embedded in $\mathfrak{A}$ by

$$(28) \qquad H_j \mapsto \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_{j-1} \times H_j \times \mathfrak{A}_{j+1} \times \ldots \times \mathfrak{A}_r \quad (H_j \in \mathfrak{H}_j, 1 \leq j \leq r-1).$$

The action ν of N on $\mathfrak{A}$ factorizes through an action ν_j of N on $\mathfrak{A}_j$ such that $\nu(n)(\psi_1, \ldots, \psi_r) = (\nu_1(n)\psi_1, \ldots, \nu_r(n)(\psi_r))$ for $(\psi_1, \ldots, \psi_r) \in \mathfrak{A}_1 \times \ldots \times \mathfrak{A}_r$.

3.4 The affine Weyl group

Let $S(\mathcal{H})$ be the set of orthogonal reflections s_H with respect to the hyperplanes $H \in \mathfrak{H}$ (26) and let $W(\mathfrak{H}) \subset \nu(N)$ be the group generated by $S(\mathcal{H})$. The group $W(\mathfrak{H})$ is normal in $\nu(N)$.

The group $W(\mathfrak{H}) = W^{aff}$ is an affine Weyl group associated to a reduced root system Σ of V^* [Bki, VI §2.1, §2.5 Prop. 8] and [BT1, 6.2.22]

(29) $$\mathfrak{H} = \{H_{\beta+n} = \varphi + \mathrm{Ker}(\beta + n) \mid \beta \in \Sigma, n \in \mathbb{Z}\}.$$

We denote $s_{H_{\beta+n}} = s_{\beta+n}$. The product $s_{\beta+n}s_{\beta+n+1}$ is the translation by the coroot $\beta^\vee$ of $\beta \in \Sigma$ [Bki, V §2.4 Prop. 5]. We have $H_{\beta+n+1} = H_{\beta+n} - (1/2)\beta^\vee$. The subgroup $\Lambda(\mathfrak{H})$ of translations in $W(\mathfrak{H})$ identifies with the $\mathbb{Z}$-module $Q(\Sigma^\vee)$ generated by the set $\Sigma^\vee$ of coroots [Bki, VI §2.1].

Two points $x, y \in \mathfrak{A}$ are called $\mathfrak{H}$-equivalent if : for all $H \in \mathfrak{H}$, either $x, y \in H$ or they are in the same connnected component of $\mathfrak{A} - H$ [Bki, V §1.2] [BT1, 1.3]. A facet $\mathfrak{F} \subset \mathfrak{A}$ is an equivalence class. A facet of $\mathfrak{F}$ is a facet contained in the closure $\overline{\mathfrak{F}}$ of $\mathfrak{F}$. A vertex is a point which is a facet. A chamber of $\mathfrak{A}$ (a connected component of $\mathfrak{A} - \cup_{H \in \mathfrak{H}} H$) is called an alcove [Bki, V.1.3 Déf.2] to avoid a confusion with the chambers relatively to $\mathfrak{H}_\varphi = \{H \in \mathfrak{H} \mid \varphi \in H\}$ that we call Weyl chambers. The group $W(\mathfrak{H})$ acts simply transitively on the alcoves of $\mathfrak{A}$ [Bki, VI.2.1].

We choose an alcove $\mathfrak{C} \subset \mathfrak{A}$ of vertex the special point φ. A face of $\mathfrak{C}$ is a facet of $\mathfrak{C}$ contained in a single $H \in \mathfrak{H}$, called its support. A wall of $\mathfrak{C}$ is an hyperplane $H \in \mathfrak{H}$ containing a face of $\mathfrak{C}$ [Bki, V. §1.4, Déf. 3]. The set

$$S(\mathfrak{C}) = \{s_H \mid H \in \mathfrak{H} \text{ wall of } \mathfrak{C}\}$$

of orthogonal reflections s_H with respect to the walls H of $\mathfrak{C}$, generates $W(\mathfrak{H})$ and $(W(\mathfrak{H}), S(\mathfrak{C}))$ is a Coxeter system. The type of a facet $\mathfrak{F}$ of $\mathfrak{C}$ is the set

(30) $$S_{\mathfrak{F}} = \{s_H \mid H \in \mathfrak{H} \text{ wall of } \mathfrak{C},\ \mathfrak{F} \subset H\}.$$

We have $S_{\mathfrak{C}} = \emptyset$. A facet $\mathfrak{F}$ of $\mathfrak{C}$ is determined by its type because

$$\mathfrak{F} = \{x \in \overline{\mathfrak{C}} \mid x \in H \Leftrightarrow \mathfrak{F} \subset H \quad \text{for any wall } H \text{ of } \mathfrak{C}\}.$$

The bijection between the facets of $\mathfrak{C}$ and their types reverses the inclusion:

$$\mathfrak{F}' \text{ is a facet of } \mathfrak{F} \ \Rightarrow\ S_{\mathfrak{F}} \subset S_{\mathfrak{F}'}.$$

The types of the facets of $\mathfrak{C}$ are the subsets of $S(\mathfrak{C})$ generating a finite subgroup. A facet of $\mathfrak{A}$ is the image by an element of $W(\mathfrak{H})$ of a unique facet of $\mathfrak{C}$ and we can define the type of any facet [BT1, 1.3.5].

Let $W_{\mathfrak{F}}$ be the group generated by $S_{\mathfrak{F}}$. Then $(W_{\mathfrak{F}}, S_{\mathfrak{F}})$ is a finite Coxeter system. As φ is a special point [Bki, V §3.10], $W(\mathfrak{H})$ is a semi-direct product [Bki, V.3.10 Prop. 9 and Def. 1] and [BT1, 1.3)]:

(31) $$W(\mathfrak{H}) \simeq \Lambda(\mathfrak{H}) \rtimes W_\varphi,$$

the groups W_φ, the Weyl group of Σ, the Weyl group of Φ, and the group $W_0 = N/Z$ are isomorphic. The group W_φ, acts simply transitively on the Weyl chambers of $\mathfrak{A}$.

The set of affine roots is the subset of automorphisms of $\mathfrak{A}$

$$\Sigma^{aff} = \{\beta + n \mid \beta \in \Sigma, n \in \mathbb{Z}\}.$$

The action of the group $W(\mathfrak{H})$ on $\mathfrak{A}$ induces an action on Σ^{aff}. For $A, A' \in \Sigma^{aff}$, we have $\operatorname{Ker} A = \operatorname{Ker} A'$ if and only if $A' = \pm A$. For $x \in \mathfrak{C}$, $A(x) \neq 0$ and the sign of $A(x)$ does not depend on the choice of x. We say that

$$A \text{ is } \mathfrak{C}\text{-positive if } A \text{ takes positive values on } \mathfrak{C}.$$

The set of affine $\mathfrak{C}$-positive roots is denoted by $\Sigma^{aff,+}$; we have $\Sigma^+ = \Sigma^{aff,-} \cap \Sigma$. We denote $\Sigma^{aff,-}, \Sigma^-$ for the $\mathfrak{C}$-negative roots. Let

$$\Delta_\Sigma^{aff} = \Delta_\Sigma^{aff}(\mathfrak{C}) = \{A \in \Sigma^{aff,+} \mid \operatorname{Ker} A \text{ is a wall of } \mathfrak{C}\}. \tag{32}$$

The set Δ_Σ^{aff} is in bijection with $S^{aff} = S(\mathfrak{C})$ by the map $A \mapsto s_A$ of inverse $s \mapsto A_s$, and with a subset $\Delta_\Sigma(\mathfrak{C})$ of Σ by the gradient map.

The pair (W^{aff}, S^{aff}) is an affine Coxeter system. We recall [Kumar, 1.3.11, $(b_4), (b_5)$, proof before (2)]:

1. $s(A_s) < 0$ and $s(A) > 0$ for $s \in S^{aff}, A \in \Sigma^{aff,+}, A \neq A_s$.
2. $w(A_s) = A_{s'} \Rightarrow wsw^{-1} = s'$ for $s, s' \in S^{aff}, w \in W^{aff}$.
3. The length ℓ of (W^{aff}, S^{aff}) satisfies for $(w, s) \in W^{aff} \times S^{aff}$,

$$\ell(ws) = \begin{cases} \ell(w) + 1 & \text{if} \quad w(A_s) > 0, \\ \ell(w) - 1 & \text{if} \quad w(A_s) < 0. \end{cases}$$

For a facet $\mathfrak{F}$ of $\mathfrak{C}$, let

$$\Sigma_{\mathfrak{F}}^{aff} = \{A \in \Sigma^{aff} \mid \operatorname{Ker} A \text{ contains } \mathfrak{F}\}, \tag{33}$$

$$\Delta_{\Sigma,\mathfrak{F}}^{aff} = \{A \in \Sigma^{aff,+} \mid \operatorname{Ker} A \text{ is a wall of } \mathfrak{C} \text{ containing } \mathfrak{F}\}. \tag{34}$$

Any element of $\Sigma_{\mathfrak{F}}^{aff,+} = \Sigma_{\mathfrak{F}}^{aff} \cap \Sigma^{aff,+}$ is a linear combination of elements of $\Delta_{\Sigma,\mathfrak{F}}^{aff}$ with unique coefficients in $\mathbb{N}$. The set $\Delta_{\Sigma,\mathfrak{F}}^{aff}$ is in bijection with the type $S_{\mathfrak{F}}$ of $\mathfrak{F}$ by the map $A \mapsto s_A$ and with a subset $\Delta_{\Sigma,\mathfrak{F}}$ of Σ by the gradient map. We have $\Delta_{\Sigma,\mathfrak{C}} = \Delta_{\Sigma,\mathfrak{C}}^{aff} = \emptyset$ and $\Delta_\Sigma = \Delta_{\Sigma,\varphi} = \Delta_{\Sigma,\varphi}^{aff}$ is a basis of Σ.

With the notations of (15), (28), $W^{aff} = W_1^{aff} \times \ldots \times W_r^{aff}$ is the direct product of the affine Weyl groups $W_j^{aff} = W(\mathfrak{H}_j)$ for $1 \leq j \leq r$; we consider W_j as a subgroup of W^{aff} with its natural action on $\mathfrak{A}_j$ and acting trivially on $\mathfrak{A}_i$ for $i \neq j$. The irreducible components of $\Sigma = \sqcup_{j=1}^r \Sigma_j$ are the reduced root systems Σ_j associated to W_j^{aff} considered as subsets of $(V_1 \times \ldots \times V_r)^*$ vanishing on V_i for $i \neq j$. The alcoves $\mathfrak{C} = \mathfrak{C}_1 \times \ldots \times \mathfrak{C}_r$ are the product of the alcoves $\mathfrak{C}_j$ of $\mathfrak{A}_j$ for $1 \leq j \leq r$, [BT1, 6.2.12], [Bki, V §3.8 Prop. 6], the walls of $\mathfrak{C}$ are the images by (28) of the walls of $\mathfrak{C}_j$ for $1 \leq j \leq r$. The set $S(\mathfrak{C}) = \sqcup_{j=1}^r S(\mathfrak{C}_j)$ is the disjoint union of the sets $S(\mathfrak{C}_j)$. The sets $\Sigma^{aff} = \sqcup_{j=1}^r \Sigma_j^{aff}$, $\Sigma^{aff,+} = \sqcup_{j=1}^r \Sigma_j^{aff,+}$, $\Sigma^+ = \sqcup_{j=1}^r \Sigma_j^+$, $\Delta_\Sigma^{aff} = \sqcup_{j=1}^r \Delta_{\Sigma_j}^{aff,+}$, $\Delta_\Sigma = \sqcup_{j=1}^r \Delta_{\Sigma_j}$ are the disjoint unions of the similar sets for $1 \leq j \leq r$. This allows often to reduce to an irreducible root system Σ.

By [Bki, VI.1.8 Prop. 25, VI.2.3 Prop.5], the alcove $\mathfrak{C}_j$ is the set of $\varphi_j + x$ for $x \in V_j$ satisfying

$$\gamma(x) > 0 \text{ for all } \gamma \in \Delta_{\Sigma_j} \text{ and } \tilde{\beta}_j(x) < 1 \;\Leftrightarrow\; 0 < \gamma(x) < 1 \text{ for all } \gamma \in \Sigma_j^+. \tag{35}$$

where $\tilde{\beta}_j = \sum_{\gamma \in \Delta_{\Sigma_j}} n_\gamma \gamma$ is the highest root of Σ_j^+ given explicitely in the tables of Bourbaki [Bki, pp. 250-275]. We have

$$\Delta_{\Sigma_j}^{aff} = \Delta_{\Sigma_j} \cup \{-\tilde{\beta}_j + 1\}.$$

Returning to Σ, we deduce

(36) $\mathfrak{C} = \{\varphi + x, \mid 0 < \gamma(x) < 1 \text{ for all } \gamma \in \Sigma^+\}, \quad \Delta_\Sigma^{aff} = \Delta_\Sigma \cup \{-\tilde{\beta}_1 + 1, \ldots, -\tilde{\beta}_r + 1\}.$

An affine root $A = \alpha + k, \alpha \in \Sigma, k \in \mathbb{Z}$, is positive if $\alpha(x) + k > 0$ for $x \in \mathcal{C}$, hence if $k \geq 0$ if $\alpha \in \Sigma^+$ and $k > 0$ if $\alpha \in \Sigma^-$

$$\Phi^{aff,+} = \{\alpha + k \mid (\alpha, k) \in (\Sigma^+ \times \mathbb{N}) \cup (\Sigma^- \times \mathbb{N}_{>0})\}.$$

The vertices of $\mathfrak{C}_j$ are $\{\varphi_j, \varphi_j + n_\beta^{-1}\omega_\beta \ (\beta \in \Delta_{\Sigma_j})\}$ where $\omega_{\beta^\vee}$ is a fundamental coweight [Bki, VI.2.3 Cor. to Prop. 5]. The vertex $\varphi + n_\beta^{-1}\omega_{\beta^\vee}$ is special if and only if $n_\beta = 1$ [Bki, VI §1.10, 2.2 Cor., §2.2 Prop. 3]. Any set of vertices of $\mathfrak{C}_j$ is the set of vertices of a facet of $\mathcal{C}$.

For $Y \subset \Delta_{\Sigma_j}$, the facet $\mathfrak{F}_{\varphi_j, Y}$ of vertices $\varphi_j, (\varphi_j + n_\beta^{-1}\omega_{\beta^\vee})_{\beta \in Y}$ is the set of $\varphi_j + x$ such that $\gamma(x) = 0$ for $\gamma \in \Delta_{\Sigma_j} - Y$, and $0 < \beta(x) < 1$ for $\beta \in Y$. The facet $\mathfrak{F}_Y$ of vertices $(\varphi_j + n_\beta^{-1}\omega_{\beta^\vee})_{\beta \in Y}$ is the set of $\varphi_j + x$ such that $\gamma(x) = 0$ for $\gamma \in \Delta_{\Sigma_j} - Y$, $\tilde{\beta}_j(x) = 1$ and $0 < \beta(x) < 1$ for $\beta \in Y$. We have

$$\Delta^{aff}_{\Sigma_j, \mathfrak{F}_{\varphi_j, Y}} = \Delta_{\Sigma_j, \varphi_j} - Y, \quad \Delta^{aff}_{\Sigma_j, \mathfrak{F}_Y} = (\Delta_{\Sigma_j, \varphi_j} - Y) \cup \{-\tilde{\beta}_j + 1\}.$$

Lemma 3.7. *The translation by $v_j \in V_j$ stabilises $\mathfrak{H}_j$ if and only if $\gamma(v_j) \in \mathbb{Z}$ for all $\gamma \in \Sigma_j$. The translation by v_j normalizes $\mathfrak{C}_j$ if and only if $v_j = 0$.*

Proof. $\gamma(x) + k = 0$ is equivalent to $\gamma(x + v_j) + k - \gamma(v_j) = 0$ and for $r \in \mathbb{R}$, $\operatorname{Ker} \gamma + r \in \mathfrak{H}_j$ if and only if $r \in \mathbb{Z}$. The image of $\mathfrak{C}_j$ by $\gamma \in \Delta_{\Sigma_j}$ is an interval $]a, b[$. The image of $\mathfrak{C}_j + v_j$ by γ is the interval $]a + \gamma(v_j), b + \gamma(v_j)[$. If $\mathfrak{C}_j + v_j = \mathfrak{C}_j$, we have $\gamma(v_j) = 0$ for all $\gamma \in \Delta_{\Sigma_j}$, hence $v_j = 0$. □

3.5 The filtration of U_α

The properties [BT1, 6.2.1] of the valuation φ imply that, for $\alpha \in \Phi$ and $r \in \mathbb{R}$, the set

(37) $$U_{\alpha + r} = \{u \in U_\alpha \mid \varphi_\alpha(u) \geq r\}$$

is a compact open subgroup of U_α (careful $U_{\alpha+0} \neq U_\alpha$), and $(U_{\alpha+r})_{r \in \Gamma_\alpha}$ is a strictly decreasing filtration of union U_α and trivial intersection. For $n \in N$ of image $w \in W_0$ we have [BT1, 6.2.10 proof of (iii))]:

$$nU_{\alpha + \varphi_\alpha(u)}n^{-1} = U_{w(\alpha) + \varphi_{w(\alpha)}(nun^{-1})}.$$

For $\alpha \in \Phi, r \in \Gamma_\alpha$, let $U_{\alpha + r_+}$ be the group $U_{\alpha + r'}$ for $r' \in \Gamma_\alpha, r' > r$, and r' minimal for these properties, and let

$$U_{\alpha + r, k} = U_{\alpha + r}/U_{\alpha + r_+}.$$

When the root system is not reduced, we make the following observation:

Lemma 3.8. *For $\alpha, 2\alpha \in \Phi, r \in (1/2)\Gamma_{2\alpha} \subset \Gamma_\alpha$, the sequence*

$$1 \to U_{2\alpha + 2r, k} \to U_{\alpha + r, k} \to U_{\alpha + r}/U_{\alpha + r_+}U_{2\alpha + 2r} \to 1$$

is exact.

Proof. We have to show that $U_{2\alpha + 2r} \cap U_{\alpha + r_+} = U_{2\alpha + (2r)_+}$. By (20), the left hand side is $U_{2\alpha + 2r'}$ where r' is the smallest element of Γ_α with $r' > r$, and for $r'' \in \Gamma_{2\alpha}$, the strict inequality $2r < r''$ is equivalent to the inequality $2r' \leq r''$. The right hand side is $U_{2\alpha + r''}$ and $U_{2\alpha + 2r'} = U_{2\alpha + r''} = U_{2\alpha + (2r)_+}$. □

For $\alpha, 2\alpha \in \Phi$, we introduce the set

$$\Gamma'_\alpha = \Gamma_\alpha - \{r \in (1/2)\Gamma_{2\alpha} \mid U_{\alpha+r} = U_{\alpha+r_+} U_{2\alpha+2r}\} = \{r \in \Gamma_\alpha \mid U_{2\alpha+2r,k} \neq U_{\alpha+r,k}\}.$$

This set is never empty [BT2, 4.2.21]. When $\alpha \in \Phi, 2\alpha \notin \Phi$ we put $\Gamma'_\alpha = \Gamma_\alpha$. The set

$$\Phi^{aff} = \cup_{\alpha\in\Phi} \alpha + \Gamma'_\alpha$$

is called the set of affine roots. We have a natural injection

$$\cup_{\alpha\in\Phi_{red}} \alpha + \Gamma_\alpha \to \Phi^{aff} \tag{38}$$

sending $\alpha + r$ to $2\alpha + 2r$ if $r \notin \Gamma'_\alpha$, and to $\alpha + r$ otherwise. Let Φ^{aff}_{red} denote the image.

For $\alpha \in \Phi, r \in \Gamma_\alpha$, we say that $\alpha + r$ is $\mathfrak{C}$-positive when $\alpha(x) + r > 0$ for $x \in \mathfrak{C}$. The map (38) respects $\mathfrak{C}$-positivity.

For $\alpha \in \Phi$, there exists a unique positive number $e_\alpha > 0$ such that the map

$$\alpha + r \mapsto e_\alpha(\alpha + r) : \cup_{\alpha\in\Phi} \alpha + \Gamma_\alpha \to \Sigma^{aff} \tag{39}$$

is surjective, respects positivity and restricts to a bijection

$$\Phi^{aff}_{red} \xrightarrow{\simeq} \Sigma^{aff}. \tag{40}$$

This bijection allows to replace Φ^{aff}_{red} by the affine root system Σ^{aff} (the filtration is hidden in the bijection).

It is obvious that $e_\alpha = e_{-\alpha}$. With (20),

$$\Gamma_\alpha = \gamma_\alpha \mathbb{Z} \quad \gamma_\alpha = e_\alpha^{-1}, \quad \text{if } \alpha \in \Phi_{red}. \tag{41}$$

When $\alpha, 2\alpha \in \Phi$, we have $e_{2\alpha} = (1/2)e_\alpha$, $\Gamma_{2\alpha}$ is a group because $0 \in \Gamma_0$ (27) [BT1, Cor. 6.2.16], there exists a unique positive integer $f_\alpha \in \mathbb{N}_{>0}$ such that

$$\Gamma_{2\alpha} = \gamma_{2\alpha}\mathbb{Z} \quad \gamma_{2\alpha} = 2f_\alpha e_\alpha^{-1}, \quad \text{if } \alpha, 2\alpha \in \Phi. \tag{42}$$

Lemma 3.9. *e_α is a positive integer for all $\alpha \in \Phi$, which is divisible by $2f_\alpha$ if $2\alpha \in \Phi$.*

Proof. By the proof of [SS, Lemma **I**.2.10], Γ_α contains $n_\alpha^{-1}\mathbb{Z}$ where $n_\alpha \in \mathbb{N}_{>0}$ for any $\alpha \in \Phi$. ☐

3.6 The adjoint building

For $x \in V$ and $\alpha \in \Phi$,

$$\text{the smallest element } r_x(\alpha) \in \Gamma_\alpha \text{ such that } \alpha(x) + r_x(\alpha) \geq 0, \tag{43}$$

depends only on the facet $\mathfrak{F}$ of $\mathfrak{A}$ containing $\varphi + x$, and is also denoted by $r_{\mathfrak{F}}(\alpha)$.

Example 3.10. *$r_\varphi(\alpha) = 0$ for all $\alpha \in \Phi$.*

$r_{\mathfrak{C}}(\alpha) = 0$ if $\alpha \in \Phi$ is $\mathfrak{C}$-positive,

$r_{\mathfrak{C}}(\alpha) = e_\alpha^{-1}$ if $\alpha \in \Phi_{red}$ is $\mathfrak{C}$-negative (41).

$r_{\mathfrak{C}}(2\alpha) = 2f_\alpha e_\alpha^{-1}$ if $\alpha, 2\alpha \in \Phi$ are $\mathfrak{C}$-negative (42).

Example 3.11. *Let $\mathfrak{F}$ be a facet contained in the wall $H_{\alpha+r}$, $\alpha \in \Phi, r \in \Gamma_\alpha$. Then $r_{\mathfrak{F}}(\alpha) = -r_{\mathfrak{F}}(-\alpha) = r$.*

Let U_x be the group generated by $\cup_{\alpha\in\Phi} U_{\alpha+r_x(\alpha)}$ and let N_x be the stabilizer of $\varphi+x$ in N. The group N_x normalizes U_x, and

$$P_x = N_x U_x \tag{44}$$

is a group (denoted by $\hat{P}_x$ in [BT1, 7.1.8]). These groups depending only on the facet $\mathfrak{F}$ containing $\varphi+x$, are also denoted by $U_{\mathfrak{F}}, N_{\mathfrak{F}}, P_{\mathfrak{F}}$. For $\alpha\in\Phi$ we have

$$P_{\mathfrak{F}} \cap U_\alpha = U_{\alpha+r_{\mathfrak{F}}(\alpha)}. \tag{45}$$

This is clear if α is not multipliable. If $\alpha, 2\alpha \in \Phi$ this is true because $U_{2\alpha+r_{\mathfrak{F}}(2\alpha)} = U_{\alpha+r_{\mathfrak{F}}(2\alpha)/2} \cap U_{2\alpha}$ is contained in $U_{\alpha+r_{\mathfrak{F}}(\alpha)}$ as $r_{\mathfrak{F}}(2\alpha)$ is the smallest element of $\Gamma_{2\alpha}$ satisfying $r_{\mathfrak{F}}(2\alpha)\geq 2r_{\mathfrak{F}}(\alpha)$ by (20) [BT1, 7.4.1]. This shows also that:

$$\text{The group } U_{\mathfrak{F}} \text{ is generated by } \cup_{\alpha\in\Phi_{red}} U_{\alpha+r_{\mathfrak{F}}(\alpha)}. \tag{46}$$

Two facets $\mathfrak{F}$ and $\mathfrak{F}'$ with $r_{\mathfrak{F}}(\alpha) = r_{\mathfrak{F}'}(\alpha)$ for all $\alpha\in\Phi^+_{red}$ are equal. Therefore two facets $\mathfrak{F}$ and $\mathfrak{F}'$ with $U_{\mathfrak{F}} = U_{\mathfrak{F}'}$ are equal.

Definition 3.12. *The adjoint building is*

$$\mathfrak{B}(G_{ad}) := G\times\mathfrak{A}/\sim \tag{47}$$

where $\sim$ is the equivalence relation on $G\times\mathfrak{A}$ defined by

$$(g,\varphi+x)\sim(h,\varphi+y) \Leftrightarrow \textit{there exists } n\in N \mid \varphi+y = n.(\varphi+x) \textit{ and } g^{-1}hn\in P_x,$$

with the natural action of G, induced by $(g,(h,\psi))\mapsto(gh,\psi)$ for $g,h\in G, \psi\in\mathfrak{A}$.

The apartments of $\mathfrak{B}(G_{ad})$ are the images by G of the apartment $\mathfrak{A}$. The facets, resp. alcoves, of $\mathfrak{B}(G_{ad})$ are the images by G of the facets, resp. alcoves, of $\mathfrak{A}$. The G-orbit of a facet contains a unique facet of the chosen alcove $\mathfrak{C}$ of $\mathfrak{A}$.

The group P_x is obviously the G-stabilizer of $(1,\varphi+x)$. The pointwise G-stabilizer (or fixator) $P_{\mathfrak{F}}$ of a facet $\mathfrak{F}$ is the intersection of the G-stabilizers of its vertices.

The map $\psi\mapsto(1,\psi):\mathfrak{A}\to\mathfrak{B}(G_{ad})$ is an N-equivariant embedding. The G-stabilizer of $\mathfrak{A}$ is N. The G-fixator of $\mathfrak{A}$ is the kernel of the homomorphism $v: Z\to V$ (18) (it is denoted by $\hat{H}$ in [BT1, 4.1.2, 6.2.11,7.4.10]). Let $\mathfrak{F}$ be a facet of $\mathfrak{A}$. The G-fixator of $\mathfrak{F}$ is the semidirect product (44) (see also [BT1, 4.1.1, 6.4.2, 7.1.3])

$$P_{\mathfrak{F}} = U_{\mathfrak{F}} \rtimes \operatorname{Ker} v$$

It acts transitively on the apartments containing $\mathfrak{F}$ [BT1, 7.4.9] hence $U_{\mathfrak{F}}$ acts also transitively on the apartments containing $\mathfrak{F}$.

We denote by U^+ the subgroup of G generated by U_α for $\alpha\in\Phi^+_{red}$. The product maps

$$\prod_{\alpha\in\Phi^+_{red}} U_\alpha \to U^+, \qquad \prod_{\alpha\in\Phi^+_{red}} U_{\alpha+r_{\mathfrak{F}}(\alpha)} \to U^+_{\mathfrak{F}} := U^+\cap U_{\mathfrak{F}}, \tag{48}$$

are homeomorphisms [BT1, 6.1.6, 6.4.9], whatever ordering we choose on Φ^+_{red}. We have a similar result for the groups $U^-_{\mathfrak{F}}, U^-$ defined with $\Phi^-_{red} = -\Phi^+_{red}$.

Remark 3.13. For $A\in\Sigma^{aff}$, we denote $U_A = U_{\alpha+r}$ where $\alpha+r$ is the antecedent of A by the bijection $\cup_{\alpha\in\Phi_{red}}\alpha+\Gamma_\alpha\to\Sigma^{aff}$ (40). We have $A = e_\alpha(\alpha+r)$ with $e_\alpha>0$.

The group $U_{\mathfrak{C}}$ is generated by all U_A for $A\in\Sigma^{aff,+}$.

The group $U_{\mathfrak{F}}$ is generated by all U_A for $A\in\Sigma^{aff}$ and $A\geq 0$ on $\mathfrak{F}$.

The group $U^0_{\mathfrak{F}}$ generated by all U_A for $A\in\Sigma^{aff}$ and $A=0$ on $\mathfrak{F}$ satisfies $U^0_{\mathfrak{C}} = \{1\}$, $U^0_\varphi = U_\varphi$.

3.7 Parahoric subgroups

We denote by F^s a maximal separable extension of F, by F^{unr} the maximal unramified extension of F contained in F^s, by $\mathcal{I} = \mathrm{Gal}(F^s/F^{unr})$ the inertia group and by $\sigma \in \mathrm{Gal}(F^{unr}/F)$ the Frobenius automorphism. Let $Z(\hat{G})$ be the center of the Langlands dual group $\hat{G}$ of G with the natural action of $\mathrm{Gal}(F^s/F)$. The F^s-character group $\pi_1(G) = X^*(Z(\hat{G}))$ of $Z(\hat{G})$ is the Borovoi algebraic fundamental group of $\mathbf{G}$. When G is semi-simple and simply connected, $\pi_1(G)$ is trivial.

Kottwitz [Ko, 7.1 to 7.4] defined a functorial surjection from G onto the σ-invariants of the $\mathcal{I}$-coinvariants of $\pi_1(G)$

(49) $$\kappa_G : G \to \pi_1(G)^{\sigma}_{\mathcal{I}}.$$

Definition 3.14. *A parahoric subgroup of G is the fixator $K_{\mathfrak{F}} = \ker \kappa_G \cap P_{\mathfrak{F}}$ in the kernel of κ_G of a facet $\mathfrak{F}$ of the building $\mathcal{B}(G_{ad})$.*

A pro-p-parahoric subgroup $K_{\mathfrak{F}}(1)$ of G is the pro-p-radical of a parahoric subgroup $K_{\mathfrak{F}}$ of G.

An Iwahori, resp. pro-p-Iwahori, subgroup of G is the parahoric, resp. pro-p-parahoric, subgroup fixing an alcove.

This definition of a parahoric subgroup $K_{\mathfrak{F}}$ by Haines and Rapoport [HRa], coincides with the definition by Bruhat and Tits, denoted by $\mathfrak{G}^0_{\mathfrak{F}}(\mathcal{O}^{\natural})$ in [BT2].

The pro-p-radical $K_{\mathfrak{F}}(1)$ of a parahoric group $K_{\mathfrak{F}}$ is the largest open normal pro-p-subgroup [HV, 3.6]. The quotient $K_{\mathfrak{F},k} = K_{\mathfrak{F}}/K_{\mathfrak{F}}(1)$ is the group of k-points of a connected reductive group over the residue field k of F.

A parahoric subgroup of G is G-conjugate to a parahoric subgroup fixing a facet of the alcove $\mathfrak{C}$ of $\mathfrak{A}$. *The Iwahori, resp. pro-p-Iwahori, sugroups of G are conjugate.*

From now on, $\mathfrak{F}$ is a facet of $\mathfrak{C}$, I is the Iwahori subgroup fixing $\mathfrak{C}$, and positive means $\mathfrak{C}$-positive.

The group Z admits a unique parahoric subgroup Z_0, which is the kernel of the Kottwitz morphism κ_Z [HR, 4.1.1]. The group Z_0 is a subgroup of finite index of the maximal compact subgroup $\tilde{Z}_0$ of Z. The group N normalizes $Z, Z_0, Z_0(1)$, and the subgroup $Z_0^{(p)}$ of elements of Z_0 of finite order prime to p. The quotient $Z_k = Z_{0,k} = Z_0/Z_0(1)$ is the group of points over k of a torus (non necessarily split). The quotient map $Z_0 \to Z_k$ restricted to $Z_0^{(p)}$ is an isomorphism, Z_0 is a semi-direct product

(50) $$Z_0 = Z_0(1) \rtimes Z_0^{(p)} \simeq Z_0(1) \rtimes Z_k.$$

The group $\Lambda = Z/Z_0$ is finitely generated and commutative, of torsion subgroup $\tilde{Z}_0/Z_0$. We have $Z_0 = \tilde{Z}_0$ when Z is a split torus or when G is unramified, or semi-simple and simply connected [HR, Section 11]. The group $\Lambda(1) = Z/Z_0(1)$ is finitely generated, of torsion subgroup $\tilde{Z}_0/Z_0(1)$ and may be not commutative.

The same considerations apply to the maximal split subtorus T of Z. The group $T_0 = Z_0 \cap T$ is the maximal subgroup of T.

Proposition 3.15. $Z \cap K_{\mathfrak{F}} = Z_0$.

Proof. [HR, Lemma 4.2.1]. □

The group $U_{\mathfrak{F}}$ generated by $U^+_{\mathfrak{F}} \cup U^-_{\mathfrak{F}}$ is normalized by $\tilde{Z}_0$. The unipotent groups $U_\alpha, \alpha \in \Phi$, being contained in $\mathrm{Ker}\,\kappa_G$, we deduce from (45)

(51) $$K_{\mathfrak{F}} \cap U_\alpha = U_{\alpha + r_{\mathfrak{F}}(\alpha)} \text{ for } \alpha \in \Phi, \quad K_{\mathfrak{F}} \cap U^+ = U^+_{\mathfrak{F}}, \quad K_{\mathfrak{F}} \cap U^- = U^-_{\mathfrak{F}}.$$

Proposition 3.16. [BT2, 5.2.4] $K_{\mathfrak{F}} = Z_0 U_{\mathfrak{F}} = U^-_{\mathfrak{F}} U^+_{\mathfrak{F}} U^-_{\mathfrak{F}} Z_0 = U^-_{\mathfrak{F}} U^+_{\mathfrak{F}} (N \cap K_{\mathfrak{F}})$.

We have [SS, Lemma I.2.1]:

$$K_{\mathfrak{F}}(1) \cap U_\alpha = U_{\alpha + r^*_{\mathfrak{F}}(\alpha)} \tag{52}$$

for $\alpha \in \Phi$, where $r^*_{\mathfrak{F}}(\alpha) = r_{\mathfrak{F}}(\alpha)_+$ if $\mathfrak{F} \subset \mathrm{Ker}(\alpha + r_{\mathfrak{F}}(\alpha))$ and $r^*_{\mathfrak{F}}(\alpha) = r_{\mathfrak{F}}(\alpha)$ otherwise.

Remark 3.17. *When $\alpha, 2\alpha \in \Phi$ and $2r_{\mathfrak{F}}(\alpha) \in \Gamma_{2\alpha}$ we have $2r_{\mathfrak{F}}(\alpha) = r_{\mathfrak{F}}(2\alpha)$, and $r^*_{\mathfrak{F}}(2\alpha) = r_{\mathfrak{F}}(2\alpha)_+$.*

We denote

$$U_{\mathfrak{F}}(1) = U_{\mathfrak{F}} \cap K_{\mathfrak{F}}(1), \quad U^+_{\mathfrak{F}}(1) = K_{\mathfrak{F}}(1) \cap U^+, \quad U^-_{\mathfrak{F}}(1) = K_{\mathfrak{F}}(1) \cap U^-.$$

The group $U_{\mathfrak{F}}(1)$ is generated by $U^+_{\mathfrak{F}}(1)$ and $U^-_{\mathfrak{F}}(1)$. As in (48), the product map

$$\prod_{\alpha \in \Phi^+_{red}} U_{\alpha + r^*_{\mathfrak{F}}(\alpha)} \to U^+_{\mathfrak{F}}(1), \tag{53}$$

is an homeomorphism whatever ordering we choose on Φ^+_{red} [BT2, 5.2.3]. We have a similar result for $U^-_{\mathfrak{F}}(1)$.

Example 3.18. Let $\alpha \in \Phi$ and let $\mathfrak{F}$ be a facet of $\mathfrak{C}$ such that φ is a vertex of $\mathfrak{F}$.
$r^*_\varphi(\alpha) \neq r_\varphi(\alpha) = 0$,
$r^*_{\mathfrak{C}}(\alpha) = r_{\mathfrak{C}}(\alpha)$,
$r^*_{\mathfrak{F}}(\alpha) \neq r_{\mathfrak{F}}(\alpha) = r_\varphi(\alpha)$ if $\mathfrak{F} \subset \mathrm{Ker}\,\alpha$,
$r^*_{\mathfrak{F}}(\alpha) = r_{\mathfrak{F}}(\alpha) = r_{\mathfrak{C}}(\alpha)$ if $\mathfrak{F} \not\subset \mathrm{Ker}\,\alpha$.
Let $U^{0+}_{\mathfrak{F}}$, resp. $V^{0+}_{\mathfrak{F}}$, be the group generated by $U_{\alpha+0}$ for $\alpha \in \Phi^+$ such that $\mathfrak{F} \subset \mathrm{Ker}\,\alpha$, resp. $\mathfrak{F} \not\subset \mathrm{Ker}\,\alpha$. Then, $U^+_\varphi(1) \subset U^+_{\mathfrak{F}}(1) \subset U^+_{\mathfrak{C}}(1) \subset U^+_{\mathfrak{C}}$ and more precisely

$$U_\varphi(1)^+ V^{0+}_{\mathfrak{F}} = U^+_{\mathfrak{F}}(1), \ U^+_{\mathfrak{F}}(1) U^{0+}_{\mathfrak{F}} = U^+_{\mathfrak{C}}(1) = U^+_{\mathfrak{C}}.$$

We have a similar result for $U^-_{\mathfrak{F}}$.

Proposition 3.19. *(Iwahori decomposition) $K_{\mathfrak{F}}(1) = U^+_{\mathfrak{F}}(1) Z_0(1) U^-_{\mathfrak{F}}(1)$ and the factors commute.*

Proof. [SS, Prop. I.2.2]. □

We denote $I^+ = U^+_{\mathfrak{C}} = U^+_{\mathfrak{C}}(1), \ I^- = U^-_{\mathfrak{C}} = U^-_{\mathfrak{C}}(1)$.

Corollary 3.20. *The Iwahori group $I = I(1)Z_0$ admits the Iwahori decomposition $I = I^- Z_0 I^+ = I^+ Z_0 I^-$, the factors commute, and the product maps*

$$\prod_{\alpha \in \Phi^+_{red}} U_{\alpha+0} \to I^+, \qquad \prod_{\alpha \in \Phi^-_{red}} U_{\alpha + e_\alpha^{-1}} \to I^-$$

are homeomorphisms.

Proof. Example 3.10. □

Corollary 3.21. *The map $\mathfrak{F} \mapsto K_{\mathfrak{F}}$ is decreasing and the map $\mathfrak{F} \mapsto K_{\mathfrak{F}}(1)$ is increasing:*

$$K_{\mathfrak{F}}(1) \subset K_{\mathfrak{F}'}(1) \subset K_{\mathfrak{F}'} \subset K_{\mathfrak{F}},$$

if $\mathfrak{F}$ is a facet of a facet $\mathfrak{F}'$.

Proof. If $\mathfrak{F}$ is a facet of $\mathfrak{F}'$, the inclusions $K_{\mathfrak{F}'}(1) \subset K_{\mathfrak{F}'} \subset K_{\mathfrak{F}}$ are clear. The inclusion $K_{\mathfrak{F}}(1) \subset K_{\mathfrak{F}'}(1)$ follows from (3.19). □

For $\alpha \in \Phi$ and $r \in \Gamma_\alpha$, let $U^*_{\alpha+r} = U_{\alpha+r} - U_{\alpha+r_+}$.

Lemma 3.22. *When $\mathfrak{F} \subset \operatorname{Ker}(\alpha + r_{\mathfrak{F}}(\alpha))$, we have $m_\alpha(U^*_{\alpha+r_{\mathfrak{F}}(\alpha)}) \subset K_{\mathfrak{F}} - K_{\mathfrak{F}}(1)$.*

Proof. Let $u \in U^*_{\alpha,r_{\mathfrak{F}}(\alpha)}$. Then $m_\alpha(u) = v'_\alpha(u)^{-1} u v''_\alpha(u)^{-1}$ with $v'_\alpha(u), v''_\alpha(u) \in U_{-\alpha,-r_{\mathfrak{F}}(\alpha)}$ by (16) and (21). By (51), $m_\alpha(u) \in K_{\mathfrak{F}}$ because $r_{\mathfrak{F}}(-\alpha) = -r_{\mathfrak{F}}(\alpha)$ by Example 3.11. The image of $m_\alpha(u)$ in $K_{\mathfrak{F},k}$ is not trivial because $r^*_{\mathfrak{F}}(\alpha) = r_{\mathfrak{F}}(\alpha)_+ \neq r_{\mathfrak{F}}(\alpha)$. □

3.8 Finite quotients of parahoric groups

For $H \in \mathfrak{H}$, the set Φ'_H of $\alpha \in \Phi$ such that $H = \operatorname{Ker}(\alpha + r)$ for $r \in \Gamma'_\alpha$ is never empty. Let

$$\Phi'_{\mathfrak{F}} = \cup_{\mathfrak{F} \subset H \in \mathfrak{H}} \Phi'_H.$$

Let $\Delta'_{\Phi,\mathfrak{F}} \subset \Phi^{aff}$ be the image of $\Delta^{aff}_{\Sigma,\mathfrak{F}} \subset \Sigma^{aff}$ (34) by the injection $\Sigma^{aff} \xrightarrow{\simeq} \Phi^{aff}_{red} \to \Phi^{aff}$ given by (38),(40). In other terms, $\Delta'_{\Phi,\mathfrak{F}}$ is the set of $\alpha + r \in \Phi^{aff}$ with $(\alpha + r)/2 \notin \Phi^{aff,+}$ such that $\operatorname{Ker}(\alpha + r)$ is a wall of $\mathfrak{C}$ containing $\mathfrak{F}$; note that $r = r_{\mathfrak{F}}(\alpha) \in \Gamma'_\alpha$ and that $\Phi'_{\mathfrak{F}}$ and $\Delta'_{\Phi,\mathfrak{F}}$ depend only on the set of affine hyperplanes $H \in \mathfrak{H}$ containing $\mathfrak{F}$.

Proposition 3.23. *The torus T_k is a maximal k-split torus of $K_{\mathfrak{F},k}$, the root system of $K_{\mathfrak{F},k}$ with respect to T_k is $\Phi'_{\mathfrak{F}}$. The set $\Delta'_{\Phi,\mathfrak{F}}$ is a basis of $\Phi'_{\mathfrak{F}}$. The root subgroup associated to $\alpha \in \Phi'_{\mathfrak{F}}$ is*

$$U_{\alpha,\mathfrak{F},k} = U_{\alpha+r_{\mathfrak{F}}(\alpha)}/U_{\alpha+r^*_{\mathfrak{F}}(\alpha)} = U_{\alpha+r_{\mathfrak{F}}(\alpha)}/U_{\alpha+r_{\mathfrak{F}}(\alpha)_+}.$$

Proof. [BT2, 5.1.31]. □

Remark 3.24. *When $\alpha, 2\alpha \in \Phi$, if 2α belongs to $\Phi'_{\mathfrak{F}}$ but not α, we have*

$$U_{2\alpha,\mathfrak{F},k} = U_{\alpha+r_{\mathfrak{F}}(\alpha)}/U_{\alpha+r_{\mathfrak{F}}(\alpha)_+},$$

because $2r_{\mathfrak{F}}(\alpha) = r_{\mathfrak{F}}(2\alpha)$, $U_{\alpha+r_{\mathfrak{F}}(\alpha)} = U_{\alpha+r_{\mathfrak{F}}(\alpha)_+} U_{2\alpha+r_{\mathfrak{F}}(2\alpha)}$ and Lemma 3.8.

A minimal parabolic subgroup of $K_{\mathfrak{F},k}$ is $B_{\mathfrak{F},k} = Z_k \rtimes U^+_{\mathfrak{F},k}$ of unipotent radical $U^+_{\mathfrak{F},k} = \prod_{\alpha \in \Phi'^+_{\mathfrak{F}}} U_{\alpha,\mathfrak{F},k}$ whatever ordering we choose on the set $\Phi'^+_{\mathfrak{F}}$ of positive roots of $\Phi'_{\mathfrak{F}}$. Let $N_{\mathfrak{F},k}$ be the subgroup of $K_{\mathfrak{F},k}$ generated by Z_k and $m_\alpha(u_k)$ for $\alpha \in \Phi'_{\mathfrak{F}}, u_k \in U^*_{\alpha,\mathfrak{F},k}$, and let $s_{\alpha,k} \in N_{\mathfrak{F},k}/Z_k$ and $s_\alpha(u_k) \in N_{\mathfrak{F},k}$ be the images of $m_\alpha(u_k)$. Note that $s_{\alpha,k}$ is independent of u_k. For $\alpha \in \Delta'_{\Phi,\mathfrak{F}}$, let

$G'_{\alpha,\mathfrak{F},k}$ the group generated by $U_{\alpha,\mathfrak{F},k}$ and $U_{-\alpha,\mathfrak{F},k}$ and $Z'_{\alpha,\mathfrak{F},k} = Z_k \cap G'_{\alpha,\mathfrak{F},k}$.

Proposition 3.25. [HV, 5.2 Lemma], [CE, 2.20, 6.3 (ii)] *The finite groups $K_{\mathfrak{F},k}$ and $G'_{\alpha,\mathfrak{F},k}$ for $\alpha \in \Delta'_{\Phi,\mathfrak{F}}$, are generated by strongly split BN-pairs of characteristic p:*
$B = B_{\mathfrak{F},k},\ N = N_{\mathfrak{F},k},\ S = \{s_{\alpha,k} \mid \alpha \in \Delta'_{\Phi,\mathfrak{F}}\}$, *for* $K_{\mathfrak{F},k}$,
$B = Z'_{\alpha,\mathfrak{F},k} U_{\alpha,\mathfrak{F},k},\ N = Z'_{\alpha,\mathfrak{F},k} \cup m_\alpha(u_k) Z'_{\alpha,\mathfrak{F},k},\ S = \{s_{\alpha,k}\}$, *for* $G'_{\alpha,\mathfrak{F},k}$.

A parabolic subgroup of $K_{\mathfrak{F},k}$ containing $B_{\mathfrak{F},k}$ is called standard.

Proposition 3.26. *Let $\mathfrak{F}, \mathfrak{F}'$ be two facets of $\mathfrak{C}$ such that $\mathfrak{F}$ is a facet of $\mathfrak{F}'$.*

We have $\Phi'_{\mathfrak{F}'} \subset \Phi'_{\mathfrak{F}}$. The group $M_{\mathfrak{F},k,\mathfrak{F}'}$ generated by Z_k and $U_{\alpha,\mathfrak{F}',k}$ for $\alpha \in \Phi'_{\mathfrak{F}'}$ is the Levi subgroup of a standard parabolic subgroup $Q_{\mathfrak{F},k,\mathfrak{F}'}$ of $K_{\mathfrak{F},k}$.

The parahoric subgroup $K_{\mathfrak{F}'}$ is the inverse image of $Q_{\mathfrak{F},k,\mathfrak{F}'}$ in $K_{\mathfrak{F}}$, the pro-p-parahoric subgroup $K_{\mathfrak{F}'}(1)$ is the inverse image of the unipotent radical of $Q_{\mathfrak{F},k,\mathfrak{F}'}$, and $K_{\mathfrak{F}',k} \simeq M_{\mathfrak{F},k,\mathfrak{F}'}$.

Proof. [BT2, 4.6.33, 5.1.32]. □

Corollary 3.27. *The reduction map $K_{\mathfrak{F}} \to K_{\mathfrak{F},k}$ induces isomorphisms*

$$K_{\mathfrak{F}'}\backslash K_{\mathfrak{F}}/K_{\mathfrak{F}'} \simeq Q_{\mathfrak{F},k,\mathfrak{F}'}\backslash K_{\mathfrak{F},k}/Q_{\mathfrak{F},k,\mathfrak{F}'}, \quad K_{\mathfrak{F}'}(1)\backslash K_{\mathfrak{F}}/K_{\mathfrak{F}'}(1) \simeq U^+_{\mathfrak{F},k,\mathfrak{F}'}\backslash K_{\mathfrak{F},k}/U^+_{\mathfrak{F},k,\mathfrak{F}'}.$$

Corollary 3.28. *The pro-p-Iwahori subgroup $I(1)$ and the Iwahori subgroup I are the inverse images in $K_{\mathfrak{F}}$ of $U^+_{\mathfrak{F},k}$ and of $B_{\mathfrak{F},k}$.*

Remark 3.29. *A pro-p-Sylow subgroup of $K_{\mathfrak{F}}$ is an open subgroup of finite index prime to p. The pro-p-Iwahori subgroup $I(1)$ is a pro-p-Sylow subgroup of $K_{\mathfrak{F}}$.*

Proof. The index of $I(1)$ in $K_{\mathfrak{F}}$ is equal to the index of $U^+_{\mathfrak{F},k}$ in the finite group $K_{\mathfrak{F},k}$, and the Bruhat decomposition implies that this index is prime to p. □

For $\alpha \in \Phi'_{\mathfrak{F}}$, the parahoric subgroup $K_{\mathfrak{F}}$ contains $U_{\alpha+r_{\mathfrak{F}}(\alpha)}$ (51) and $r^*_{\mathfrak{F}}(\alpha) = r_{\mathfrak{F}}(\alpha)_+$. By Prop. 3.23, the reduction map $K_{\mathfrak{F}} \to K_{\mathfrak{F},k}$ induces an isomorphism

$$u \mapsto u_k : U^*_{\alpha+r_{\mathfrak{F}}(\alpha)} = U_{\alpha+r_{\mathfrak{F}}(\alpha)} - U_{\alpha+r_{\mathfrak{F}}(\alpha)_+} \to U^*_{\alpha,\mathfrak{F},k} = U_{\alpha,\mathfrak{F},k} - \{1\},$$

and sends $m_\alpha(u) \in K_{\mathfrak{F}}$ to $m_\alpha(u_k) \in K_{\mathfrak{F},k}$. We denote by $s_\alpha \in N/Z = W_0$, $s_{\alpha+r_{\mathfrak{F}}(\alpha)} \in N/Z_0, s_\alpha(u) \in N/Z_0(1)$ the images of $m_\alpha(u)$. For $u, u' \in U^*_{\alpha+r_{\mathfrak{F}}(\alpha)}$ we have $m_\alpha(u')^{-1}m_\alpha(u) \in Z \cap K_{\mathfrak{F}}$ and $Z \cap K_{\mathfrak{F}} = Z \cap I = Z_0$ by Prop. 3.15. We have bijective maps

$$B_{\mathfrak{F},k}m_\alpha(u_k)B_{\mathfrak{F},k}/B_{\mathfrak{F},k} \simeq U^+_{\mathfrak{F},k}m_\alpha(u_k)U^+_{\mathfrak{F},k}/U^+_{\mathfrak{F},k} \simeq U_{\alpha,\mathfrak{F},k}.$$

Corollary 3.30. *For $\alpha \in \Phi'_{\mathfrak{F}}$ and $u \in U^*_{\alpha,\mathfrak{F}}$, we have*

$$Im_\alpha(u)I/I \simeq U_{\alpha,\mathfrak{F},k} \simeq I(1)m_\alpha(u)I(1)/I(1).$$

As $U_{\alpha,\mathfrak{F},k}$ does not depend on the choice of $u \in U^*_{\alpha,\mathfrak{F}}$, the integer $|Im_\alpha(u)I/I| = |U_{\alpha,\mathfrak{F},k}|$ is the same for all $u \in U^*_{\alpha,\mathfrak{F}}$.

For $s \in S(\mathfrak{C})$ and $\mathfrak{F}_s$ the face of $\mathfrak{C}$ contained in the wall H_s fixed by s, the set $\Delta'_{\mathfrak{F}_s}$ has a single element α_s. We denote $q_s := |Im_{\alpha_s}(u)I/I|, q_{s,k} := |U_{\mathfrak{F}_s,k,\alpha_s}|$. Cor. 3.30 implies:

Corollary 3.31. *For $s \in S(\mathfrak{C})$, $q_s = q_{s,k}$.*

3.9 The Iwahori Weyl groups

We recall that the group G' generated by $\cup_{\alpha\in\Phi}U_\alpha$ (section 3.2) and we define the subgroup G^{aff} of G generated by the parahoric subgroups of G. The subgroups G' and G^{aff} of G are normal and (Prop. 3.16)

$$G = ZG', \quad G^{aff} = Z_0G'.$$

For a subgroup X of G, we set $X' = X \cap G', X^{aff} = X \cap G^{aff}$. We have $(Z^{aff}, N^{aff}) = (Z_0Z', Z_0N')$.

Definition 3.32. *We call*

$$W_0 = N/Z,\ W^{aff} = N^{aff}/Z_0,\ W = N/Z_0,\ W^{aff}(1) = N^{aff}/Z_0(1),\ W(1) = N/Z_0(1)$$

the finite, affine, Iwahori, pro-p-affine, pro-p-Iwahori, Weyl groups of G.

We note that $W_0 \simeq W_0^{aff} \simeq W'_0$ and $W^{aff} \simeq W', W^{aff}(1) \simeq W'(1)$ for the natural definitions $W_0^{aff} = N^{aff}/Z^{aff}, W'_0 = N'/Z', W' = N'/Z'_0, W'(1) = N'/Z'_0(1)$. The action of N^{aff} on $\mathfrak{A}$ identifies $W(\mathfrak{H})$ with W^{aff} and $S(\mathfrak{C})$ with a subset S^{aff} of W^{aff}. The group W_0 identifies with a subgroup of $W(\mathfrak{H})$ hence of W^{aff} and $S = S^{aff} \cap W_0$.

Most of the properties of this section are encapsulated in an important theorem of the theory of Bruhat-Tits [BT2, 5.2.12]:

Theorem 3.33. (G^{aff}, I, N^{aff}) *is a double Tits system of Coxeter systems*

$$(W^{aff}, S^{aff}), \quad (W_0, S)$$

and the inclusion $G^{aff} \subset G$ is $I - N^{aff}$-adapted of connected type.

We recall that by the first assertion, (G^{aff}, I, N^{aff}) and $(G^{aff}, B^{aff}, N^{aff})$ are Tits systems [BT1, 1.2.6, 5.1.1] hence satisfy the properties:

(T1) $I \cup N^{aff}$ generates G^{aff} and $I \cap N^{aff}$ is normal in N^{aff}.

(T3) For all $s \in S^{aff}, w \in W^{aff}$, we have $sIw \subset IwI \cup IswI$.

(T4) For all $s \in S^{aff}$, we have $sIs \neq I$.

and the same properties for (S, W_0, B^{aff}) instead of (S^{aff}, W^{aff}, I).

By the second assertion see the definitions [BT1, 1.2.13, 4.1.3].

Proposition 3.34. Bruhat Decompositions for G^{aff} [BT1, 1.2.7]. *We have*

$$G^{aff} = B^{aff} N^{aff} B^{aff} = I N^{aff} I = I(1) N^{aff} I(1).$$

The maps $n \mapsto B^{aff} n B^{aff}, n \mapsto InI, n \mapsto I(1)nI(1)$ induce bijections

$$W_0 \simeq B^{aff} \backslash G^{aff} / B^{aff}, \quad W^{aff} \simeq I \backslash G^{aff} / I, \quad W^{aff}(1) \simeq I(1) \backslash G^{aff} / I(1).$$

Proof. The assertions $IN^{aff}I = I(1)N^{aff}I(1)$, $W^{aff}(1) \simeq I(1)\backslash G^{aff}/I(1)$ involving $I(1)$ use that $I = I(1)Z_0$ and $Z_0 \subset N^{aff}$ for the equality, and the N-equivariant semi-direct product $Z_0 = Z_0(1) \rtimes Z_0^{(p)}$ (50) with the disjoint decomposition

$$InI = \sqcup_{t \in Z_0^{(p)}} I(1)tnI(1) \quad \text{for all } n \in N,$$

for the isomorphism. □

Note that the equality $[InI : I] = [I(1)nI(1) : I(1)]$ of indices for $n \in N$ (Cor. 3.30) follows easily from $I = \sqcup_{t \in Z_0^{(p)}} I(1)t, \ InI = \sqcup_{t \in Z_0^{(p)}} I(1)tnI(1)$.

We have a similar Bruhat decomposition for G. Let $B = ZU^+$.

Proposition 3.35. Bruhat Decompositions for G. *We have*

$$G = BNB = INI = I(1)NI(1).$$

The maps $n \mapsto BnB, n \mapsto InI, n \mapsto I(1)nI(1)$ induce bijections

$$W_0 \simeq B\backslash G/B, \ W \simeq I\backslash G/I, \ W(1) \simeq I(1)\backslash G/I(1), \ InI/I \simeq I(1)nI(1)/I(1).$$

Proof. For B and I, the equalities and the isomorphism with W_0 follow from $G = G^{aff}Z, N = N^{aff}Z, B = B^{aff}Z$ and from Prop. 3.34. For the isomorphism with W_0 see also [BT1, 5.1.32]. The isomorphism with W follows from [BT1, 4.2.2 (iii)] where W, Z_0 are denoted by $\hat{W}, H$.

We deduce $G = I(1)NI(1)$ from $G = INI$, $I = Z_0I(1)$ and $Z_0 \subset N$. We have $InI/I \simeq I(1)nI(1)/I(1)$ because

$$I/(I \cap nIn^{-1}) = I(1)Z_0/(I(1) \cap nI(1)n^{-1})Z_0 \simeq I(1)/(I(1) \cap nI(1)n^{-1}).$$

If $I(1)nI(1) = I(1)n'I(1)$ we have $InI = In'I$ and the images w, w' of n, n' in W are equal. As $I = Z_0I(1)$, the double coset $InI = I(1)nZ_0I(1)$ is a disjoint union of $I(1)nzI(1)$ for $z \in Z_0/Z_0(1) = Z_k$. This implies that the images of n, n' in $W(1)$ are equal. □

The Iwahori decomposition of I implies that

$$N \cap I = Z_0. \tag{54}$$

Let $\mathrm{Norm}_{\mathfrak{C}}$ be the N-stabilizer of the alcove $\mathfrak{C}$. We denote by $\Omega \subset W$ and $\Omega(1) \subset W(1)$ the images of $\mathrm{Norm}_{\mathfrak{C}}$. We have

$$N = N^{aff}\,\mathrm{Norm}_{\mathfrak{C}}, \quad \mathrm{Norm}_{\mathfrak{C}} \cap N^{aff} = \mathrm{Norm}_{\mathfrak{C}} \cap I = Z_0, \quad \mathrm{Norm}_{\mathfrak{C}} \cap Z = \mathrm{Ker}\, v, \tag{55}$$

because W^{aff} acts simply transitively on the set of alcoves of $\mathfrak{A}$, a translation normalizing $\mathfrak{C}$ is trivial (Lemma 3.7), $z \in Z$ acts on $\mathfrak{A}$ by translation by $-v(z)$ (18), and (54).

The G-stabilizer of the alcove $\mathfrak{C}$ is also the G-normalizer of I because I is the fixator of $\mathfrak{C}$ in the kernel of the Kottwitz morphism κ_G (Def. 3.14) and $\mathfrak{C}$ is the only alcove of the (adjoint) Bruhat-Tits building fixed by I (this follows from the comment after (46) and $G' \subset \mathrm{Ker}\,\kappa_G$). The G-stabilizer of the apartment $\mathfrak{A}$ is N (after Def. 3.12). The Iwahori group I acts transitively on the apartments of the (adjoint) Bruhat-Tits building containing $\mathfrak{C}$, as $U_{\mathfrak{C}}$ has this property (paragraph above (48)) and $I = Z_0 U_{\mathfrak{C}}$ (Prop. 3.16). We deduce that the apartments containing $\mathfrak{C}$ are in bijection with I/Z_0 and

$$\mathrm{Norm}_{\mathfrak{C}}\, I = \text{the } G\text{-normalizer of } I = \text{the } G\text{-stabilizer of } \mathfrak{C}.$$

Proposition 3.36. *1. The groups*

$$G/G^{aff} \simeq Z/Z^{aff} \simeq N/N^{aff} \simeq W/W^{aff} \simeq \mathrm{Norm}_{\mathfrak{C}}/Z_0 \simeq \Omega \tag{56}$$

are commutative and finitely generated.

2. $G = \mathrm{Norm}_{\mathfrak{C}}\, G^{aff} = \mathrm{Norm}_{\mathfrak{C}}\, G'$ *and* $\mathrm{Norm}_{\mathfrak{C}}$ *normalizes* I, N^{aff} *and* I', N'.

3. G' *satisfies the Bruhat decompositions* $G' = B'N'B' = I'N'I'$.
The maps $n \mapsto B'nB', n \mapsto I'nI'$ *induce isomorphisms*

$$W_0 \simeq B'\backslash G'/B', \;\; W^{aff} \simeq I'\backslash G'/I'.$$

Proof. 1. The isomorphisms (56) are clear. The groups are commutative and finitely generated because $Z_0 \subset Z^{aff}$ and we recall that the group $\Lambda = Z/Z_0$ is commutative and finitely generated (above Prop. 3.15).

2. The equalities for G follow from (56) and $G^{aff} = Z_0 G'$ with $Z_0 \subset \mathrm{Norm}_{\mathfrak{C}}$.

$\mathrm{Norm}_{\mathfrak{C}}$ normalizes I by the remarks made before the proposition; it normalizes N^{aff} because G normalizes G^{aff} (beginning of this section); it normalizes the intersections I', N' of I, N with G' because G' is normal in G (beginning of this section).

3. This follows from

$$(G^{aff}, B^{aff}, N^{aff}, I, I(1)) = (Z_0 G', Z_0 B', Z_0 N', Z_0 I', Z_0(1) I'(1))$$

and the Bruhat decompositions of G^{aff} (Prop. 3.34). We have

$$W_0 \simeq B^{aff}\backslash G^{aff}/B^{aff} = Z_0 B'\backslash Z_0 G'/Z_0 B' \simeq B'\backslash G'/B',$$
$$W^{aff} \simeq I\backslash G^{aff}/I = Z_0 I'\backslash Z_0 G'/Z_0 I' \simeq I'\backslash G'/I',$$

because $Z_0 \cap G' \subset I' \cap B'$ for the right isomorphisms. □

We note that (55) implies

$$W = W^{aff} \rtimes \Omega, \quad W(1) = W^{aff}(1)\Omega(1), \quad W^{aff}(1) \cap \Omega(1) = Z_k. \tag{57}$$

The extension $W(1) \to W$ of kernel Z_k does not split in general [VigMA].

Remark 3.37. [HRa, Lemma 17] [**?**, Lemma 1.3] *The kernel of the Kottwitz morphism* κ_G *is* G^{aff}. *By Prop. 3.36, the image of* κ_G *is isomorphic to* Ω. *By* (55)*, the* G^{aff}*-stabilizer of* $\mathfrak{C}$ *is* I *and the* G'*-stabilizer of* $\mathfrak{C}$ *is* I'.

The set S^{aff} is invariant by conjugation by Ω, hence the length ℓ of the Coxeter group (W^{aff}, S^{aff}) is invariant by conjugation by Ω. The length extends to a map on W and $W(1)$, still called a length and denoted by ℓ :

$$\ell(\tilde{w}) = \ell(w) = \ell(w'), \tag{58}$$

for $\tilde{w} \in W(1)$ lifting $w \in W$ and $w' \in W^{aff}$ such that $w = w'u, u \in \Omega$. The set of elements of length 0 in W, resp. $W(1)$, is Ω, resp. $\Omega(1)$.

The cardinal q_w of $|InI/I| = |I(1)nI(1)/I(1)|$ for $n \in N$ of image $w \in W$ or $W(1)$ (Prop. 3.35, Cor. 3.31), can be explicitely computed from the q_s for $s \in S^{aff}$:

Proposition 3.38. (Braid relations for W) *For* $w_1, w_2 \in W$, $q_{w_1w_2} = q_{w_1}q_{w_2}$ *is equivalent to* $\ell(w_1) + \ell(w_2) = \ell(w_1w_2)$.

Proof. The map $n \mapsto [I : I \cap nIn^{-1}]$ is invariant by conjugation by $N_{\mathfrak{C}}$ and $\ell(w), q_w$ are invariant by conjugation by Ω.

The braid relations for W^{aff} imply the braid relations for W, and follow from the properties of the affine Tits system (G^{aff}, I, N^{aff}). This is well known but the only reference that I am aware of is [Bki, IV §2 Exercices. 3,8,23].

□

4 Iwahori-Masumoto presentations

4.1 Generalities on Hecke rings

In this preliminary subsection, G is an arbitrary locally profinite group containing a compact open subgroup I, and R is a commutative ring.

The Hecke R-algebra $\mathcal{H}_R(G, I)$ of I in G is the ring of I-bi-invariant compactly supported functions from G to R, with the convolution product $*$. The value at I is an isomorphism from the intertwining algebra $\mathrm{End}_{RG} R[I\backslash G]$ onto $\mathcal{H}_R(G, I)$. We have

$$\mathcal{H}_R(G, I) = R \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}}(G, I).$$

We call $\mathcal{H}_{\mathbb{Z}}(G, I)$ the Hecke ring of I in G.

For $g \in G$, the characteristic function of IgI is denoted by T_g. The Hecke R-algebra $\mathcal{H}_R(G, I)$ is a free R-module of basis $(T_g)_{g \in I\backslash G/I}$.

For $g, h \in G$, the convolution product $T_g * T_h$ is

$$T_g * T_h = \sum_{x \in I\backslash IgIhI/I} (T_g * T_h)(x) T_x, \tag{59}$$

where $(T_g * T_h)(x)$ is the cardinal of $(IgI \cap xIh^{-1}I)/I$ [Viglivre, I.3.4 (3)], or equivalently,

$$(T_g * T_h)(x) \text{ is the cardinal of } \{u \in Y_g \mid u^{-1}x \in gIhI\}, \tag{60}$$

where Y_g is a system of representatives of the cosets $I/(gIg^{-1} \cap I)$. A system of representatives of the coset IgI/I is $Y_g g$. The number of elements of IgI/I is denoted by q_g. The linear map

$$d : \mathcal{H}_R(G, I) \to R, \quad T_g \mapsto q_g \quad (g \in G) \tag{61}$$

respects the product [Viglivre, I.3.5]. For $g, h \in G$, the formula (59) implies

$$q_g q_h = \sum_{x \in I \backslash IgIhI/I} (T_g \circ T_h)(x) q_x$$

For $x \in IgIhI$, $(T_g * T_h)(x)$ is a positive integer $\leq \min(q_g, q_h)$ and we have $(T_g * T_h)(gh) \geq 1$. Therefore $q_g q_h \geq q_{gh}$ and

(62) $$q_g q_h = q_{gh} \quad \text{is equivalent to } T_g * T_h = T_{gh}.$$

We have

(63) $$T_g * T_h = T_{gh} \quad \text{if } g \text{ or } h \text{ normalizes } I.$$

4.2 Iwahori-Matsumoto presentation

The Hecke ring of the Iwahori subgroup I, resp. pro-p-Iwahori subgroup $I(1)$, in the reductive group G is called the Iwahori Hecke ring $\mathcal{H}$, resp. the pro-p-Iwahori Hecke ring $\mathcal{H}(1)$, of G. For $n \in N$ of image w in W or in $W(1)$, we write $T_n = T_w$ in $\mathcal{H}$ or in $\mathcal{H}(1)$.

Proposition 4.1. *The Iwahori Hecke ring $\mathcal{H}$, resp. pro-p-Iwahori Hecke ring $\mathcal{H}(1)$, is a free $\mathbb{Z}$-module with basis $(T_w)_{w \in W}$, resp. $(T_w)_{w \in W(1)}$, satisfying the braid relations*

$$T_w * T_{w'} = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w').$$

Proof. For the basis, Prop. 3.35. For the braid relations, (62), Prop. 3.35 and 3.38. □

Let $s \in S^{aff}$. We denote by $S_s^{aff}(1)$ the inverse image of s in $W(1)$. Let $\tilde{s} \in S_s^{aff}(1)$. The elements T_s in $\mathcal{H}$ and $T_{\tilde{s}}$ in $\mathcal{H}(1)$ satisfy quadratic relations. It is possible to prove them using Bruhat-Tits theory as in [VigMA] when $\mathbf{G}$ is split. But we will obtain them, using Prop. 3.25, by following the carefully written proofs of the quadratic relations in the Hecke algebras of finite groups with a strongly split BN-pair of characteristic p over a large field of characteristic p, by Cabanes and Enguehard [CE, Chapter 6].

We denote by H_s the wall of the alcove $\mathfrak{C}$ fixed by s. A facet $\mathfrak{F}$ of $\mathfrak{C}$ contained in H_s is either the face $\mathfrak{F}_s$ or is a facet of $\mathfrak{F}_s$. Let $\alpha \in \Phi^+_{red}$ be the reduced $\mathfrak{C}$-positive root and $r \in \Gamma_\alpha$ such that $H_s = \mathrm{Ker}(\alpha + r)$. If $r \notin \Gamma'_\alpha$, then $2\alpha \in \Phi$. The affine root $A_s = \alpha_s + r_s$

$$\alpha_s + r_s = \alpha + r \text{ when } r \in \Gamma'_\alpha \text{ and } \alpha_s + r_s = 2\alpha + 2r \text{ when } r \notin \Gamma'_\alpha,$$

belongs to $\Delta'_{\Phi, \mathfrak{F}}$ (subsection 3.8). For $u \in U^*_{A_s} = U_{A_s} - U_{A_s+}$ we denote by $m_s(u)$ the unique element of $N \cap U_{-\alpha_s} u U_{-\alpha_s}$. We have $m_s(u) \in U_{-A_s} u U_{-A_s}$. The elements $u, m_s(u)$ belong to $K_{\mathfrak{F}}$. The image $s(u)$ of $m_s(u)$ in $W(1)$ belongs to $S_s^{aff}(1)$. The Hecke operator $T_s \in \mathcal{H}$ belongs to the Hecke subring $\mathcal{H}(K_{\mathfrak{F}}, I)$ by Lemma 3.22 and $T_{s(u)} \in \mathcal{H}(1)$ belongs to the Hecke subring $\mathcal{H}(K_{\mathfrak{F}}, I(1))$.

The Iwahori subgroup I is the inverse image by the reduction map $K_{\mathfrak{F}} \to K_{\mathfrak{F},k}$ of a minimal Borel subgroup $B_{\mathfrak{F},k}$ of $G_{\mathfrak{F},k}$ and $I(1)$ is the inverse image of the unipotent radical $U^+_{\mathfrak{F},k}$ of $B_{\mathfrak{F},k}$ (Cor. 3.30). The Hecke rings $\mathcal{H}(K_{\mathfrak{F}}, I)$ and $\mathcal{H}_{\mathfrak{F},k} = \mathcal{H}(K_{\mathfrak{F},k}, B_{\mathfrak{F},k})$ are isomorphic and the Hecke rings $\mathcal{H}(K_{\mathfrak{F}}, I(1))$ and $\mathcal{H}_{\mathfrak{F},k}(1) = \mathcal{H}(K_{\mathfrak{F},k}, U^+_{\mathfrak{F},k})$ are isomorphic.

The finite group $K_{\mathfrak{F},k}$ is a strongly split BN-pair of characteristic p with $B = B_{\mathfrak{F},k}, N = N_{\mathfrak{F},k}$ (Prop. 3.25). The root group of $K_{\mathfrak{F},k}$ defined by A_s is $U_{A_s,k} = U_{A_s}/U_{A_s,+}$. The root group of $K_{\mathfrak{F},k}$ defined by $-A_s$ is $U_{-A_s,k} = U_{-A_s}/U_{-A_s,+}$. The reduction u_k of u in $K_{\mathfrak{F},k}$

belongs to $U^*_{A_s,k} = U_{A_s,k} - \{1\}$ and the reduction $m_s(u_k)$ of $m_s(u)$ is the unique element of $N_{\mathfrak{F},k} \cap U_{-A_s,k} u_k U_{-A_s,k}$. We have

$$s(u)^2 = m_s(u_k)^2 \in Z_k.$$

We still denote by s the image of $m_s(u_k)$ in $W_{\mathfrak{F},k} = N_{\mathfrak{F},k}/Z_k$. By Cor. 3.30,

$$q_s = |U_{A_s,k}|.$$

The quadratic relations satisfied by $T_s \in \mathfrak{H}, T_s \in \mathcal{H}(K_{\mathfrak{F}}, I), T_s \in \mathcal{H}_{\mathfrak{F},k}$ are the same.

Proposition 4.2. (Quadratic relations in $\mathfrak{H}$)

$$T_s * T_s = q_s + (q_s - 1)T_s \quad \textit{in} \ \ \mathfrak{H}.$$

Proof. In $\mathbb{Z}[1/p] \otimes \mathcal{H}_{\mathfrak{F},k}$ [CE, Thm. 3.3]. The quadratic relation holds true in $\mathcal{H}_{\mathfrak{F},k}$ because all the elements belong to $\mathcal{H}_{\mathfrak{F},k}$. □

By a general property of Hecke algebras [Viglivre, I.3.5] :

Corollary 4.3. *The linear map $\mathcal{H} \to \mathbb{Z}$:*

$$T_w = T_{s_1} \dots T_{s_{\ell(w)}} T_u \mapsto q_w = q_{s_1} \dots q_{s_{\ell(w)}} \quad \text{if} \quad w = s_1 \dots s_{\ell(w)} u \quad (s_i \in S^{aff}, u \in \Omega),$$

is a ring homomorphism.

The quadratic relations satisfied by $T_{s(u)} \in \mathfrak{H}(1), T_{s(u)} \in \mathcal{H}(K_{\mathfrak{F}}, I(1)), T_{m_s(u_k)} \in \mathcal{H}_{\mathfrak{F},k}(1)$ are the same. We need more notation. The group $G'_{s,k}$ generated by $U_{A_s,k} \cup U_{-A_s,k}$ is a strongly split BN-pair of characteristic p with $B = Z_{k,s} U_{A_s,k}, \ N = Z_{k,s} \cup m_{s(u_k)} Z_{k,s}$ and $Z_{k,s} := G'_{A_s,k} \cap Z_k$ (Prop. 3.25). We have $m_s(u_k) \in G'_{s,k}$ and $m_s(u_k)^2 = s(u)^2$ is in $Z_{k,s}$. Let

$$c_s = (q_s - 1)|Z_{k,s}|^{-1} \sum_{t \in Z_{k,s}} t.$$

We identify $\mathbb{Z}[Z_k]$ with a subring of $\mathcal{H}_{\mathfrak{F},k}(1)$ by the map $\sum_{t \in Z_k} c(t)t \mapsto \sum_{t \in Z_k} c(t)T_t$, using the braid relations.

Proposition 4.4. (Quadratic relations in $\mathfrak{H}(1)$)

$$T_{s(u)} * T_{s(u)} = q_s s(u)^2 + c_{s(u)} T_{s(u)} \quad \textit{in} \ \ \mathfrak{H}(1),$$

where $c_{s(u)} = c_s$ if the order of $Z_{k,s}$ is $q_s - 1$ (for example if $\mathbf{G}$ is F-split). In general,

$$c_{s(u)} = \sum_{t \in Z_{k,s}} c_{s(u)}(t)t$$

for positive integers $c_{s(u)}(t)$ constant on the coset $t\{xs(x)^{-1} \,|x \in Z_k\}$, $c_{s(u)}(t) = c_{s(u)}(t^{-1})$ for $t \in Z_{k,s}$, of sum $q_s - 1$, and

$$c_{s(u)} \equiv c_s \quad \textit{modulo} \ \ p.$$

The proof is divided in three steps following the proof given by Cabanes and Enguehard in [CE, Chapter 6] for the quadratic relations in the Hecke algebra of a p-Sylow subgroup of a finite reductive group over a field of characteristic p (not over the ring R).

Step 1 *The intersections*

$$(64) \qquad m_s(u_k) U_{A_s,k} m_s(u_k) \ \cap \ U_{A_s,k} m_s(u_k) t U_{A_s,k}$$

for $t \in Z_{k,s}$ are disjoint and exhaust $m_s(u_k) U^*_{A_s,k} m_s(u_k)$ (where $U^*_{A_s,k} = U_{A_s,k} - \{1\}$).

Proof. By the Bruhat decompositions of the strongly split BN pair of characteristic p generating the group $G'_{s,k}$ (Prop. 3.25), we have

$$G'_{s,k} = \sqcup_{n \in N_{k,s}} U_{A_s,k} n U_{A_s,k}$$

where $N_{k,s} = Z_{k,s} \sqcup m_s(u_k) Z_{k,s}$. The disjointness of the sets (64) follows from the disjointness of the $U_{A_s,k} m_s(u_k) t U_{A_s,k}$ for $t \in Z_{k,s}$. The union $m_s(u_k) U_{A_s,k} m_s(u_k) \cap U_{A_s,k} m_s(u_k) Z_{k,s} U_{A_s,k}$ of the sets (64) does not contain $m_s(u_k)^2 \in Z_{k,s}$. It is equal to $m_s(u_k) U^*_{A_s,k} m_s(u_k)$ because $m_s(u_k) U^*_{A_s,k} m_s(u_k) = U^*_{-A_s,k} m_s(u_k)^2$ is contained in $U_{A_s,k} m_s(u_k) Z_{k,s} U_{A_s,k}$. □

Step 2 *Let $c_{s(u)}(t)$ denote the cardinality of* (64) *and $c_{s(u)} = \sum_{t \in Z_{k,s}} c_{s(u)}(t) t$. We have $T_{s(u)} * T_{s(u)} = q_s s(u)^2 + c_{s(u)} T_{s(u)}$ in $\mathcal{H}(1)$.*

Proof. Proof of Prop. 6.8 (iii) in [CE]. □

Step 3 *The integers $c_{s(u)}(t)$ for $t \in Z_{k,s}$ satisfy:*

1. *Their sum is $q_s - 1$.*
2. *$c_{s(u)}(t) = c_{s(u)}(tt')$ for $t \in Z_{k,s}, t' \in \{s(x)x^{-1} \mid x \in Z_k\}$.*
3. *$c_{s(u)}(t) = c_{s(u)}(t^{-1})$ for $t \in Z_{k,s}$.*
4. *They are constant modulo p.*
5. *They are positive.*
6. *When the order of $Z_{k,s}$ is $q_s - 1$, in particular when $\mathbf{G}$ is F-split, $c_{s(u))} = c_s = \sum_{t \in Z_{k,s}} t$.*

Proof. 1. The sets (64) of cardinal $c_{s(u)}(t)$ for $t \in Z_{k,s}$, form a partition of $U_{A_s,k} - \{1\}$ by Step 1. Therefore

$$\sum_{t \in Z_{k,s}} c_{s(u)}(t) = |U_{A_s,k}| - 1 = q_s - 1.$$

2. The group $\{s(x)x^{-1} = sxs^{-1}x^{-1} \mid x \in Z_k\}$ is contained in $Z_{k,s}$ because $m_s(u_k)$ normalizes Z_k and Z_k normalizes $G'_{s,k}$. We have

$$c_{m_s(u_k)}(t) = c_{m_s(u_k)}(tt') \quad \text{for } t' \in \{s(x)x^{-1} \mid x \in Z_k\}$$

because $x \in Z_k$ commutes with $c_{s(u)} T_{s(u)}$ by the quadratic relations, $x T_{s(u)} = T_{s(u)} s(x)$, x commutes $T_{s(u)} * T_{s(u)}$ and Z_k is commutative.

3. The set (64) mutiplied by $m_s(u_k)^{-2} \in Z_{k,s}$, equal to $U_{A_{-s},k} \cap U_{A_s,k} m_s(u_k)^{-1} t U_{A_s,k}$, has $c_{s(u)}(t)$ elements. Its image by the inverse map has also $c_{s(u)}(t)$ elements. It is equal to $U_{A_{-s},k} \cap U_{A_s,k} t^{-1} m_s(u_k) U_{A_s,k} = U_{A_{-s},k} \cap U_{A_s,k} m_s(u_k) s(t^{-1}) U_{A_s,k}$ which has $c_{s(u)}(s(t)^{-1})$ elements. By 2, $c_{s(u)}(s(t)^{-1}) = c_{s(u)}(s(t)^{-1} t')$ for $t' = s(t) t^{-1}$ hence $c_{s(u)}(t) = c_{s(u)}(t^{-1})$.

4. The function $t \mapsto c_{m_s(u_k)}(t) : Z_{k,s}/\{s(x)x^{-1} \mid x \in Z_k\} \to \mathbb{Z}$ is constant modulo p [CE, Prop. 6.10 (i) and (ii)]. We repeat the arguments. If the function is not constant modulo p, there is some non-trivial character ψ_0 of $Z_{k,s}$ with values in k^* such that

$$b = \sum_{t \in Z_{k,s}} c_{m_s(u_k)}(t) \psi_0(t) \neq 0.$$

The Hecke k-algebra $\mathcal{H}_k(G'_{s,k}, U_{A_s,k})$ is equal to $k[Z_{k,s}] + k[Z_{k,s}] T_{m_s(u_k)}$ with the braid and quadratic relations

$$T_{m_s(u_k)} t = s(t) T_{m_s(u_k)}, \quad T^2_{m_s(u_k)} = c_{m_s(u_k)} T_{m_s(u_k)}.$$

One may define $|Z_{k,s}|+2$ characters of $\mathcal{H}_k(G'_{s,k},U_{A_s,k})$:

ξ_0 equal to ψ_0 on $t\in Z_{k,s}$ and equal to b on $T_{m_s(u_k)}$.

ξ equal to ψ on $t\in Z_{k,s}$ and equal to 0 on $T_{m_s(u_k)}$ for every character ψ of $Z_{k,s}$.

ξ_{st} trivial on $Z_{k,s}$ and equal to -1 on $T_{m_s(u_k)}$ because $\sum_{t\in Z_{k,s}} c_{m_s(u_k)}(t)=q_s-1$ is congruent to -1 modulo p.

The Hecke algebra $\mathcal{H}_k(G'_{s,k},U_{A_s,k})$ is isomorphic to $\mathrm{End}_{k[G'_{s,k}]}$ c-$\mathrm{Ind}_{U_{A_s,k}}^{G'_{s,k}} 1$. The simple modules of $\mathcal{H}_k(G'_{s,k},U_{A_s,k})$ are in bijection with the isomorphism classes of indecomposable summands of the k-representation c-$\mathrm{Ind}_{U_{A_s,k}}^{G'_{s,k}} 1$ [CE, Proof of (i) in Thm. 6.10]. This representation is a direct sum over the characters ψ of $Z_{k,s}$ inflated to $B'_{s,k}=Z_{k,s}U_{A_s,k}$, of the representations c-$\mathrm{Ind}_{B'_{s,k}}^{G'_{s,k}}\psi$. The restriction of c-$\mathrm{Ind}_{B'_{s,k}}^{G'_{s,k}}\psi$ to $B'_{s,k}$ is isomorphic to the direct sum of ψ and of the natural action of $B'_{s,k}$ on the indecomposable regular module $k[U_{A_s,k}]$. The only character of $G'_{s,k}$ is trivial. We deduce that the representation c-$\mathrm{Ind}_{U_{A_s,k}}^{G'_{s,k}} 1$ is the direct sum of $|Z_{k,s}|+1$ indecomposable subrepresentations [CE, Lemma 6.4]. The Hecke algebra $\mathrm{End}_{k[G'_{s,k}]}$ c-$\mathrm{Ind}_{U_{A_s,k}}^{G'_{s,k}} 1$ cannot have $|Z_{k,s}|+2$ characters. This implies that we could not make our assumption, in other terms the integers $c_{m_s(u_k)}(t)$ for $t\in Z_{k,s}$ are constant modulo p.

5. Their sum q_s-1 is not divisible by p, hence they are not divisible by p, in particular they are not 0, they are positive.

6. They are all equal to 1 if and only the order of $Z_{k,s}$ is q_s-1 if and only if $c_{s(u)}=c_s=\sum_{t\in Z_{k,s}} t$.

When the group $\mathbf{G}$ is F-split, the order of $Z_{k,s}$ is q_s-1 and $q_s=q$ the order q of the residue field k of F [VigMA, section 2.2].

This ends the proof of Prop. 4.4. □

Remark *The positive integers $c_{s(u)}(t)$ for $t\in Z_{k,s}$ are all equal if and only if $c_{s(u)}=c_s$. When this is the case, the order of $Z_{k,s}$ divides q_s-1.*

Example where the order of $Z_{k,s}$ does not divide q_s-1 [KoXu, Remark 3.8]: we suppose that q is odd. Let G denote the F-rational points of the unramified unitary group $U(2,1)(E/F)$ where E/F is a quadratic unramified extension of residue field k_E. Then Z_k identifies with $k_E^*\times\mathbf{U}(1)(k_E/k)$ and we have $S=\{s,s'\}$ where

$q_{s'}=q$, the group $Z_{k,s'}$ identifies with $k^*=k-\{0\}$ of order $q_{s'}-1$ and $c_{s'(u)}=c_{s'}$,

$q_s=q^3$, the group $Z_{k,s}$ identifies with $k_E^*=k_E-\{0\}$ of order q^2-1 not dividing q_s-1, the positive integers $c_{s(u)}(t)=1$ for $t\in k^*$ and $c_{s(u)}(t)=q+1$ for $t\in k_E^*-k^*$ are not constant. We note that

$$c_{s(u)}=c_s-qc_{s'}\equiv c_s \mod q$$

Lemma 4.5. *The set $\{m_s(u')\mid u'\in U^*_{A_s,k}\}$ is a coset $Z_{k,s}m_s(u_k)$.*

Proof. By the Bruhat decomposition $G'_{s,k}=Z_{k,s}U_{A_s,k}\sqcup U_{A_s,k}m_s(u_k)Z_{k,s}U_{A_s,k}$, there exists a map $t:U^*_{A_s,k}\to Z_{k,s}$ defined by

$$m_s(u_k)u'm_s(u_k)=xm_s(u_k)t(u')y \quad \text{for some } x,y\in U_{A_s,k}.$$

Remembering the definition of $m_s(u')$ (16), we note that $m_s(u')=t(u')m_s(u_k)^{-1}$. The lemma means that $t(U^*_{A_s,k})=Z_{k,s}$.

Recalling Step 1 we have the disjoint union

$$m_s(u_k)U^*_{A_s,k}m_s(u_k)=\sqcup_{t\in t(U^*_{A_s,k})}(m_s(u_k)U_{A_s,k}m_s(u_k)\ \cap\ U_{A_s,k}m_s(u_k)tU_{A_s,k}).$$

By Step 2, we have

$$c_{m_s(u_k)} = \sum_{u' \in U^*_{A_s,k}} T_{t(u')}$$

The lemma follows from the part 4 of Step 3. $\square$

The set

$$\tilde{S}^{aff}_s = \{s(u') \mid u' \in U^*_{A_s,k}\} = Z_{k,s}s(u) = s(u)Z_{k,s}$$

is a coset of $Z_{k,s}$ in $S^{aff}_s(1) = Z_k s(u) = s(u)Z_k$, which depends only on s.

Lemma 4.6. *Let $\tilde{s} \in S^{aff}_s(1)$. We have*
$T_{\tilde{s}} * T_{\tilde{s}} = q_s \tilde{s}^2 + c_{\tilde{s}} T_{\tilde{s}}$ *where* $c_{\tilde{s}} = c_s s(u)^{-1}\tilde{s}$,
$c_{\tilde{s}} = c_s$ *if and only if* $\tilde{s} \in \tilde{S}^{aff}_s$.

Proof. We have for $t \in Z_k$,
$T_{s(u)t} * T_{s(u)t} = T_{s(u)} * T_{s(u)} s(t)t = (q_s s(u)^2 + c_s T_{s(u)})s(t)t = q_s(s(u)t)^2 + c_s t T_{s(u)t}$. $\square$

4.3 Generic algebra

Let R be a commutative ring, let

$$W^{aff}, S^{aff}, \Omega, W, Z_k, W(1), \tag{65}$$

satisfying:

- a1 (W^{aff}, S^{aff}) is a Coxeter system.
- a2 Ω is a group acting on W^{aff} and stabilizing S^{aff}.
- a3 W is the semi-direct product $W^{aff} \rtimes \Omega$.
- a3 Z_k is a commutative group.
- a4 $1 \to Z_k \to W(1) \to W \to 1$ is an extension of W by Z_k.

For a subset X of W, we denote by $X(1)$ the inverse image of X in $W(1)$.

The length ℓ of (W^{aff}, S^{aff}) being invariant by conjugation by Ω, extends to a length ℓ of W constant on the double cosets of Ω, and inflates to a length on $W(1)$, still denoted by ℓ. The subgroup of elements of length 0 in W is Ω, and in $W(1)$ is $\Omega(1)$. The inverse image of W^{aff} in $W(1)$ is a normal subgroup $W^{aff}(1)$ such that $Z_k = W^{aff}(1) \cap \Omega(1)$ and $W(1) = W^{aff}(1)\Omega(1)$ as in (57).

For $w \in W(1)$ and $t \in Z_k$, $w(t) = wtw^{-1}$ depends only on the image of w in W because Z_k is commutative. By linearity the conjugation defines an action

$$(w, c) \mapsto w \bullet c : W(1) \times R[Z_k] \to R[Z_k]$$

of $W(1)$ on $R[Z_k]$ factorizing through the map $W(1) \to W$. For $s, s' \in S^{aff}$ we write $s \sim s'$ if $s, s' \in S^{aff}$ are conjugate in W; if $s, s' \in S^{aff}(1)$ we write $s \sim s'$ if their image in S^{aff} are W-conjugate.

Theorem 4.7. *Let $(q_s, c_s) \in R \times R[Z_k]$ for all $s \in S^{aff}(1)$. We have, for all $s \sim s'$ in $S^{aff}(1)$, $w \in W(1)$, $ws'w^{-1}s^{-1}, t \in Z_k$,*

- *a5* $q_s = q_{st} = q_{s'}$,
- *a6* $c_{st} = c_s t$ *and* $w \bullet c_{s'} = c_{ws'w^{-1}}$.

Then the R-free module of basis $(T_w)_{w\in W(1)}$ admits a unique R-algebra structure satisfying

$$\textit{the braid relations}: \quad T_wT_{w'} = T_{ww'} \quad \textit{for } w,w'\in W(1),\ \ell(w)+\ell(w')=\ell(ww'),$$
$$\textit{the quadratic relations}: \quad T_s^2 = q_sT_{s^2}+c_sT_s \quad \textit{for } s\in S^{aff}(1),$$

where $c_s=\sum_{t\in Z_k} c_s(t)t\in R[Z_k]$ is identified with $\sum_{t\in Z_k} c_s(t)T_t$.

This algebra is denoted by $\mathcal{H}_R(q_s,c_s)$ and called the R-algebra of $W(1)$ with parameters (q_s,c_s).

We will prove that the conditions a5

a6' $c_{st}=c_st$ and $w\bullet c_{s'}=ws'w^{-1}s^{-1}c_s$ if $\ell(sw)>\ell(w)$, $q_sw\bullet c_{s'}=q_{s'}ws'w^{-1}s^{-1}c_s$ if $\ell(sw)<\ell(w)$,

are necessary for the existence of the algebra. We will not prove that a5 and a6' are sufficient although the same proof than in [Schmidt, Thm. 3.1.5] should work.

Remark 4.8. When a5 is satisfied and for all $s\in S^{aff}(1)$, $x\in R$, $q_sx=0$ implies $x=0$, the conditions a6' and a6 are equivalent because $q_s=q_{s'}$ by a5, $ws'w^{-1}s^{-1}c_s=c_{ws'w^{-1}}$ by $c_{st}=c_st$ and the commutativity of Z_k, and can simplify by q_s as $q_sx=0$ implies $x=0$.

Proof. 1) We show that the conditions a5 on (q_s) and a6' on (c_s) are necessary. The braid relations identify $R[Z_k]$ with a subalgebra of $\mathcal{H}_R(q_s,c_s)$ and $T_wt=T_{wt}=wtw^{-1}T_w$ for $w\in W(1), t\in Z_k$ hence

$$T_wc=(w\bullet c)T_w \quad (c\in R[Z_k], w\in W(1)). \tag{66}$$

The equalities $q_s=q_{st}$ and $c_{st}=c_st$ follow from

$$q_s(st)^2+c_stT_{st}=(T_sT_s)s^{-1}tst=T_stT_st=T_{st}T_{st}=q_{st}(st)^2+c_{st}T_{st}.$$

The equalities $q_{s'}=q_s$ and $w\bullet c_{s'}=ws'w^{-1}s^{-1}c_s$ for $s,s'\in S^{aff}(1), w\in W(1), swz=ws'$ for some $z\in Z_k$, follow from the associativity of the product

$$T_s(T_wT_{s'})=(T_sT_w)T_{s'}. \tag{67}$$

a) Case $\ell(sw)=\ell(ws')=\ell(w)+1$. By the braid and quadratic relations,

$$T_s(T_wT_{s'})=T_sT_{ws'}=T_sT_{swz}=T_sT_sT_{wz}=q_ss^2T_{wz}+c_sT_{swz}.$$
$$(T_sT_w)T_{s'}=T_{sw}T_{s'}=T_{ws'z^{-1}}T_{s'}=ws'z^{-1}(ws')^{-1}T_{ws'}T_{s'}$$
$$=ws'z^{-1}(ws')^{-1}T_wT_{s'}T_{s'}=ws'z^{-1}(ws')^{-1}T_w(q_{s'}{s'}^2+c_{s'}T_{s'}).$$

We compute $ws'z^{-1}(ws')^{-1}T_w{s'}^2=ws'z^{-1}(ws')^{-1}w{s'}^2w^{-1}T_w=s^2wzw^{-1}T_w=s^2T_{wz}$. This implies

$$(T_sT_w)T_{s'}=q_{s'}s^2T_{wz}+ws'z^{-1}(ws')^{-1}(w\bullet c_{s'})T_{ws'}$$

We compare and deduce $q_{s'}=q_s,\ w\bullet c_{s'}=ws'z(ws')^{-1}c_s=ws'w^{-1}s^{-1}c_s$.

b) Case $\ell(sw)=\ell(ws')=\ell(w)-1$. We expand first $T_wT_{s'}$ and T_sT_w using $T_w=T_{ws'^{-1}}T_{s'}=T_sT_{s^{-1}w}$ by the braid relations. By the quadratic relations,

$$T_wT_{s'}=T_{ws'^{-1}}(q_{s'}{s'}^2+c_{s'}T_{s'})=q_{s'}T_{ws'}+(w{s'}^{-1}\bullet c_{s'})T_{ws'^{-1}}T_{s'}=q_{s'}T_{ws'}+T_w({s'}^{-1}\bullet c_{s'}),$$
$$T_sT_w=(q_ss^2+c_sT_s)T_{s^{-1}w}=q_sT_{sw}+c_sT_w.$$

Recalling $sw = ws'z^{-1}$ we have $\ell(sws') = \ell(ws') + 1$, we compute

$$T_s(T_wT_{s'}) = q_{s'}T_{sws'} + T_sT_w({s'}^{-1} \bullet c_{s'}) = q_{s'}T_{sws'} + (q_sT_{sw} + c_sT_w)({s'}^{-1} \bullet c_{s'}),$$
$$(T_sT_w)T_{s'} = q_sT_{sws'} + c_sT_wT_{s'} = q_sT_{sws'} + q_{s'}c_sT_{ws'} + c_sT_w({s'}^{-1} \bullet c_{s'}).$$

We compare to get $q_{s'} = q_s,\ q_{s'}c_sT_{ws'} = q_sT_{sw}({s'}^{-1} \bullet c_{s'})$. Writing

$$T_{sw}({s'}^{-1} \bullet c_{s'}) = sws'^{-1}w^{-1}T_{ws'}({s'}^{-1} \bullet c_{s'}) = sws'^{-1}w^{-1}(w \bullet c_{s'})T_{ws'},$$

we obtain $q_s(w \bullet c_{s'}) = q_{s'}ws'w^{-1}s^{-1}c_s$.

2) The algebra is unique if it exists because the expansion of the product $T_wT_{w'}$, for $w, w' \in W(1)$, in the basis $(T_{w''})_{w''\in W(1)}$ is uniquely determined by the braid and quadratic relations. This is clear if $\ell(ww') = \ell(w)+\ell(w')$ by the braid relations. Otherwise, let $w = s_1 \dots s_{\ell(w)}u, w' = s'_1 \dots s'_{\ell(w')}u'$ with $s_i, s'_j \in S^{aff}(1), u, u' \in \Omega(1)$ be two reduced decompositions of w, w'. We note that $s''_j := us'_ju^{-1}$ lies in $S^{aff}(1)$ as $\Omega(1)$ normalizes $S^{aff}(1)$. Using the braid relations we compute

$$\begin{aligned} T_wT_{w'} &= T_{s_1} \dots T_{s_{\ell(w)}}T_uT_{s'_1} \dots T_{s'_{\ell(w')}}T_{u'} = T_{s_1} \dots T_{s_{\ell(w)}}T_{s''_1} \dots T_{s''_{\ell(w')}}T_{uu'} \\ &= T_{w_1}T_{s''_{j+1}} \dots T_{s''_{\ell(w')}}T_{uu'} = T_{w_1s''_{j+1}}T^2_{s''_{j+1}}T_{s''_{j+2}} \dots T_{s''_{\ell(w')}}T_{uu'}. \end{aligned}$$

where $w_1 := s_1 \dots s_{\ell(w)}s''_1 \dots s''_j$ and $\ell(w_1s''_{j+1}) = \ell(w_1) - 1 = \ell(w) + j - 1$. Using the quadratic relations we compute

$$T_{w_1s''_{j+1}}T^2_{s''_{j+1}} = q_{s''_{j+1}}T_{w_1s''_{j+1}} + T_{w_1s''_{j+1}}c_{s''_{j+1}}T_{s''_{j+1}} = q_{s''_{j+1}}T_{w_1s''_{j+1}} + (w_1s''_{j+1}) \bullet c_{s''_{j+1}}T_{w_1}.$$

After finitely many steps we obtain the coefficients $T_wT_{w'}$ in the basis $(T_{w''})_{w''\in W(1)}$.

3) The unicity and existence of the R-algebra is proved in [Schmidt, Thm. 3.1.5] when the quadratic relations are replaced by

$$T^2_{\tilde{s}} = a_sT_{(\tilde{s})^2} + b_sT_{\tilde{s}} \quad (s \in S^{aff}),$$

where $\tilde{s}$ is a fixed lift of of s, and the parameters $(a_s, b_s) \in R \times R[Z_k]$ for $s \in S^{aff}$ satisfy: for $s, s' \in S^{aff},\ w \in W(1),\ w\tilde{s}'w^{-1}(\tilde{s})^{-1} \in Z_k$,

a5" $a_{\tilde{s}} = a_{\tilde{s}'}$

a6" $w \bullet b_{\tilde{s}'} = w\tilde{s}'w^{-1}(\tilde{s})^{-1}b_{\tilde{s}}$.

Recalling that the map $(t, s) \mapsto t\tilde{s} : Z_k \times S^{aff} \to S^{aff}(1)$ is bijective, we define a map $(a_s, b_s)_{s\in S^{aff}} \mapsto (q_s, c_s)_{s\in S^{aff}(1)}$ such that $q_{t\tilde{s}} := a_{\tilde{s}}, c_{t\tilde{s}} := tb_{\tilde{s}}$. Then $(a_s, b_s)_s \in S^{aff}$ satisfies a5", a6" if and only if $(q_s, c_s)_{s\in S^{aff}(1)}$ satisfies a5, a6. Noting that the braid relations imply $T_{t\tilde{s}} = T_tT_{\tilde{s}}$, the braid and quadratic relations in Schmidt are equivalent to our braid and quadratic relations. □

Remark 4.9. a) When (q_s) satisfies a5, $(c_s = q_s - 1)$ satisfies a6.

b) a6 implies:

$s \bullet c_s = c_s$ (take $w = s \in S^{aff}(1)$ in a6'). This means that T_s commutes with c_s.

$c_s sts^{-1} = c_st$ for $t \in Z_k$ (use $st \bullet c_{st} = c_{st},\ s \bullet c_s = c_s, c_{st} = c_st$ and the commutativity of Z_k). Hence, if $c_s \neq 0$, the group $\{tst^{-1}s^{-1} \mid t \in Z_k\}$ is finite.

c) By [Bki, VI.1.3 Prop. 3, VI.4.3 Thm. 4], [Borel, 3.3], the number of W^{aff}-conjugacy classes of $S(\mathcal{H})$ is the number of connected components of the graph obtained by erasing the multiple edges of the Coxeter graph $\operatorname{Cox} S^{aff}$ of S^{aff}; this number is

1 if $\operatorname{Cox} S$ is of type $(A_n)_{n\geq 2}, (D_n)_{n\geq 3}, (E_n)_{n=6,7,8}$,

2 if $\operatorname{Cox} S$ is of type $A_1, (B_n)_{n\geq 2}, G_2, F_4$,

3 if $\operatorname{Cox} S$ is of type $(C_n)_{n\geq 3}$.

d) Note that the Hecke R-algebra of a Coxeter system (W,S) with parameters $(q_s,c_s)_{s\in S}$ in $R\times R$, constant on the intersections with S of the conjugacy classes of W, was introduced in [Bki, Ex. 23].

e) We can extend the parameters $(q_s,c_s)_{s\in S^{aff}(1)}$ in $R\times R[Z_k]$ satisfying the conditions (a5, a6) to parameters $(q(\tau),c(\tau))_{\tau\in S(\mathcal{H})(1)}$ in $R\times R[Z_k]$ satisfying $q_s=q(s), c_s=c(s)$ if $s\in S^{aff}(1)$ and

a5 $q(\tau)$ is constant on the conjugacy classes of $W(1)$ and $q(\tau t)=q(\tau)$ for $t\in Z_k$.

a6 $c(w\tau w^{-1})=w\bullet c(\tau)$ for $w\in W(1)$ and $c(\tau t)=c(\tau)t$ for $t\in Z_k$,

f) We can also choose indeterminates satisfying a5. They are denoted by bold face letters $\mathbf{q}_s,\mathbf{q}(\tau)$. We will also consider indeterminates $\mathfrak{q}_s$ and $\mathfrak{q}(\tau)$ satisfying $\mathfrak{q}_s=\mathfrak{q}(s)$, a5 and of square $\mathfrak{q}(\tau)^2=\mathbf{q}(\tau)$.

Example 4.10. *1. The Iwahori Hecke ring is $\mathcal{H}=\mathcal{H}_{\mathbb{Z}}(q_s,q_s-1)$ with q_s given by Cor. 3.31 and $Z_k=\{1\}, W=W(1)$.*

2. The pro-p-Iwahori Hecke ring $\mathcal{H}(1)=\mathcal{H}_{\mathbb{Z}}(q_s,c_s)$ with q_s given by Cor. 3.31, c_s as in Prop. 4.4, for $s\in S^{aff}(1)$.

3. The group algebra $R[W(1)]=\mathcal{H}_R(1,0)$ with $q_s=1, c_s=0$ for all $s\in S^{aff}(1)$.

4. The Lusztig affine Hecke R-algebras with parameters (q_s) with q_s an invertible square in R [Lusztig] *are examples of R-algebras $\mathcal{H}_R(q_s,q_s-1)$ with $Z_k=\{1\}, W=W(1)$.*

The R-algebra $\mathcal{H}_R^{aff}(q_s,c_s)$ of $W^{aff}(1)$ is a subalgebra of the R-algebra $\mathcal{H}_R(q_s,c_s)$ of $W(1)$. By the braid relations, the R-linear map such that $u\mapsto T_u$ for $u\in\Omega(1)$, embeds the group R-algebra $R[\Omega(1)]$ of $\Omega(1)$ in $\mathcal{H}_R(q_s,c_s)$. The intersection $R[\Omega(1)]\cap\mathcal{H}_R^{aff}(q_s,c_s)$ is the group R-algebra $R[Z_k]$ of Z_k.

Proposition 4.11. *The R-algebra $\mathcal{H}_R(q_s,c_s)$ is isomorphic to the twisted tensor product*

$$R[\Omega(1)]\hat{\otimes}_{R[Z_k]}\mathcal{H}_R^{aff}(q_s,c_s)$$

with the product $(T_u\hat{\otimes}T_w)(T_{u'}\hat{\otimes}T_{w'})=T_{uu'}\hat{\otimes}T_{u'^{-1}wu'}T_{w'}$ for $u,u'\in\Omega(1), w,w'\in W^{aff}(1)$.

Proof. Clear. □

Lemma 4.12. *Let $T_s^*=T_s-c_s$ for $s\in S^{aff}(1)$. The quadratic relation in $\mathcal{H}_R(q_s,c_s)$ is*

$$T_s^*T_s=T_sT_s^*=q_ss^2 \quad \text{or} \quad T_{s^{-1}}^*T_s=T_sT_{s^{-1}}^*=q_s.$$

For $u\in\Omega(1)$, we have $c_{u^{-1}su}=T_u^{-1}c_sT_u,\quad T_{u^{-1}su}^*=T_u^{-1}T_s^*T_u.$

Proof. We have $T_s^*T_s=T_sT_s-c_sT_s=q_ss^2$ and $T_ss^{-2}=s^{-2}T_s=T_{s^{-1}},\ c_{s^{-1}}=c_ss^{-2}$, because $s^2\in Z_k$. The product $T_s^*T_s$ commutes because c_s and s^2 commute with T_s. Comparing the quadratic relation for $T_{usu^{-1}}=T_uT_sT_u^{-1}$ with the quadratic relation for T_s multiplied on the left by T_u and on the right by T_u^{-1} we obtain $c_{usu^{-1}}=T_uc_sT_u^{-1}$. □

Let $w=s_1\dots s_{\ell(w)}u$ with $u\in\Omega(1)$ and $s_i\in S^{aff}(1)$ for $1\leq i\leq\ell(w)$, and let

$$T_{w^{-1}}^*:=T_u^{-1}T_{s_{\ell(w)}^{-1}}^*\dots T_{s_1^{-1}}^*.$$

Proposition 4.13. *1) T_w is invertible in $\mathcal{H}_R(q_s,c_s)$ of inverse $q_w^{-1}T_{w^{-1}}^*$, if q_s is invertible in R for all $s\in S^{aff}$.*

2) $T_w^*=T_{s_1}^*\dots T_{s_{\ell(w)}}^*T_u$ does not depend on the decomposition of w.

3) $T_{w^{-1}}^*T_w=T_wT_{w^{-1}}^*=q_w$.

4) $T^*_w\ c = (w \bullet c)\ T^*_w$ *for* $c \in R[Z_k]$.

5) $T^*_w\ T_u = T^*_{wu} = T_u T^*_{u^{-1}wu}$ *for* $w \in W(1), u \in \Omega(1)$.

6) If $w = s_1 \dots s_n \in W^{aff}(1)$ *is a reduced decomposition and* $u \in \Omega(1)$*, the elements* $q_{wu} := q_{s_1} \dots q_{s_{\ell(w)}}$ *and* $c_w := c_{s_1} \dots c_{s_{\ell(w)}}$ *are well defined.*

Proof. By Remark 4.12, $T_w = T_{s_1} \dots T_{s_{\ell(w)}} T_u$ (by the braid relations, it is independent on the decomposition) is invertible of inverse

$$T_w^{-1} = q_w^{-1} T_{u^{-1}} T^*_{s^{-1}_{\ell(w)}} \dots T^*_{s_1^{-1}} = q_w^{-1} T^*_{w^{-1}},$$

with $w^{-1} = u^{-1} s^{-1}_{\ell(w)} \dots s_1^{-1}$. Replacing w^{-1} by $w = u(u^{-1}s_1u)\dots(u^{-1}s_nu)$ with $n = \ell(w) = \ell(w^{-1})$ and $u^{-1}s_iu \in S^{aff}$, we obtain $T^*_w = T_u T^*_{u^{-1}s_1u} \dots T^*_{u^{-1}s_nu}$. By Remark 4.12, $T^*_w = T^*_{s_1} \dots T^*_{s_n} T_u$. As $T_w = T_{s_1} \dots T_{s_{\ell(w)}} T_u$ was independent of the decomposition, the same is true for $T^*_w = T^*_{s_1} \dots T^*_{s_{\ell(w)}} T_u$.

From $T^*_w = T^*_{s_1} \dots T^*_{s_{\ell(w)}} T_u$ and

a) $T_s T^*_s = T^*_s T_s = q_s$ (Remark 4.12), we deduce $T_w T^*_w = T^*_w T_w = q_w$.

b) $(T_s - c_s)t = T_{st} - c_s sts^{-1} = sts^{-1}(T_s - c_s)$ for $t \in Z_k$ (use that Z_k is commutative and Remark 4.9 b)), we deduce $T^*_w\ c = (w \bullet c) T^*_w$ for $c \in R[Z_k]$,

c) $T^*_s T_u = T^*_{su} = T_u T^*_{u^{-1}su}$ (Remark 4.12), we deduce $T^*_w\ T_u = T^*_{wu} = T_u T^*_{u^{-1}wu}$ for $u \in \Omega(1)$.

The braid relations and 2) imply that q_w and c_w are well defined in 6). □

4.4 $\mathbf{q}_w \mathbf{q}_{w'} = \mathbf{q}_{ww'} \mathbf{q}^2_{w,w'}$

To a sequence $(s_1, \dots, s_n)$ in S^{aff}, we associate the sequence

(68) $$\mathfrak{T}(s_1, \dots, s_n) = (\tau_1 := s_1, \tau_2 := s_1 s_2 s_1^{-1}, \dots, \tau_n := s_1 \dots s_{n-1} s_n (s_1 \dots s_{n-1})^{-1})$$

in $S(\mathcal{H})$. We consider parameters which are indeterminates (Remark 4.9 e), f)).

Definition 4.14. *For* $w = s_1 \dots s_n u$ *with* $s_i \in S^{aff}$ *for* $1 \leq i \leq n$ *and* $u \in \Omega$*, and* $\tilde{w} \in W(1)$ *lifting* w*, let* $\mathfrak{T}_{\tilde{w}} = \mathfrak{T}_w = \mathfrak{T}_{s_1 \dots s_n}$ *be the set of elements of odd multiplicity in* $\mathfrak{T}(s_1, \dots, s_n)$ *and let*

$$\mathbf{q}_{\tilde{w}} = \mathbf{q}_w = \prod_{\tau \in \mathfrak{T}_w} \mathbf{q}(\tau).$$

For $\tilde{w}, \tilde{w}' \in W(1)$ *lifting* $w, w' \in W$*, let*

$$\mathbf{q}_{\tilde{w}, \tilde{w}'} = \mathbf{q}_{w,w'} = (\mathbf{q}_w \mathbf{q}_{w'} \mathbf{q}^{-1}_{ww'})^{1/2}.$$

When $w \in W^{aff}$, $\mathfrak{T}_w$ consists of the elements $w' s w'^{-1}$ for all triples $(w', w'', s) \in W^{aff} \times W^{aff} \times S^{aff}$ such that $w = w' s w''$ and $\ell(w) = \ell(w') + \ell(w'') + 1$.

When the decomposition $w = s_1 \dots s_{\ell(w)} u$ is reduced, $\mathfrak{T}(s_1, \dots, s_{\ell(w)}) = \mathfrak{T}_w$ [Bki, IV.1.4 Lemme 2, Remarque], [Kumar, 1.3.14],

$$\mathbf{q}_w = \mathbf{q}_{s_1} \dots \mathbf{q}_{s_{\ell(w)}},$$

and $\mathfrak{T}_w, \mathbf{q}_w$ depend only on w.

Remark 4.15. *When* $w \in W$*,* $\mathfrak{T}_w$ *contains* $s \in S^{aff}$ *if and only if* $\ell(sw) < \ell(w)$.

Lemma 4.16. *Let* $w, w' \in W$*. Then* $\mathbf{q}_{w,w'} = 1$ *if and only if* $\ell(w) + \ell(w') = \ell(ww')$.

Proof. By the braid relations $\mathbf{q}_w \mathbf{q}_{w'} = \mathbf{q}_{ww'}$ if and only if $\ell(w) + \ell(w') = \ell(ww')$. □

Lemma 4.17. *Let $w, w' \in W^{aff}$. We have*

$$\mathfrak{T}_{ww'} = (\mathfrak{T}_w \cup \mathfrak{T}_{ww'w^{-1}}) - (\mathfrak{T}_w \cap \mathfrak{T}_{ww'w^{-1}}).$$

Proof. If $w = s_1 \ldots s_n, w' = s'_1 \ldots s'_m$ are reduced decomposition of w, w' then the multiset $\mathfrak{T}(s_1, \ldots, s_n, s'_1 \ldots s'_m)$ is a union of $\mathfrak{T}_w$ and of $w\mathfrak{T}_{w'}w^{-1} = \mathfrak{T}_{ww'w^{-1}}$. The elements of $\mathfrak{T}_w \cap \mathfrak{T}_{ww'w^{-1}}$ have multiplicity 2, the other ones have multiplicity 1. This implies the formula for $\mathfrak{T}_{ww'}$. □

Remark 4.18. *Let $w, w' \in W^{aff}$. Then, $\ell(ww') = \ell(w) + \ell(w') - 2\,\mathrm{Card}(\mathfrak{T}_w \cap \mathfrak{T}_{ww'w^{-1}})$.*

The computation of $\mathbf{q}_{w,w'}$ can be done using the following lemma.

Lemma 4.19. *Let $w, w' \in W^{aff}$ and $u, u' \in \Omega$. We have*

$$\mathbf{q}_{w,w'} = \prod_{\tau \in \mathfrak{T}_w \cap \mathfrak{T}_{ww'w^{-1}}} \mathbf{q}(\tau), \quad \mathbf{q}_{wu,w'u'} = \mathbf{q}_{w,uw'u^{-1}}.$$

Proof. The formula for $\mathbf{q}_{w,w'}$ follows from Lemma 4.19. The group Ω normalizes S^{aff} and $w, uw'u^{-1}, wuw'u' = wuw'u^{-1}uu'$ belong to W^{aff}. We compute:
$\mathbf{q}^2_{wu,w'u'} = \mathbf{q}_{wu}\mathbf{q}_{w'u'}\mathbf{q}^{-1}_{wuw'u'} = \mathbf{q}_w\mathbf{q}_{w'}\mathbf{q}^{-1}_{wuw'u^{-1}uu'} = \mathbf{q}_w\mathbf{q}_{uw'u^{-1}}\mathbf{q}^{-1}_{wuw'u^{-1}} = \mathbf{q}^2_{w,uw'u^{-1}}$. □

Example 4.20. Let w, w' in W.
$\mathbf{q}_w = \mathbf{q}_{w^{-1}}$, $\mathbf{q}_{w,w^{-1}} = \mathbf{q}_w$.
$\mathbf{q}_{w^{-1},ww'w^{-1}} = \prod_{\tau \in \mathfrak{T}_w \cap \mathfrak{T}_{w'}} \mathbf{q}(\tau)$ is equal to $\mathbf{q}_w = \prod_{\tau \in \mathfrak{T}_w} \mathbf{q}(\tau)$ if and only if $\mathfrak{T}_w \subset \mathfrak{T}_{w'}$ if and only if $\ell(w'w) = \ell(w') - \ell(w)$.

4.5 Reduction to $q_s = 1$

We explain a method to reduce the proof of a property of the R-algebra $\mathcal{H}_R(q_s, c_s)$ (Thm. 4.7) to the simpler case where $q_s = 1$ for all s.

We consider indeterminates $\mathfrak{q}_s, \mathfrak{q}(\tau)$ satisfying a5 and of square $\mathfrak{q}(\tau)^2 = \mathbf{q}(\tau)$ and elements $c_s, c(\tau)$ in $R[Z_k]$ satisfying a6 as before (Remark 4.9 e), f)). For $w \in W(1)$, let $\mathfrak{q}_w = \prod_{\tau \in \mathfrak{T}_w} \mathfrak{q}(\tau)$ as in Def. 4.14.

The "generic" algebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$ is a $R[(\mathbf{q}_s)]$-subalgebra of the $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-algebra $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$,

(69) $$\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s) \subset \mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s).$$

In $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$, the elements

(70) $$\tilde{T}_w := \mathfrak{q}_w^{-1} T_w \quad (w \in W(1)).$$

form a $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-basis satisfying the braid relations and the quadratic relations with parameters $(1, \mathfrak{q}_s^{-1} c_s)$:

(71) $$(\tilde{T}_s)^2 = s^2 + \mathfrak{q}_s^{-1} c_s \, \tilde{T}_s \quad (s \in S^{aff}(1)).$$

Applying Thm. 4.7, we obtain :

Proposition 4.21. *The $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-linear map*

(72) $$T_w \mapsto \tilde{T}_w : \mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1} c_s) \to H_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)$$

is an algebra isomorphism.

We can often reduce to the case $q_s = 1$ by considering:

1) The $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-algebra $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(1, \mathfrak{q}_s^{-1} c_s)$.
2) The $R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]$-algebra isomorphism (72).
3) The generic $R[(\mathbf{q}_s)]$-subalgebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q_s}, c_s) \subset \mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q_s}, c_s)$.
4) The specialisation $\mathcal{H}_R(q_s, c_s) = R \otimes_{R[(\mathbf{q}_s)]} \mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$ sending $\mathbf{q}_s$ to q_s for all $s \in S^{aff}/\sim$.

We give an example:

Proposition 4.22. *The properties 2) to 5) of Prop. 4.13 are valid in $\mathcal{H}_R(q_s, c_s)$ even when q_s is not invertible.*

Proof. By Prop. 4.13, the properties 2) to 5) of the proposition are true in the algebra $\mathcal{H}_{R[(\mathbf{q}_s,\mathbf{q}_s^{-1})]}(\mathbf{q}_s, c_s)$. They are relations between elements of the generic $R[(\mathbf{q}_s)]$-subalgebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$. They remain true in the algebra $\mathcal{H}_R(q_s, c_s) = R \otimes_{R[(\mathbf{q}_s)]} \mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$ obtained by the specialisation sending $\mathbf{q}_s$ to q_s for $s \in S^{aff}/\sim$. ☐

Proposition 4.23. *The R-linear map ι of $\mathcal{H}_R(q_s, c_s)$ defined by*

$$\iota(T_w) = (-1)^{\ell(w)} T_w^* \quad \text{for } w \in W(1),$$

is an involutive automorphism.

Proof. $\iota^2(T_s) = \iota(-T_s + c_s) = -\iota(T_s) + c_s = T_s$, hence if ι is involutive.

$(\iota(T_s))^2 = (T_s - c_s)^2 = T_s^2 - 2c_s T_s + c_s^2 = q_s s^2 - c_s T_s + c_s^2 = q_s s^2 - c_s(T_s - c_s) = q_s s^2 + c_s \iota(T_s)$, hence ι respects the quadratic relations.

Obviously the braid relations are respected. ☐

Remark 4.24. *The reduction to $q_s = 1$ is not possible in the classical framework of algebras with parameters $(q_s, q_s - 1)$.*

5 Alcove walk bases and Bernstein relations

Let $R, W^{aff}, S^{aff}, \Omega, W, Z_k, W(1), (q_s, c_s)$ as in subsection 4.3, satisfying a1 to a6, and the following hypotheses:

b1 W^{aff} is the affine Weyl group of a reduced root system Σ, generated by the orthogonal reflections with respect to a set of affine hyperplanes

$$\mathfrak{H} = \{\mathrm{Ker}(\beta + k) \mid \beta \in \Sigma, k \in \mathbb{Z}\}$$

in an euclidean real vector space V, and S^{aff} is the set of orthogonal reflections with respect to the walls of an alcove $\mathfrak{C}$ of vertex 0 in V.

b2 The action of W^{aff} on V extends to an action of W such that for any $w \in W$, an element $w_0^{-1} w$ acts by a translation respecting $\mathfrak{H}$, for some w_0 in the stabilizer W_0 of 0 in W^{aff}.

b3 For $s, s' \in S^{aff}$ such that ss' has finite order $n(s, s')$, there exist $s(1), s'(1) \in S^{aff}(1)$ above s, s' such that $s(1)s'(1)s(1)\ldots = s'(1)s(1)s'(1)\ldots$ where the two products have $n(s, s')$ factors.

We use the notations of section 3.4, without Σ in the index because Σ is now the unique root system (there is no Φ). The group Ω which normalizes S^{aff} is the stabilizer of $\mathfrak{C}$ in $W = W^{aff} \rtimes \Omega$.

We denote by Λ, resp. Λ^{aff}, the subgroup of W, resp. W^{aff}, acting by translations on V and by

$$\nu : \Lambda \to V$$

the homomorphism such that $\lambda \in \Lambda$ acts by translations by $\nu(\lambda)$. The group Λ is normalized by $w_0 \in W_0$: $w_0 \lambda w_0^{-1}$ acts by translation by $w_0.\nu(\lambda)$, the homomorphism ν is W_0-equivariant : $\nu(w_0 \lambda w_0^{-1}) = w_0.\nu(\lambda)$, and

$$W = \Lambda \rtimes W_0. \tag{73}$$

The lattice $Q(\Sigma^\vee)$ generated by the set $\Sigma^\vee$ of coroots of Σ is equal to $\nu(\Lambda^{aff})$ and

$$Q(\Sigma^\vee) \subset \nu(\Lambda) \subset P(\Sigma^\vee),$$

where $P(\Sigma^\vee)$ is the lattice of weights of $\Sigma^\vee$, that is, the elements $v \in V$ such that $\alpha(v) \in \mathbb{Z}$ for all $\alpha \in \Sigma$.

The action of W on V inflates to an action of $W(1)$ trivial on Z_k and the homomorphism ν inflates to an homomorphism $\nu : \Lambda(1) \to V$ vanishing on Z_k, where $\Lambda(1)$ is the inverse image of Λ in $W(1)$. We have

$$W(1) = \Lambda(1) W_0(1), \tag{74}$$

where $W_0(1)$ is the inverse image of W_0 in $W(1)$, $\Lambda(1) \cap W_0(1) = Z_k$ and $\Lambda(1)$ is normal in $W(1)$.

Remark 5.1. *Note that for the data arising from (R, F, G) satisfies bj for $j = 1, 2, 3$ and that $\Lambda = Z/Z_0$, $\Lambda(1) = Z/Z_0(1)$, where the extension $\Lambda(1) \to \Lambda$ of kernel Z_k does not split in general.*

Proof. The property b3 follows from Prop. 3.4 applied to the root data generating the finite quotients of the parahoric sugroups of G (Thm. 3.1). See the subsection 3.3 for b1 and b2. For the last assertion, see [VigMA]. □

5.1 Length

We denote $\Sigma^{aff} := \{\beta + k \ \mid \beta \in \Sigma, k \in \mathbb{Z}\}$. The set $\{s_{\beta+k} \mid \beta + k \in \Sigma^{aff}\}$ is equal to $S(\mathcal{H})$ defined in Subsection 4.4.

Lemma 5.2. *For any $\beta + k \in \Sigma^{aff}, \lambda \in \Lambda$, we have $\lambda s_{\beta+k} \lambda^{-1} = s_{\beta+k-\beta\circ\nu(\lambda)}$.*

Proof. Let $x \in V$. The action of W^{aff} on V is faithful. We have

$$\begin{aligned} s_{\beta+k}(x) &= x - (\beta(x) + k)\beta^\vee = s_\beta(x) - k\beta^\vee, \\ (\lambda s_{\beta+k} \lambda^{-1})(x) &= s_{\beta+k}(x - \nu(\lambda)) + \nu(\lambda) = x - (\beta(x - \nu(\lambda)) + k)\beta^\vee \\ &= s_\beta(x) - (k - \beta \circ \nu(\lambda))\beta^\vee. \end{aligned}$$

□

For $\beta \in \Sigma$ we have $s_{\beta-1} = s_\beta s_{\beta+1} s_\beta$. The element of Λ^{aff}

$$\mu_\beta := s_{\beta+1} s_\beta = s_\beta s_{\beta-1}.$$

satisfying $\nu(\mu_\beta) = -\beta^\vee$, appears often in this work. The conjugation by μ_β sends $s_{\beta+k}$ to $s_{\beta+k+2}$ for $k \in \mathbb{Z}$.

Remark 5.3. *$\beta\circ\nu(\Lambda)=\delta\mathbb{Z}$ with $\delta\in\{1,2\}$ as $\beta(\beta^\vee)=2$. Lemma 5.2 implies that the set of Λ-conjugates of s_β is $\{s_{\beta+k}\mid k\in\delta\mathbb{Z}\}$.*

For $\tau\in S(\mathcal{H})$, let H_τ be the affine hyperplane fixed pointwise by τ. When two facets of V are not contained in the connected component of $V-H_\tau$, we say that H_τ is separates them. By [Bki, IV. 1 Ex. 16 h)], *for $w\in W^{aff}$, the set of hyperplanes of $\mathfrak{H}$ separating the alcoves $\mathfrak{C}$ and $w(\mathfrak{C})$ is*

$$\mathfrak{H}_w=\{H_\tau\mid\tau\in\mathfrak{T}_w\},\tag{75}$$

where the finite set $\mathfrak{T}_w$ of cardinal $\ell(w)$ is defined in Subsection 4.4.

Example 5.4. *Let $s\in S^{aff},w\in W^{aff}$. Then $\ell(sw)=\ell(w)+1$ means that $w(\mathfrak{C})$ and $\mathfrak{C}$ are on the same side of the wall H_s of $\mathfrak{C}$ fixed by s.*

Definition 5.5. *Let $x\in\mathfrak{C}$, $\beta\in\Sigma$, $\tilde{w}\in W(1)$ above $w\in W$. We define $\ell_\beta(\tilde{w})=\ell_\beta(w)\in\mathbb{Z}$ as the integer such that*

$$\ell_\beta(w)<\beta(w(x))<\ell_\beta(w)+1\tag{76}$$

The integer $\ell_\beta(w)$ does not depend on the choice of $x\in\mathfrak{C}$, and depends only on the action of w on V. Note that $\beta(w(x))=(w^{-1}(\beta))(x)$ where $w^{-1}(\beta)\in\Sigma^{aff}$.

Let Σ^+,Σ^- be the set of positive, negative, roots of Σ (we say positive instead of $\mathfrak{C}$-positive). When $\beta\in\Sigma^+$, $0<\beta(x)<1$ by (35). If $w^{-1}(\beta)\in\Sigma^+$ then $\ell_\beta(w)=0$.

Lemma 5.6. *For $w\in W$, $\beta\in\Sigma^+$ and $k\in\mathbb{Z}$, the hyperplane $\mathrm{Ker}(\beta+k)$ separates the alcoves $\mathfrak{C}$ and $w(\mathfrak{C})$ if and only if*

$$k\in[0,-\ell_\beta(w)-1] \text{ and } \ell_\beta(w)\leq-1\ ,\text{ or } k\in[-\ell_\beta(w),-1] \text{ and } \ell_\beta(w)\geq 1.$$

Proof. Then $\mathrm{Ker}(\beta+k)\in\mathfrak{H}$ separates $\mathfrak{C}$ and $w(\mathfrak{C})$ if and only if $\beta(x)+k$ and $\beta(w(x))+k$ have a different sign.

Let $\beta\in\Sigma^+$. Then $k<\beta(x)+k<1+k$ and $\ell_\beta(w)+k<\beta(w(x))+k<\ell_\beta(w)+k+1$. Hence $\beta(x)+k$ is positive if and only if $k\geq0$ and $\beta(w(x))+k$ is negative if and only if $\ell_\beta(w)+1+k\leq0$. This holds if and only if $\ell_\beta(w)\leq-1$ and $k\in[0,-\ell_\beta(w)-1]$. Similarly $\beta(x)+k$ negative and $\beta(w(x))+k$ positive is equivalent to $k\in[-\ell_\beta(w),-1]$ and $\ell_\beta(w)\geq1$. □

Proposition 5.7. *The length of $w\in W$ or $W(1)$ is $\ell(w)=\sum_{\beta\in\Sigma^+}|\ell_\beta(w)|$.*

Proof. Let $w\in W^{aff}$. The length of w is the cardinal of $\mathfrak{T}_w$. Use (75) and Lemma 5.6. The number of $\mathrm{Ker}(\beta+k)\in\mathfrak{H}$ with $\beta\in\Sigma^+,k\in\mathbb{Z}$ separating $\mathfrak{C}$ and $w(\mathfrak{C})$ is $|\ell_\beta(w)|$. This remains valid for $w\in W$ because Ω normalizes $\mathfrak{C}$, and for $w\in W(1)$ because Z_k acts trivially. □

Example 5.8. 1) When w acts trivially, $\ell_\beta(w)=0$ if $\beta\in\Sigma^+$ and $\ell_\beta(w)=-1$ if $\beta\in\Sigma^-$.

2) Let $w\in W_0$ and $\beta\in\Sigma$. We have $\beta(w(x))=w^{-1}(\beta)(x)$. Hence $\ell_\beta(w)=0$ if $w^{-1}(\beta)\in\Sigma^+$, and $\ell_\beta(w)=-1$ if $w^{-1}(\beta)\in\Sigma^-$. The hyperplane $\mathrm{Ker}\,\beta$ separates $\mathfrak{C}$ and $w(\mathfrak{C})$ if and only if $w^{-1}(\beta)\in\Sigma^-$. The length $\ell(w)$ of $w\in W_0$ is the number of $\beta\in\Sigma^+$ such that $w^{-1}(\beta)\in\Sigma^-$.

3) Let $\gamma+k\in\Sigma^{aff}$. For $\beta\in\Sigma^+$ and $x\in\mathfrak{C}$ we have

$$\beta(s_{\gamma+k}(x))=\beta(s_\gamma(x)-k\gamma^\vee)=s_\beta(\gamma)(x)-kn_{\beta,\gamma},$$

Proposition 5.9. *Let $\beta\in\Sigma,\lambda\in\Lambda,w\in W_0$. We have*

1) $\ell_\beta(\lambda)$ equals $\beta\circ\nu(\lambda)$ if $\beta\in\Sigma^+$ and $\beta\circ\nu(\lambda)-1$ if $\beta\in\Sigma^-$.

2) $\ell_\beta(\lambda w)$ equals $\beta\circ\nu(\lambda)$ if $\beta\in w(\Sigma^+)$ and $\beta\circ\nu(\lambda)-1$ if $\beta\in w(\Sigma^-)$.

3) $\ell_\beta(w\lambda)$ *equals* $w^{-1}(\beta)\circ\nu(\lambda)$ *if* $\beta\in w(\Sigma^+)$ *and* $w^{-1}(\beta)\circ\nu(\lambda)-1$ *if* $\beta\in w(\Sigma^-)$.

Proof. Let $x\in\mathfrak{C}$. We recall that $\beta\circ\nu(\lambda)$ is an integer.

When β is positive we have $0<\beta(x)<1$ and $\beta\circ\nu(\lambda)<\beta(x+\nu(\lambda))<1+\beta\circ\nu(\lambda)$.

When β is negative, $-1<\beta(x)<0$ and $-1+\beta\circ\nu(\lambda)<\beta(x+\nu(\lambda))<\beta\circ\nu(\lambda)$.

We have $\lambda w(x)=w(x)+\nu(\lambda)$, $w\lambda(x)=w(x+\nu(\lambda))$,

$\beta(\lambda w(x))=\beta(w(x))+\beta\circ\nu(\lambda)=w^{-1}(\beta)(x)+\beta\circ\nu(\lambda)$,

$\beta(w\lambda(x))=w^{-1}(\beta)(x+\nu(\lambda))=w^{-1}(\beta)(x)+w^{-1}(\beta)\circ\nu(\lambda)$.

□

Corollary 5.10. *We have for* $(\lambda,w)\in\Lambda\times W_0$,

$$\ell(\lambda w)=\sum_{\beta\in\Sigma^+\cap w(\Sigma^+)}|\beta\circ\nu(\lambda)|+\sum_{\beta\in\Sigma^+\cap w(\Sigma^-)}|\beta\circ\nu(\lambda)-1|,$$
$$\ell(w\lambda)=\sum_{\beta\in\Sigma^+\cap w^{-1}(\Sigma^+)}|\beta\circ\nu(\lambda)|+\sum_{\beta\in\Sigma^+\cap w^{-1}(\Sigma^-)}|\beta\circ\nu(\lambda)+1|.$$

Proof. Prop. 5.7, 5.9 imply the above equality of $\ell(\lambda w)$ and

$$\ell(w\lambda)=\sum_{\beta\in\Sigma^+\cap w(\Sigma^+)}|w^{-1}(\beta)\circ\nu(\lambda)|+\sum_{\beta\in\Sigma^+\cap w(\Sigma^-)}|w^{-1}(\beta)\circ\nu(\lambda)-1|.$$

Replace $w^{-1}(\beta)$ by β in the first sum and by $-\beta$ in the second sum. □

Corollary 5.11. *We have* $\Lambda\cap\Omega=\operatorname{Ker}\nu$ *and for* $\lambda\in\Lambda, w\in W_0$,

$$\ell(\lambda)=\sum_{\beta\in\Sigma^+}|\beta\circ\nu(\lambda)|,$$
$$\ell(w)=\ell(w^{-1})=|\Sigma^+\cap w(\Sigma^-)|,$$
$$\ell(w\lambda)=\begin{cases}\ell(\lambda)+\ell(w) \text{ if and only if } \beta\circ\nu(\lambda)\geq 0 \text{ for } \beta\in\Sigma^+\cap w^{-1}(\Sigma^-),\\ \ell(\lambda)-\ell(w) \text{ if and only if } \beta\circ\nu(\lambda)<0 \text{ for } \beta\in\Sigma^+\cap w^{-1}(\Sigma^-),\end{cases}$$
$$\ell(\lambda w)=\begin{cases}\ell(\lambda)+\ell(w) \text{ if and only if } \beta\circ\nu(\lambda)\leq 0 \text{ for } \beta\in\Sigma^+\cap w(\Sigma^-),\\ \ell(\lambda)-\ell(w) \text{ if and only if } \beta\circ\nu(\lambda)>0 \text{ for } \beta\in\Sigma^+\cap w(\Sigma^-),\end{cases}$$
$$\ell(\lambda w)=0 \text{ if and only if } \beta\circ\nu(\lambda)=\begin{cases}0 \text{ for } \beta\in\Sigma^+\cap w(\Sigma^+),\\ 1 \text{ for } \beta\in\Sigma^+\cap w(\Sigma^-).\end{cases}$$

Compare with [Vig] Appendice.

A Weyl chamber of V is the open set of $x\in V$ with $\beta(x)>0$ for all β in a basis of Σ. A closed Weyl chamber is the closure of a Weyl chamber.

Example 5.12. *For* $\lambda,\lambda'\in\Lambda$, $\ell(\lambda\lambda')=\ell(\lambda)+\ell(\lambda')$ *if* $\nu(\lambda),\nu(\lambda')$ *belong to the same closed Weyl chamber.*

Proof. If $x,x'\in V$ belong to the same closed Weyl chamber, then $\beta(x)\beta(x')\geq 0$ for all $\beta\in\Sigma$. Then $|\beta(x+x')|=|\beta(x)|+|\beta(x')|$. Apply $\sum_{\beta\in\Sigma}|\beta\circ\nu(\lambda)|=2\ell(\lambda)$. □

For $w\in W$, we recall $\mathbf{q}_w$ from Def. 4.14.

Proposition 5.13. *For* $\lambda\in\Lambda$, *the element* $\mathbf{q}_\lambda$ *depends only on the* W*-orbit of* λ. *In particular,* $\ell(\lambda)$ *depends only on the* W*-orbit of* λ.

The length equality follows from the $\mathbf{q}_*$-equality, but is also a consequence of

$$\beta \circ \nu(w\lambda w^{-1}) = w^{-1}(\beta) \circ \nu(\lambda), \quad \text{for } \beta \in \Sigma, w \in W_0, \tag{77}$$

which implies $2\ell(w\lambda w^{-1}) = \sum_{\beta \in \Sigma} |w^{-1}(\beta) \circ \nu(\lambda)| = \sum_{\beta \in \Sigma} |\beta \circ \nu(\lambda)| = 2\ell(\lambda)$.

If the elements of S^{aff} are W^{aff}-conjugate (Remark 4.9 c)) the length equality implies the $\mathbf{q}_*$-equality.

Proof. Let $\alpha \in \Delta_\Sigma$. We denote $s_\alpha(\lambda) = s_\alpha \lambda s_\alpha \in \Lambda$. We will prove that $\mathbf{q}_\lambda = \mathbf{q}_{s_\alpha(\lambda)}$. This implies the proposition because the group W_0 is generated by s_α for $\alpha \in \Delta_\Sigma$ and the W_0-orbit of λ is equal to the W-orbit of $\lambda \in \Lambda$, as $W = \Lambda \rtimes W_0$ and Λ is commutative.

We have $\mathbf{q}_\lambda = \prod_{\tau \in \mathfrak{T}_\lambda} \mathbf{q}(\tau)$ (Def. 4.14).

We apply Lemma 5.6 to compare $\mathfrak{T}_\lambda$ and $\mathfrak{T}_{s_\alpha(\lambda)}$. Let $\beta \in \Sigma^+, k \in \mathbb{Z}$. If $\beta \neq \alpha$, then $s_\alpha(\beta) \in \Sigma^+$ and $\ell_\beta(s_\alpha(\lambda)) = \ell_{s_\alpha(\beta)}(\lambda)$ (Prop. 5.9 1)). The affine hyperplane $\operatorname{Ker}(\beta+k) \in \mathfrak{H}$ separates $\mathfrak{C}$ from $\lambda(\mathfrak{C})$ if and only if $\operatorname{Ker}(s_\alpha(\beta)+k)$ separates $\mathfrak{C}$ from $s_\alpha(\lambda)(\mathfrak{C})$. Hence $s_{\beta+k} \in \mathfrak{T}_\lambda$ if and only if $s_\alpha(\beta)+k \in \mathfrak{T}_{s_\alpha(\lambda)}$. We have $s_{s_\alpha(\beta)+k} = s_\alpha s_{\beta+k} s_\alpha$.

We have $\ell_\alpha(\lambda) = \alpha \circ \nu(\lambda)$ (Prop. 5.9 1)).

We suppose first $\ell_\alpha(\lambda) = \alpha \circ \nu(\lambda) \neq 0$. We have $\ell_\alpha(s_\alpha(\lambda)) = \ell_{-\alpha}(\lambda) = -\ell_\alpha(\lambda)$.

$\operatorname{Ker}(\alpha+k)$ separates $\mathfrak{C}$ from $\lambda.\mathfrak{C}$ if and only if k in $[0, -\ell_\alpha(\lambda)-1]$ or in $[-\ell_\alpha(\lambda), -1]$, depending on the sign of $\ell_\alpha(\lambda)$.

$\operatorname{Ker}(\alpha+k')$ separates $\mathfrak{C}$ from $s_\alpha(\lambda)(\mathfrak{C})$ if and only if k' in $[0, \ell_\alpha(\lambda)-1]$ or in $[\ell_\alpha(\lambda), -1]$, depending on the sign of $\ell_\alpha(\lambda)$.

We have $s_\alpha s_{\alpha+k} s_\alpha = s_{\alpha-k}$ and $\operatorname{Ker}(\alpha+k)$ separates $\mathfrak{C}$ from $\lambda(\mathfrak{C})$ if and only if $-k$ in $[1, \ell_\alpha(\lambda)]$ or in $[\ell_\alpha(\lambda)+1, 0]$, depending on the sign of $\ell_\alpha(\lambda)$.

Hence if $\alpha \circ \nu(\lambda) \neq 0$,

$$\mathfrak{T}_\lambda = \mathfrak{T}'_\lambda \sqcup \{s\}, \ \mathfrak{T}_{s_\alpha(\lambda)} = \mathfrak{T}'_{s_\alpha(\lambda)} \sqcup \{s'\}, \ \mathfrak{T}'_{s_\alpha(\lambda)} = s_\alpha \mathfrak{T}'_\lambda s_\alpha,$$

$$(s, s') = (s_\alpha, s_{\alpha+\ell_\alpha(\lambda)}) \ \text{ if } \ \ell_\alpha(\lambda) > 0, \quad (s, s') = (s_{\alpha+\ell_\alpha(\lambda)}, s_\alpha) \ \text{ if } \ \ell_\alpha(\lambda) > 0.$$

If $\alpha \circ \nu(\lambda) = 0$, no affine hyperplane $\operatorname{Ker}(\alpha+k)$ separates $\mathfrak{C}$ from $\lambda(\mathfrak{C})$ or from $s_\alpha(\lambda)(\mathfrak{C})$. Setting $\mathfrak{T}_\lambda = \mathfrak{T}'_\lambda$ we deduce

$$\mathfrak{T}'_{s_\alpha(\lambda)} = s_\alpha \mathfrak{T}'_\lambda s_\alpha.$$

In general $\mathbf{q}_\lambda$ is equal to

$$\mathbf{q}'_\lambda \text{ if } \alpha \circ \nu(\lambda) = 0, \ \mathbf{q}'_\lambda \mathbf{q}(s_\alpha) \text{ if } \alpha \circ \nu(\lambda) > 0, \ \mathbf{q}'_\lambda \mathbf{q}(s_{\alpha+\alpha\circ\nu(\lambda)}) \text{ if } \alpha \circ \nu(\lambda) < 0.$$

where $\mathbf{q}'_\lambda := \prod_{\tau \in \mathfrak{T}'_\lambda} \mathbf{q}(\tau)$ and similarly $\mathbf{q}_{s_\alpha(\lambda)}$ is equal to

$$\mathbf{q}'_{s_\alpha(\lambda)} \text{ if } \alpha \circ \nu(\lambda) = 0, \ \mathbf{q}'_{s_\alpha(\lambda)} \mathbf{q}(s_{\alpha-\alpha\circ\nu(\lambda)}) \text{ if } \alpha \circ \nu(\lambda) > 0, \ \mathbf{q}'_{s_\alpha(\lambda)} \mathbf{q}(s_\alpha) \text{ if } \alpha \circ \nu(\lambda) < 0.$$

as $\alpha \circ \nu(s_\alpha(\lambda)) = -\alpha \circ \nu(\lambda)$. We recall that $\mathbf{q}(\tau)$ depends only on the W-orbit of τ (Remark 4.9 e), f)). Applying Lemma 5.2 and $\mathfrak{T}'_{s_\alpha(\lambda)} = s_\alpha \mathfrak{T}'_\lambda s_\alpha$ we obtain $\mathbf{q}_\lambda = \mathbf{q}_{s_\alpha(\lambda)}$. □

Proposition 5.14. *Let $\beta \in \Delta_{\Sigma_j}$. Then $(\beta \circ \nu)(\Lambda^{aff}) = 2\mathbb{Z}$ when Σ_j has rank 1, or when β is a long root of Σ_j and Σ_j has type $C_n, n \geq 2$.*

Otherwise $(\beta \circ \nu)(\Lambda^{aff}) = \mathbb{Z}$.

Proof. The translation subgroup Λ^{aff} of W^{aff} is generated by $s_\gamma s_{\gamma+1}$ for $\gamma \in \Sigma^+$, and $\beta \circ \nu(s_\gamma s_{\gamma+1}) = \beta(\gamma^\vee) = n(\beta, \gamma)$ is a Cartan integer. The group $\beta \circ \nu(\Lambda^{aff})$ is generated by the Cartan integers $n(\beta, \gamma)$ for $\gamma \in \Sigma$ and contains $2 = n(\beta, \beta)$. When γ does not belong to the irreducible component Σ_j we have $0 = n(\beta, \gamma)$.

On the Cartan matrix [Bki, VI.Planches], we see that $n(\beta, \gamma) \in 2\mathbb{Z}$ for all $\gamma \in \Sigma_j$ if and only if Σ_j has a single element, or Σ_j is of type C_n and β is a long root. □

We recall that for $\beta \in \Sigma$, $(\beta \circ \nu)(\Lambda) = \delta\mathbb{Z}$ with $\delta \in \{1,2\}$ and $\delta = 1 \Leftrightarrow s_{\beta-1}$ is conjugate to s_β by Λ (Remark 5.3). For $\beta \in \Sigma_j$ as in Proposition 5.14, $\delta = 2$ implies that Σ_j has rank 1, or β is a long root of Σ_j and Σ_j has type $C_n, n \geq 2$.

We recall the element $\mu_\beta = s_{\beta+1}s_\beta = s_\beta s_{\beta-1} \in \Lambda^{aff}$ (after Lemma 5.2).

Lemma 5.15. *Let $\beta \in \Delta_{\Sigma_j}$ such that $(\beta \circ \nu)(\Lambda^{aff}) = 2\mathbb{Z}$. Let $\tilde{\beta} \in \Sigma_j$ be the highest positive root of Σ_j.*

There exists $s' \in S^{aff}, w \in W_0$ such that

$$s_{\beta-1} = ws'w^{-1}, \quad \ell(\mu_\beta) = 2\ell(w) + 2.$$

If Σ_j has rank 1, then $w = 1$ and $s' = s_{\beta-1} = s_{\tilde{\beta}-1}$.

If Σ_j has type $C_n, n \geq 2$, and β is a long root of Σ_j, then $s' = s_{\tilde{\beta}-1}$ and $w(\tilde{\beta}) = \beta$.

Proof. By Lemma 5.6, the hyperplanes $\mathrm{Ker}(\beta + k)$, for $k \in \mathbb{Z}$, separating $\mathfrak{C}$ and $\mu_\beta(\mathfrak{C})$ are $\mathrm{Ker}(\beta)$ and $\mathrm{Ker}(\beta + 1)$. As $\mathfrak{C}$ and $\mu_\beta(\mathfrak{C})$ are not on the same side of $\mathrm{Ker}\,\beta$, we have $\ell(s_{\beta-1}) < \ell(\mu_\beta)$ [Bki] (V.3.2 Thm.1) hence

$$\ell(\mu_\beta) = \ell(s_{\beta-1}) + 1.$$

We choose a reduced decomposition $s_{\beta-1} = s_1 \ldots s_n$ with $s_j \in S^{aff}$ for $1 \leq j \leq n = \ell(s_{\beta-1})$. By the strong exchange condition ([Kumar] Thm. 1.3.11 c)), there exists a unique integer i such that $1 = s_1 \ldots \hat{s_i} \ldots s_n$. Set

$$s' = s_i, \quad w = s_1 \ldots s_{i-1} = (s_{i+1} \ldots s_n)^{-1}.$$

Then

$$s_{\beta-1} = ws'w^{-1} \text{ with } \ell(s_{\beta-1}) = 2\ell(w) + 1.$$

We deduce $\mu_\beta = s_\beta ws'w^{-1}$, $\ell(\mu_\beta) = 2\ell(w) + 2$.

If Σ_j has rank 1, $w = 1$, $s' = s_{\beta-1}$ and $\ell(\mu_\beta) = 2$.

If Σ_j has type $C_n, n \geq 2$, the long roots of Σ_j are W_0-conjugate to the highest positive root $\tilde{\beta}$. If β is a long root, let $w \in W_0$ such that $w(\tilde{\beta}) = \beta$. Then $w(s_{\tilde{\beta}-1}) = s_{w(\tilde{\beta}-1)} = s_{\beta-1}$. No element of $S^{aff} - \{s_{\tilde{\beta}-1}\}$ is conjugate to $_{\tilde{\beta}-1}$ in W^{aff} ([Borel] 3.3 and [Bki] VI PLanche III). We deduce $s' = s_{\tilde{\beta}-1}$. □

5.2 Alcove walk

By ([Gortz] Definition 2.3.1), an orientation o of $(V, \mathfrak{H})$ is given by distinguishing for each affine hyperplane $H \in \mathfrak{H}$, a positive half-space among the two half-spaces which form the complement of H in V (the non-positive half-space is called negative) such that for all $H \in \mathfrak{H}$, either 1) or 2) holds:

1) For any finite subset of $\mathfrak{H}$, the intersection of the negative half-spaces is non-empty.

2) For any finite subset of $\mathfrak{H}$, the intersection of the positive half-spaces is non-empty.

The group $W(1)$ acts on the orientations of $(V, \mathfrak{H})$. The image by $w \in W(1)$ of an orientation o is the orientation $o \bullet w$ is such that the $o \bullet w$-positive side of $H \in \mathfrak{H}$ is the image by w^{-1} of the o-positive side of $w(H)$. The action of $W(1)$ factorizes through an action of W.

The group W_0 acts simply transitively on the Weyl chambers (the connected components of $V - \cup_{\beta \in \Sigma} \mathrm{Ker}\,\beta$), hence on the bases of Σ, and on the alcoves (the connected components of $V - \cup_{(\beta,k) \in \Sigma^{aff}} \mathrm{Ker}(\beta + k)$) of vertex 0. The basis $\Delta_{\mathfrak{D}}$ associated to the Weyl chamber $\mathfrak{D}$ is the set of $\beta \in \Sigma$ taking positive values on $\mathfrak{D}$ such that s_β is a wall of $\mathfrak{D}$. The Weyl chamber $\mathfrak{D}$ is called $\Delta_{\mathfrak{D}}$-dominant. The action of W_0 inflates to an action of $W_0(1)$.

Definition 5.16. *Let Δ' be a basis of Σ associated to a Weyl chamber $\mathfrak{D}_{\Delta'}$.*

For $H \in \mathfrak{H}$ there exists a unique pair $(\beta, k) \in \Sigma^{aff}$ with

$$H = \operatorname{Ker}(\beta + k) \quad \textit{and } \beta \textit{ is positive on } \Delta'.$$

The orientation $o_{\Delta'}$ such that the $o_{\Delta'}$-positive side of H is the set of $x \in V$ where $\beta(x) + k > 0$ for all $H \in \mathfrak{H}$, is called a spherical orientation.

The $o_{\Delta'}$-negative side of H is the $o_{-\Delta'}$-positive side of H. The most $o_{\Delta'}$-negative point of V lies infinitely far in the Δ'-antidominant Weyl chamber $\mathfrak{D}_{-\Delta'} = -\mathfrak{D}_{\Delta'}$. The $o_{\Delta'}$-negative side of H contains a quartier of the form $y + \mathfrak{D}_{-\Delta'}$.

The spherical orientations o_{Δ_Σ} and $o_{-\Delta_\Sigma}$ are respectively called dominant and antidominant, as the bases $\Delta_\Sigma = \Delta_{\mathfrak{D}^+}, -\Delta_\Sigma = \Delta_{\mathfrak{D}^-}$ of Σ are respectively associated to the dominant and antidominant Weyl chambers $\mathfrak{D}^+$ (containing $\mathfrak{C}$) and $\mathfrak{D}^- = -\mathfrak{D}^+$.

Proposition 5.17. *A spherical orientation $o_{\Delta'}$ is fixed by $\Lambda(1)$ and $o_{\Delta'} \bullet w = o_{w^{-1}(\Delta')}$ for $w \in W_0(1)$.*

Conversely one can prove that an orientation fixed by $\Lambda(1)$ is a spherical orientation.

Proof. Let $\lambda \in \Lambda, x \in V, \beta \in \Sigma, k \in \mathbb{Z}$.

We suppose that β is Δ'-positive. Then x belongs to the $o_{\Delta'} \bullet \lambda$-positive side of $\operatorname{Ker}(\beta + k)$ if $x + \nu(\lambda)$ belongs to the $o_{\Delta'}$-positive side of $\operatorname{Ker}(\beta + k) + \nu(\lambda)$. We have $\operatorname{Ker}(\beta + k) + \nu(\lambda) = \operatorname{Ker}(\beta + k - (\beta \circ \nu)(\lambda))$ and $\beta(x + \nu(\lambda)) + k - (\beta \circ \nu)(\lambda) = \beta(x) + k$. Therefore $o_{\Delta'} \bullet \lambda = o_{\Delta'}$.

We suppose that β is $w^{-1}(\Delta')$-positive, that is, $w(\beta)$ is Δ'-positive, and that x belongs to the $o_{w^{-1}(\Delta')}$-positive side of $\operatorname{Ker}(\beta + k)$. We have $\beta(x) + k > 0$ and $\beta(x) = w(\beta)(w.x)$. Hence $w.x$ belongs to the $o_{\Delta'}$-positive side of $\operatorname{Ker}(w(\beta) + k)$. We have $\operatorname{Ker}(w(\beta) + k) = w. \operatorname{Ker}(\beta + k)$. Hence x belongs to the $o_{\Delta'} \bullet w$-positive side of $\operatorname{Ker}(\beta + k)$. ☐

Let o be an orientation of $(V, \mathfrak{H})$. We say that we cross $H \in \mathfrak{H}$ in the o-positive direction if we go from the o-negative side to the o-positive side (in the o-negative direction otherwise). Let $(w, s) \in W \times S^{aff}$. When we walk from the alcove $w.\mathfrak{C}$ to the alcove $ws.\mathfrak{C}$, we cross the affine hyperplane $H_{wsw^{-1}} \in \mathfrak{H}$ fixed by wsw^{-1}.

Definition 5.18. *Let o be an orientation of $(V, \mathfrak{H})$ and let $(w, s) \in W \times S^{aff}$. Let*

- *$\epsilon_o(w, s) = 1$ if $w.\mathfrak{C}$ belongs to the o-negative side of $H_{wsw^{-1}}$,*
- *$\epsilon_o(w, s) = -1$ if $w.\mathfrak{C}$ belongs to the o-positive side of $H_{wsw^{-1}}$.*

Let $\epsilon_o(\tilde{w}, \tilde{s}) = \epsilon_o(w, s)$ for $\tilde{w}, \tilde{s} \in W(1)$ lifting w, s.

When we walk from $w.\mathfrak{C}$ to $ws.\mathfrak{C}$, we cross $H_{wsw^{-1}}$ in the o-positive, resp. o-negative, direction if $\epsilon_o(w, s) = 1$, resp. -1. We say that we cross $H_{wsw^{-1}}$ in the $\epsilon_o(w, s)$ direction with respect to o.

Let $s_1, \ldots, s_n$ in S^{aff}. The walk from $\mathfrak{C}$ to $s_1 \ldots s_n(\mathfrak{C})$ following the gallery $\mathfrak{C}$, $s_1.\mathfrak{C}$, $s_1 s_2.\mathfrak{C}, \ldots, s_1 \ldots s_n.\mathfrak{C}$, crosses the hyperplanes

$$H_{s_1} = H_{\tau_1},\ s_1.H_{s_2} = H_{\tau_2},\ \ldots,\ s_1 s_2 \ldots s_{n-1}.H_{s_n} = H_{\tau_n},$$

where $\mathfrak{T}(s_1, \ldots, s_n) = (\tau_1, \ldots, \tau_n)$ (68), in the

$$\epsilon_o(1, s_1), \epsilon_o(s_1, s_2), \ldots, \epsilon_o(s_1 \ldots s_{i-1}, s_i), \ldots \epsilon_o(s_1 \ldots s_{n-1}, s_n) \tag{78}$$

directions with respect to o.

Example 5.19. *For $J \subset \Delta$, let $S_J = \{s_\beta \mid \beta \in J\}$, let $W_J \subset W_0$ be the subgroup generated by S_J and let w_J be the element of maximal length in W_J.*

For $w \in W_J, s \in S_J$, we have $\epsilon_{o_{w_J(\Delta)}}(w, s) = 1$ if and only if $\ell(ws) = \ell(w) + 1$.

Proof. Let $\Sigma_J \subset \Sigma$ be the root system generated by J. Let $\beta \in J$ such that $s = s_\beta$. We have $H_{wsw^{-1}} = \operatorname{Ker} w(\beta)$. Let $x \in \mathfrak{C}$. The alcove $w.\mathfrak{C}$ is contained in the $o_{w_J(\Delta)}$-negative side of $H_{sws^{-1}}$ if and only if $w(\beta)$ is $w_J(\Delta)$-negative because $w(\beta)(w.x) = \beta(x)$ is positive. The root $w(\beta)$ belongs to Σ_J; hence $w(\beta)$ is $w_J(\Delta)$-negative, if and only if $w_J w(\beta)$ is Δ-negative, if and only if $w(\beta)$ is Δ-positive, if and only if $\ell(ws_\beta) > \ell(w)$. □

Lemma 5.20. *Let $s, s' \in S^{aff}$ with ss' of finite order $n(s,s')$. Then the sequences with $n(s,s')$ terms*

$$(\epsilon_o(1,s), \epsilon_o(s,s'), \epsilon_o(ss',s), \ldots) \text{ and } (\epsilon_o(1,s'), \epsilon_o(s',s), \epsilon_o(s's,s'), \ldots)$$

are equal to $(1,1\ldots,1,-1,-1\ldots-1)$ and $(-1,-1\ldots,-1,1,1\ldots1)$,
or to $(-1,-1\ldots,-1,1,1\ldots1)$ *and* $(1,1\ldots,1,-1,-1\ldots-1)$,
where $(1,1\ldots,1)$ *have the same length* k, $0 \le k \le n(s,s')$, *in both sequences.*

Proof. ([Gortz] Proof of Thm. 3.3.1). □

Lemma 5.21. *For (w,w',s,u) in $W \times W \times S^{aff} \times \Omega$, we have $\epsilon_o(ws,s) \neq \epsilon_o(w,s)$ and*

$$\epsilon_{o\bullet w}(w',s) = \epsilon_o(ww',s), \quad \epsilon_o(wu,s) = \epsilon_o(w,usu^{-1}), \quad \epsilon_{o\bullet u}(w,s) = \epsilon_o(uwu^{-1},u^{-1}su).$$

In particular $\epsilon_o(1,s) \neq \epsilon_{o\bullet s}(1,s)$.

Proof. 1) $ws(\mathfrak{C})$ and $w(\mathfrak{C})$ are in different sides of $H_{wsw^{-1}}$.

2) $\epsilon_{o\bullet w}(w',s) = 1$ if and only if $w'.\mathfrak{C}$ is contained in the $o \bullet w$-negative side of $H_{w'sw'^{-1}}$. The $o \bullet w$-negative side of $H_{w'sw'^{-1}}$ is the image by w^{-1} of the o-negative side of $H_{ww'sw'^{-1}w^{-1}}$. Hence $\epsilon_{o\bullet w}(w',s) = 1$ if and only if $ww'.\mathfrak{C}$ is contained in the o-negative side of $H_{ww'sw'^{-1}w^{-1}}$, if and only if $\epsilon_o(ww',s) = 1$.

3) We have $u.\mathfrak{C} = \mathfrak{C}$. We have $\epsilon_o(wu,s) = 1$ if and only if $wu.\mathfrak{C} = w.\mathfrak{C}$ is contained in the o-negative side of $H_{wusu^{-1}w^{-1}}$ if and only if $\epsilon_o(w,usu^{-1}) = 1$.

4) We compute $\epsilon_{o\bullet u}(w,s) = \epsilon_o(uw,s) = \epsilon_o(uwu^{-1},u^{-1}su)$.

5) $\epsilon_{o\bullet s}(1,s) = \epsilon_o(s,s) \neq \epsilon_o(1,s)$.

□

5.3 Alcove walk bases

Notations as in Thm. 4.7 and Def. 5.16. We will associate to any orientation o of $(V,\mathfrak{H})$ a basis $(E_o(w))_{w\in W(1)}$ of the R-algebra $\mathcal{H}_R(q_s,c_s)$ of $W(1)$ with parameters $(q_s,c_s)_{s\in S^{aff}(1)}$.

Definition 5.22. *For $(w,s) \in W(1) \times S^{aff}(1)$ and an orientation $o \in (V,\mathfrak{H})$, and for $T_s \in \mathcal{H}_R(q_s,c_s)$ we set:*

$$T_s^{\epsilon_o(w,s)} = T_s \text{ if } \epsilon_o(w,s) = 1, \quad T_s^{\epsilon_o(w,s)} = T_s^* = T_s - c_s \text{ if } \epsilon_o(w,s) = -1 \tag{79}$$

where $\epsilon_o(w,s)$ is defined in Def. 5.18. For $s_1, \ldots, s_n$ in $S^{aff}(1), u, u' \in \Omega(1)$, we set:

$$E_o(u,s_1,\ldots,s_n,u') = T_u T_{s_1}^{\epsilon_o(u,s_1)} \ldots T_{s_i}^{\epsilon_o(us_1\ldots s_{i-1},s_i)} \ldots T_{s_n}^{\epsilon_o(us_1\ldots s_{n-1},s_n)} T_{u'}. \tag{80}$$

We remark that $E_o(s) = T_s^{\epsilon_o(1,s)}$, $E_{o\bullet s}(s) = T_s^{\epsilon_{o\bullet s}(1,s)}$,

$$E_{o\bullet s}(s) = E_o(s) + \epsilon_{o\bullet s}(1,s)c_s, \quad E_o(s)E_{o\bullet s}(s) = q_s s^2. \tag{81}$$

(Remark 4.12 and Lemma 5.21).

We suppose first that $q_s = 1$ for all $s \in S^{aff}/\sim$. In this case $T_s^* = T_{s^{-1}}^{-1}$.

Proposition 5.23. *When $q_s = 1$ for all $s \in S^{aff}/\sim$, $E_o(w) = E_o(u, s_1, \ldots, s_n, u')$ depends only on the product $w = us_1 \ldots s_n u'$.*

Proof. [Gortz, Thm. 3.3.1], [Schmidt, Thm. 3.3.19].

a) Let $s, s' \in S^{aff}$ of finite order $n(s, s')$ and $s(1), s'(1) \in S^{aff}(1)$ satisfying b3 (this is the only place where b3 is needed). We show

$$E_o(s(1), s'(1), \ldots) = E_o(s'(1), s(1), \ldots).$$

By symetry we can suppose that the sequences in Lemma 5.20 are $(1, 1 \ldots, 1, *, * \ldots *)$ and $(*, * \ldots, *, 1, 1 \ldots 1)$ with k terms equal to 1. We decompose accordingly the products $s(1)s'(1) \ldots = w_k w_{n(s,s')-k}$, $s'(1)s(1) \ldots = w'_{n(s,s')-k} w'_k$. By the braid relations,

$$E_o(s(1), s'(1), \ldots) = T_{w_k} T^{-1}_{w^{-1}_{n(s,s')-k}} \text{ and } E_o(s'(1), s(1), \ldots) = T^{-1}_{w'^{-1}_{n(s,s')-k}} T_{w'_k}.$$

The element $w'^{-1}_{n(s,s')-k} w_k = w'_k w^{-1}_{n(s,s')-k}$ has length $n(s, s')$ because $w'^{-1}_{n(s,s')-k}$ ends by $s'(1)^{-1}$ while w_k begins with $s(1)$. The additivity of the lengths is satisfied and by the braid relations $T_{w'^{-1}_{n(s,s')-k}} T_{w_k} = T_{w'_k} T_{w^{-1}_{n(s,s')-k}}$. We deduce $E_o(s(1), s'(1), \ldots) = E_o(s'(1), s(1), \ldots)$.

b) Let $t_1 \ldots t_n \in Z_k$ such that $s_1 \ldots s_n = s'_1 \ldots s'_n$ where $s'_i = s_i t_i$ for $1 \leq i \leq n$. The equality

$$E_o(s_1, \ldots, s_n, u) = E_o(s'_1, \ldots, s'_n, u).$$

is obvious by the braid relations using that the elements of Z_k have length 0, act trivially on V, and that Z_k is normal in $W(1)$.

c) We suppose that $s_{i+1} = s_i^{-1}$. We have

$$E_o(s_1, \ldots, s_n, u) = E_o(s_1, \ldots s_{i-1}, s_{i+2}, \ldots, s_n, u),$$

because $T_s^{\epsilon_o(w,s)} T_{s^{-1}}^{\epsilon_o(ws,s^{-1})} = 1$. This follows from $\epsilon_o(w, s) \neq \epsilon_o(ws, s)$ (Lemma 5.21); we recall that $\epsilon_o(ws, s^{-1}) = \epsilon_o(ws, s)$ and that $q_s = 1$.

d) As the elements of the group $\Omega(1)$ normalize $S^{aff}(1)$ and have length 0, the equality

$$E_o(u, s_1, \ldots, s_n, u') = E_o(us_1u^{-1}, \ldots, us_nu^{-1}, uu')$$

follows from $T_u T_s^* = T^*_{usu^{-1}} T_u$ and $\epsilon_{o \bullet u}(w, s) = \epsilon_o(uwu^{-1}, usu^{-1})$ for $s \in S^{aff}(1), w \in W(1), u \in \Omega(1)$ (Prop. 4.13 5)) .

e) The proposition follows from a), b), c), d) and [Bki, IV.1.5 Prop. 5].

□

Proposition 5.24. *When $q_s = 1$ for all $s \in S^{aff}/\sim$, we have the product formula*

$$E_o(ww') = E_o(w) E_{o \bullet w}(w') \quad (w, w' \in W(1)). \tag{82}$$

Proof. Let $w = s_1 \ldots s_n u, w' = s'_1 \ldots s'_m u'$ with $s_i, s'_j \in S^{aff}(1), u' \in \Omega(1)$. From Prop. 5.23 we have

$$E_o(ww') = E_o(s_1, \ldots, s_n, us'_1u^{-1}, \ldots us'_mu^{-1}, uu').$$

From $\epsilon_{o \bullet w}(1, s) = \epsilon_o(w, s)$ for $(w, s) \in W(1) \times S^{aff}(1)$ (Lemma 5.21), the right hand side is equal to

$$E_o(s_1 \ldots s_n) E_{o \bullet w}(us'_1u^{-1}, \ldots us'_mu^{-1}, uu')$$

and $E_{o \bullet w}(us'_1u^{-1}, \ldots us'_mu^{-1}, uu') = E_{o \bullet w}(uw') = T_u E_{o \bullet w}(w')$ from Prop. 5.23. Hence $E_o(ww') = E_o(s_1 \ldots s_n) T_u E_{o \bullet w}(w') = E_o(w) E_{o \bullet w}(w')$. □

We recall $q_{w,w'} = (q_w q_{w'} q^{-1}_{ww'})^{1/2}$ for $w, w' \in W(1)$ (Def. 4.14).

Theorem 5.25. *Let o be an orientation of $(V,\mathfrak{H})$, let $w,w'\in W(1)$, and let a reduced decomposition $w=s_1\dots s_{\ell(w)}u$ $u\in\Omega(1)$, $s_i\in S^{aff}(1)$ for $1\le i\le \ell(w)$. Then,*

$$E_o(w)=E_o(s_1,\dots,s_{\ell(w)},u)\in\mathcal{H}_R(q_s,c_s)$$

depends only on w and $E_o(w)E_{o\bullet w}(w')=q_{w,w'}E_o(ww')$.

Proof. As in the subsection (4.5), the $R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]$-algebra $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(1,\mathfrak{q}_s^{-1}c_s)$ satisfies Prop. 5.23 and 5.24. Let $s\in S^{aff}(1)$ and $u\in\Omega(1)$. The elements $T_s,T_u,T_s^*=T_s-\mathfrak{q}_s^{-1}c_s$ and $E_o(w)$ in $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(1,\mathfrak{q}_s^{-1}c_s)$ are sent by the isomorphism $h\mapsto\tilde{h}$ (72) to elements of $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s,c_s)$ equal to

$$\tilde{T_s}=\mathfrak{q}_s^{-1}T_s,\quad \tilde{T_u}=T_u,\quad \tilde{(T_s^*)}=\tilde{T}_s-\mathfrak{q}_s^{-1}c_s=\mathfrak{q}_s^{-1}(T_s-c_s)=\mathfrak{q}_s^{-1}T_s^*,$$
$$\tilde{E}_o(w)=\tilde{(T_{s_1}^{\epsilon_o(1,s_1)})}\dots\tilde{(T_{s_i}^{\epsilon_o(s_1\dots s_{i-1},s_i)})}\dots\tilde{(T_{s_{\ell(w)}}^{\epsilon_o(s_1\dots s_{n-1},s_{\ell(w)})})}\tilde{T}_u.$$

We have
$$\tilde{E}_o(w)=\mathfrak{q}_w^{-1}E_o(s_1,\dots,s_{\ell(w)},u).$$

In $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s,c_s)$, the product $\mathfrak{q}_w\tilde{E}_o(w)$ depends only on w hence the same is true for $E_o(s_1,\dots,s_{\ell(w)},u)$. The product formula in $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(1,\mathfrak{q}_s^{-1}c_s)$ implies the product formula

$$E_o(w)E_{o\bullet w}(w')=\mathbf{q}_{w,w'}E_o(ww'),\quad \mathbf{q}_{w,w'}=\mathfrak{q}_w\mathfrak{q}_{w'}\mathfrak{q}_{ww'}^{-1},\tag{83}$$

in $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s,c_s)$. The product formula holds true in the generic $R[\mathbf{q}_s]$-subalgebra $\mathcal{H}_{R[\mathbf{q}_s]}(\mathbf{q}_s,c_s)$. By specialisation, it holds true in $\mathcal{H}_R(q_s,c_s)$.

□

The Bruhat partial order $<$ on (W^{aff},S^{aff}) extends to $W=W^{aff}\rtimes\Omega: wu<w'u'$ for $w,w'\in W^{aff},u,u'\in\Omega$ if $w<w'$ and $u=u'$. The extended Bruhat partial order $<$ on W inflates to $W(1):\tilde{w}<\tilde{w}'$ for $\tilde{w},\tilde{w}'\in W(1)$ lifting $w,w'\in W$ if $w<w'$.

Corollary 5.26. *$(E_o(w))_{w\in W(1)}$ is an R-basis of $\mathcal{H}_R(q_s,c_s)$ satisfying the triangular decomposition: for $w\in W(1)$, $E_o(w)=T_w+\sum_{w'<w}a_{w'}T_{w'}$ with $w'\in W(1),a_{w'}\in R$.*

Proof. $E_o(w)-T_w$ is a finite sum of elements $T_{w'}$ where $w'=s'_1\dots s'_r tu$ for $t\in Z_k,u\in\Omega(1)$, and $(s'_1,\dots,s'_r)$ extracted from the sequence $(s_1,\dots,s_n)$ with $r<n$. □

Corollary 5.27. *Let $w=s_1\dots s_r u$ with $s_i\in S^{aff}(1),u\in\Omega(1)$ be a reduced decomposition. Then,*
$$E_o(w)=E_o(s_1)E_{o\bullet s_1}(s_2),\dots E_{o\bullet s_1s_2\dots s_{r-1}}(s_r)T_u.$$

Proof. $q_{w,w'}=1$ when $\ell(w)+\ell(w')=\ell(ww')$ (Lemma 4.16). □

One does not need to change the orientation o in the product formula in $E_o(\Lambda(1))$ when $\Lambda(1)$ fixes o. By Prop. 5.17:

Corollary 5.28. *Let o be a spherical orientation of $(V,\mathfrak{H})$. The R-submodule of $\mathcal{H}_R(q_s,c_s)$ of basis $(E_o(\lambda))_{\lambda\in\Lambda(1)}$ is an R-subalgebra $\mathcal{A}_o$ of $\mathcal{H}_R(q_s,c_s)$, with product*

$$E_o(\lambda)E_o(\lambda')=q_{\lambda,\lambda'}E_o(\lambda\lambda')\quad(\lambda,\lambda'\in\Lambda(1)).$$

The R-algebras $\mathcal{A}_o, \mathcal{A}_{o'}$ associated to two spherical orientations o, o', are isomorphic by the linear map sending $E_o(\lambda)$ to $E_{o'}(\lambda)$ for all $\lambda \in \Lambda(1)$.

For each (open) Weyl chamber $\mathfrak{D}$ of V, let $\Lambda_{\overline{\mathfrak{D}}}(1)$ the monoid of elements $\lambda \in \Lambda(1)$ such that $\nu(\lambda)$ belongs to the closure $\overline{\mathfrak{D}}$ of $\mathfrak{D}$. For $w \in W_0(1)$, the R-linear map $\lambda \mapsto w(\lambda) := w\lambda w^{-1}$ is an isomorphism $R[\Lambda(1)_{\overline{\mathfrak{D}}}] \to R[\Lambda(1)_{w(\overline{\mathfrak{D}})}]$.

Let $\lambda, \lambda' \in \Lambda(1)$. We have $\mathbf{q}_{\lambda,\lambda'} = 1$ if and only if $\ell(\lambda\lambda') = \ell(\lambda) + \ell(\lambda')$(Lemma 4.16) if and only if $\nu(\lambda), \nu(\lambda')$ belong to the same closed Weyl chamber (Example 5.12). We deduce:

Corollary 5.29. *Let o be a spherical orientation of $(V, \mathfrak{H})$ and let $\mathfrak{D}$ be a Weyl chamber of V. Then, the monoid R-algebra $R[\Lambda_{\overline{\mathfrak{D}}}(1)]$ embeds in $\mathcal{H}_R(q_s, c_s)$ by the linear map such that*

$$\lambda \mapsto E_o(\lambda) \quad (\lambda \in \overline{\mathfrak{D}}(1)).$$

Example 5.30. *Let Δ' be a basis of Σ and $\lambda \in \Lambda, t \in \operatorname{Ker}\nu$ of lift $\tilde{\lambda}, \tilde{t} \in \Lambda(1)$. Then*
$E_{o_{\Delta'}}(\tilde{\lambda}) = T_{\tilde{\lambda}}$ if $\nu(\lambda)$ belongs to the closed Δ'-dominant Weyl chamber,
$E_{o_{\Delta'}}(\tilde{\lambda}) = T^*_{\tilde{\lambda}}$ if $\nu(\lambda)$ belongs to the closed $-\Delta'$-dominant Weyl chamber.
$E_{o_{\Delta'}}(\tilde{\lambda}\tilde{t}) = E_{o_{\Delta'}}(\tilde{\lambda})T_{\tilde{t}} = T_{\tilde{\lambda}\tilde{t}\tilde{\lambda}^{-1}}E_{o_{\Delta'}}(\tilde{\lambda})$.

Proof. When we walk from the alcove $\mathfrak{C}$ to the alcove $\mathfrak{C} + \nu(\lambda)$ we cross hyperplanes in the $o_{\Delta'}$- positive, resp. negative, direction because we walk away from, resp. towards to, the most $o_{\Delta'}$-negative point of V which lies infinitely deep in the $-\Delta'$ dominant Weyl chamber. When $t \in \operatorname{Ker}\nu$ we have $q_{\lambda t} = q_{\lambda t} = q_{\lambda,t} = q_{t.\lambda} = 1$ because $\operatorname{Ker}\nu \subset \Omega$ (Cor. 5.11) and $q_{wu} = q_{uw}$ for $w \in W, u \in \Omega$ (Def. 4.14). ☐

We recall the involutive automorphism ι of $\mathcal{H}_R(q_s, c_s)$ such that $\iota(T_w) = (-1)^{\ell(w)}T^*_w$ for $w \in W(1)$.

Lemma 5.31. $\iota(E_{o_{\Delta'}}(w)) = (-1)^{\ell(w)}E_{o_{-\Delta'}}(w)$ *for $w \in W(1)$.*

Proof. The $o_{\Delta'}$-positive side and the $o_{-\Delta'}$-negative side of any hyperplane $H \in \mathfrak{H}$ are equal. We have $\iota(T_s) = -T^*_s$ and $\iota(T^*_s) = -T_s$ for $s \in S^{aff}(1)$. ☐

Example 5.32. *With the notations of Example 5.19, for $J \subset \Delta$ and $w \in W_J(1)$ we have*

$$E_{o_{w_J(\Delta)}}(w) = T_w, \quad E_{o_{w_J(\Delta)} \bullet w^{-1}}(w) = T^*_w, \quad E_{o_{w_J(\Delta)}}(w)E_{o_{w_J(\Delta)} \bullet w}(w^{-1}) = q_w.$$

In particular, for $w \in W_0(1)$,

$$E_{o_{-\Delta}}(w) = E_{o_\Delta \bullet w^{-1}}(w) = T_w, \ E_{o_{-\Delta} \bullet w^{-1}}(w) = E_{o_\Delta}(w) = T^*_w.$$

Proof. 1) We have $\epsilon_{o_{w_J(\Delta)}}(w, s) = 1$ for $(w, s) \in W_J \times S_J$ with $\ell(ws) > \ell(w)$ (Example 5.19). Applying this to $(s_1 \ldots s_{i-1}, s_i)$ if $w = s_1 \ldots s_{\ell(w)}$ is a reduced expression of w, we get $E_{o_{w_J(\Delta)}}(w) = T_w$.

2) $\epsilon_{o_{w_J(\Delta)} \bullet w^{-1}}(s_1 \ldots s_{i-1}, s_i) = \epsilon_{o_{w_J(\Delta)}}(s_n \ldots s_i, s_i) \neq \epsilon_{o_{w_J(\Delta)}}(s_n \ldots s_{i+1}, s_i) = 1$ (Lemma 5.21).

3) Prop. 4.13, 4.22. ☐

Example 5.33. *For $s \in S(1), \tilde{s} \in S^{aff}(1) - S(1), w \in W_0(1)$, we have:*

$$E_{o_{-\Delta}}(\tilde{s}) = T^*_{\tilde{s}}, \quad E_{o_{-\Delta} \bullet w}(s) = T_s \ \text{ if and only if } \ell(ws) > \ell(w).$$

Proof. a) Let $x \in \mathfrak{C}$. We have $\epsilon_o(1, \tilde{s}) = -1$ if and only if x belongs in the $o_{-\Delta}$-positive part of $H_{\tilde{s}}$. We have $H_{\tilde{s}} = \operatorname{Ker}(-\tilde{\beta}_j + 1)$ where $\tilde{\beta}_j$ is the longest root of an irreducible component Δ_j of Δ, and $-\tilde{\beta}_i(x) + 1 > 0$. As $-\tilde{\beta}_i$ is $-\Delta$ positive, x belongs in the $o_{-\Delta}$-positive part of $\operatorname{Ker}(-\tilde{\beta}_j + 1)$.

b) $\epsilon_{o_{-\Delta} \bullet w}(1, s) = \epsilon_{o_{-\Delta}}(w, s)$ by Lemma 5.21. Then, Example 5.19. ☐

Lemma 5.34. *Let Δ' be a basis of Σ, $\beta \in \Sigma^+ \cap \pm\Delta'$ and $w \in W(1)$ such that $\ell_\beta(w) = 0$. Then,*

$$E_{o_{\Delta'} \bullet s_\beta}(w) = E_{o_{\Delta'}}(w).$$

Proof. If $\beta \in \pm\Delta'$ and $\alpha \in \Sigma - \{\pm\beta\}$, then the $o_{\Delta'}$-positive and $o_{\Delta'} \bullet s_\beta$-positive sides of $\operatorname{Ker}(\alpha + n)$, for all $n \in \mathbb{Z}$, are equal.

If $\beta \in \Sigma^+$, then $\ell_\beta(w) = 0$ if and only if the alcoves $\mathfrak{C}, w(\mathfrak{C})$ are on the same side of $H_{\beta+n}$ for all $n \in \mathbb{Z}$ (Lemma 5.6). This means that no $H_{\beta+n}$ for $n \in \mathbb{Z}$ belongs to $\mathfrak{T}_w$ (Def. 4.14).

We deduce that if $\beta \in \Sigma^+ \cap \pm\Delta', \ell_\beta(w) = 0$, then the $o_{\Delta'}$-positive side of H_τ is equal to the $o_{\Delta'} \bullet s_\beta$-positive side of H_τ for all $\tau \in \mathfrak{T}_w$, or equivalently, $E_{o_{\Delta'} \bullet s_\beta}(w) = E_{o_{\Delta'}}(w)$. ☐

Until the end of this article, we consider only spherical orientations of $(V, \mathfrak{H})$.

5.4 Bernstein relations

Let o be a spherical orientation of Weyl chamber $\mathfrak{D}_o$ associated to the basis Δ_o of Σ formed by the set of $\beta \in \Sigma$ positive on $\mathfrak{D}_o$ and such that $\operatorname{Ker}\beta$ is a wall of $\mathfrak{D}_o$ (Def. 5.16). We denote by (W_o, S_o) where $S_o := \{s_\beta \mid \beta \in \Delta_o\}$ the corresponding the Coxeter system. We have $S_o = S$, when o is dominant or antidominant. For $w \in W(1)$ and $\lambda \in \Lambda(1)$ we have $w(\lambda) := w\lambda w^{-1} \in \Lambda(1)$.

Proposition 5.35. *$E_o(s)E_{o\bullet s}(\lambda) = E_o(s\lambda s^{-1})E_o(s)$ if $\lambda \in \Lambda(1), s \in W_o(1)$ and $s^2 \in Z_k$.*

Proof. As $s^2 \in Z_k$, the product formula implies (Thm. 5.25) :

$$E_o(s)E_{o\bullet s}(\lambda) = q_{s,\lambda}E_o(s\lambda), \quad E_o(s(\lambda))E_o(s) = q_{s(\lambda),s}E_o(s\lambda),$$

and

$$q_{s,\lambda} = q_{s(\lambda),s}$$

because $q^2_{s,\lambda} = q_s q_\lambda q^{-1}_{s\lambda}$ and $q^2_{s\lambda s^{-1},s} = q_{s\lambda s^{-1}} q_s q^{-1}_{s\lambda}$ by Definition 4.14, and $q_{s\lambda s^{-1}} = q_\lambda$ as $s \in W_o(1)$ by Prop. 5.13. ☐

We denote by $\Lambda^s(1)$ the group of $\lambda \in \Lambda(1)$ such that $\nu(\lambda)$ is fixed by $s \in W_o(1)$; note that if s lifts $s_\beta, \beta \in \Sigma$, then $\lambda \in \Lambda^s(1)$ is equivalent to $(\beta \circ \nu)(\lambda) = 0$.

Definition 5.36. *Let $\lambda \in \Lambda(1), s \in W_o(1), s^2 \in Z_k$. Then,*

$$E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda)) = E_o(s\lambda s^{-1})E_o(s) - E_o(s)E_o(\lambda) \tag{84}$$

will be called a Bernstein element. Note that when $\lambda \in \Lambda^s(1), s \in S_o(1)$, the Bernstein element vanishes because $E_{o\bullet s}(\lambda) - E_o(\lambda) = 0$ by Lemma 5.34.

When $s \in (S \cap S_o)(1)$ we will show that the Bernstein element (84) belongs to the subalgebra $\mathcal{A}_o$ (Cor. 5.28); its explicit expansion in the alcove walk basis $(E_o(w))_{w\in W(1)}$ is called a Bernstein relation.

Notation 5.37. We suppose $s \in s_\beta(1), \beta \in \Delta, (\beta \circ \nu)(\lambda) \neq 0$. We denote by $\epsilon_\beta(\lambda) \in \{1, -1\}$ the sign of $\beta \circ \nu(\lambda)$. By Cor. 5.10,

$$\epsilon_\beta(\lambda) = -1 \Leftrightarrow \ell(s\lambda) < \ell(\lambda), \quad \epsilon_\beta(\lambda) = 1 \Leftrightarrow \ell(s\lambda) > \ell(\lambda).$$

The image of $\beta \circ \nu$ is $\delta\mathbb{Z}$ with $\delta \in \{1, 2\}$ (Remark 5.3). Let $n_\beta(\lambda)$ be the positive integer such that

$$\beta \circ \nu(\lambda) = \epsilon_\beta(\lambda)\delta n_\beta(\lambda).$$

We choose $\lambda_s \in \Lambda(1)$ with $\beta \circ \nu(\lambda_s) = -\delta$. If $\delta = 1$ there is no other condition on λ_s. If $\delta = 2$, we suppose that λ_s is a lift of $\mu_\beta = s_{\beta+1} s_\beta \in \Lambda^{aff}$ as in Lemma 5.15. Hence

$$\lambda_s = sw\tilde{s}w^{-1}, \ \ell(\lambda_s) = 2\ell(w) + 2,$$

where $w \in W^{aff}(1)$, $\tilde{s} \in S^{aff}(1)$ lifts $s_{-\tilde{\beta}+1}$ for the highest root $\tilde{\beta}$ of the irreducible component Σ_j of Σ containing β. Note that $\nu(\mu_\beta) = -\beta^\vee$ and that the image of $sw\tilde{s}w^{-1}s^{-1}$ in W is $s_\beta s_{\beta-1} s_\beta = s_{\beta+1}$.

We define elements $B_{o,n} \in \mathcal{A}_o$ for $n \in \mathbb{N}_{>0}$. For $n = 1$,

$$B_{o,1} := (\delta - 1)(w \bullet c_{\tilde{s}}) s^2 E_o(\lambda_s^{-1}) + c_s, \tag{85}$$

where the term containing $(w \bullet c_{\tilde{s}})$ appears only when $\delta = 2$. For $n \geq 2$,

$$B_{o,n} := \sum_{k=0}^{n-1} E_o(s(\lambda_s^k)) B_{o,1} E_o(\lambda_s^{-k}). \tag{86}$$

For $n \geq 1$, the inner k-th term

$$E_o(s(\lambda_s^k)) B_{o,1} E_o(\lambda_s^{-k}) = E_o(s(\lambda_s^k))((\delta - 1)(w \bullet c_{\tilde{s}}) s^2 E_o(\lambda_s^{-1}) + c_s) E_o(\lambda_s^{-k})$$

is the sum of $\delta = 1, 2$ terms

$$\begin{aligned} &(\delta - 1)c(2k+1)E_o(\mu(2k+1)) + c(\delta k)E_o(\mu(\delta k)) \\ =&(\delta - 1)E_o(\mu(2k+1))c'(2k+1) + E_o(\mu(\delta k))c'(\delta k). \end{aligned}$$

where $\mu(k) \in \Lambda(1)$ and $c(k), c'(k)$ in $R[Z_k]$ depend on λ_s and are defined, for $k \in \mathbb{Z}$, by :

$$\mu(\delta k) = s(\lambda_s^k)\lambda_s^{-k}, \ c(\delta k) = s(\lambda_s^k) \bullet c_s, \ c'(\delta k) = \lambda_s^k \bullet c_s,$$

and if $\delta = 2$,

$$\mu(2k+1) = s(\lambda_s^k)\lambda_s^{-k-1}, \ c(2k+1) = s(\lambda_s^k) \bullet c_\tau, \ c'(2k+1) = \lambda_s^{k+1} \bullet c_\tau$$

where $\tau = w \bullet c_{\tilde{s}} s^2 = \tilde{w} c_{\tilde{s}} s^2 \tilde{w}^{-1}$ for any $\tilde{w} \in W(1)$ lifting w.
We obtain the expansions

$$B_{o,n} = \sum_{k=0}^{\delta n-1} c(k)E_o(\mu(k)) = \sum_{k=0}^{\delta n-1} E_o(\mu(k))c'(k),$$

We recall the signs $\epsilon_o(1, s) \neq \epsilon_{o\bullet s}(1, s)$ (Lemma 5.21).

The Bernstein relations when $q_s = 1$ for $s \in S^{aff}$ are:

Theorem 5.38. *We suppose that $q_s = 1$ for $s \in S^{aff}$.*
Let $s \in (S \cap S_o)(1)$ and $\lambda \in \Lambda(1) - \Lambda^s(1)$. The Bernstein element (84) *is equal to*

$$\epsilon_{o\bullet s}(1, s)B_{o,n_\beta(\lambda)} \ E_o(\lambda) \ \textit{if} \ \epsilon_\beta(\lambda) = -1, \quad \epsilon_o(1, s) \ E_o(s\lambda s^{-1})B_{o,n_\beta(\lambda)} \ \textit{if} \ \epsilon_\beta(\lambda) = 1,$$

where $s \in s_\beta(1), \beta \in \Delta$.

The proof is divided in steps given as lemmas. We use the notations (5.37).

Lemma 5.39. *We suppose that $q_s = 1$ for $s \in S^{aff}$.*
When $\delta = 1, s \in S(1)$, we have

$$E_o(s)(E_{o\bullet s}(\lambda_s) - E_o(\lambda_s)) = \epsilon_{o\bullet s}(1, s)c_s E_o(\lambda_s).$$

Proof. We have $\beta \in s_\beta(\Sigma^-)$ because β is positive. The hypothesis $\beta \circ \nu(\lambda_s) = -1$ implies $\ell_\beta(s\lambda_s) = -\beta \circ \nu(\lambda_s) - 1 = 0$ by Prop. 5.9 3) and $E_o(s\lambda_s) = E_{o\bullet s}(s\lambda_s)$ by Lemma 5.34. Applying the product formula,

$$\begin{aligned} &E_o(s)(E_{o\bullet s}(\lambda_s) - E_o(\lambda_s)) = E_o(s\lambda_s) - E_o(s)E_o(\lambda_s) = E_{o\bullet s}(s\lambda_s) - E_o(s)E_o(\lambda_s) \\ &= E_{o\bullet s}(s)E_o(\lambda_s) - E_o(s)E_o(\lambda_s) = (E_{o\bullet s}(s) - E_o(s))E_o(\lambda_s) = \epsilon_{o\bullet s}(1,s)c_s E_o(\lambda_s). \end{aligned}$$

The last equality follows from $s \in S(1)$ and (81). ☐

Lemma 5.40. *We suppose that $q_s = 1$ for $s \in S^{aff}$.*
When $\delta = 2, s \in (S \cap S_o)(1)$, we have

$$E_o(s)(E_{o\bullet s}(\lambda_s) - E_o(\lambda_s)) = \epsilon_{o\bullet s}(1,s)\ [(w \bullet c_{\tilde{s}})s^2 + c_s E_o(\lambda_s)].$$

Proof. Applying the product formula

$$\begin{aligned} E_o(\lambda_s) &= E_o(sw\tilde{s}w^{-1}) = E_o(s)E_{o\bullet s}(w)E_{o\bullet sw}(\tilde{s})E_{o\bullet sw\tilde{s}}(w^{-1}), \\ E_{o\bullet s}(\lambda_s) &= E_{o\bullet s}(s)E_o(w)E_{o\bullet w}(\tilde{s})E_{o\bullet w\tilde{s}}(w^{-1}). \end{aligned}$$

We have $\ell_\beta(w) = 0$ because $w^{-1}(\beta) = \tilde{\beta} \in \Sigma^+$ (Lemma 5.15 and remark before Lemma 5.6). We note that $o \bullet sw\tilde{s} = o \bullet w$ because $sw\tilde{s} = \lambda_s w$ and the orientation o is spherical. We deduce (the first equality follows from Lemma 5.34):

$$\begin{aligned} &E_{o\bullet s}(w) = E_o(w),\ E_{o\bullet sw\tilde{s}}(w^{-1}) = E_{o\bullet w}(w^{-1}), \\ &E_o(\lambda_s) = E_o(s)E_o(w)E_{o\bullet sw}(\tilde{s})E_{o\bullet w}(w^{-1}), \end{aligned}$$

and $E_{o\bullet s}(\lambda_s) - E_o(\lambda_s)$ is the sum of $(E_{o\bullet s}(s) - E_o(s))E_o(w)E_{o\bullet sw}(\tilde{s})E_{o\bullet w}(w^{-1})$ and of $E_{o\bullet s}(s)E_o(w)(E_{o\bullet w}(\tilde{s}) - E_{o\bullet sw}(\tilde{s}))E_{o\bullet w}(w^{-1})$. By (81) and $s \in S(1)$,

$$\begin{aligned} &E_{o\bullet s}(\lambda_s) - E_o(\lambda_s) = \epsilon_{o\bullet s}(1,s)c_s E_{o\bullet s}(w\tilde{s}w^{-1}) + \epsilon_{o\bullet w}(1,\tilde{s})E_{o\bullet s}(sw)c_{\tilde{s}}E_{o\bullet w}(w^{-1}) \\ &= \epsilon_{o\bullet s}(1,s)(c_s E_{o\bullet s}(w\tilde{s}w^{-1}) + (sw \bullet c_{\tilde{s}})E_{o\bullet s}(sw)E_{o\bullet w}(w^{-1})) \\ &= \epsilon_{o\bullet s}(1,s)(c_s E_{o\bullet s}(w\tilde{s}w^{-1}) + (sw \bullet c_{\tilde{s}})E_{o\bullet s}(s)). \end{aligned}$$

For the second line, we note that $s_\beta \in S_o$ permutes the o-positive roots of Σ different from $\pm\beta$. The o-positive side is equal to the $o \bullet s$-positive side for the affine hyperplanes of the form $\mathrm{Ker}(\gamma + k), k \in \mathbb{Z}, \gamma \in \Sigma - \{\pm\beta\}$, between $\mathfrak{C}, \lambda_s(\mathfrak{C})$. The affine hyperplanes of the form $\mathrm{Ker}(\beta + k), k \in \mathbb{Z}$, separating $\mathfrak{C}, \lambda_s(\mathfrak{C})$ are $\mathrm{Ker}\,\beta, \mathrm{Ker}(\beta + 1)$ (Lemma 5.6). We cross the hyperplanes $\mathrm{Ker}\,\beta$ and $\mathrm{Ker}(\beta + 1)$ in the same sense when we go from $\mathfrak{C}$ to $\lambda_s(\mathfrak{C}) = \mathfrak{C} + \nu(\lambda_s)$, hence $\epsilon_{o\bullet s}(1,s) = \epsilon_{o\bullet w}(1,\tilde{s})$.

Multiplying on the left side by $E_o(s)$ we obtain

$$E_o(s)(E_{o\bullet s}(\lambda_s) - E_o(\lambda_s)) = \epsilon_{o\bullet s}(1,s)(c_s E_o(\lambda_s) + (w \bullet c_{\tilde{s}})s^2),$$

using that the product formula, $E_o(s)c = (s \bullet c)E_o(s)$ for $c \in R[Z_k]$, $s \bullet c_s = c_s$ and (81). ☐

We summarize Lemma 5.34, 5.39, 5.40: for $y \in \Lambda^s(1), x = \lambda_s$,

$$\begin{aligned} &E_{o\bullet s}(y) - E_o(y) = 0, \\ &E_o(s)(E_{o\bullet s}(x) - E_o(x)) = \epsilon_{o\bullet s}(1,s)((\delta - 1)(w \bullet c_{\tilde{s}})s^2 + c_s E_o(x)) = \epsilon_{o\bullet s}(1,s)B_{o,1}E_o(x), \end{aligned}$$

with $B_{o,1}$ defined by (85). By a formal computation we will deduce the expansion of the Bernstein element at any $\lambda \in \Lambda(1)$. First, we take $\lambda = x^{-1}$.

Lemma 5.41. *We suppose that $q_s = 1$ for $s \in S^{aff}$. When $s \in (S \cap S_o)(1)$,*

$$E_o(s)(E_{o\bullet s}(x^{-1}) - E_o(x^{-1})) = \epsilon_o(1,s)E_o(s(x^{-1}))B_{o,1}.$$

Proof. We recall that $E_o(w)$ is invertible for $w \in W(1)$ because $q_s = 1$ for $s \in S^{aff}/\sim$ and that the inverse of $E_o(\lambda)$ is $E_o(\lambda^{-1})$ because the orientation o is spherical. We multiply the equality $E_o(s)(E_{o\bullet s}(x) - E_o(x)) = \epsilon_{o\bullet s}(1,s)B_{o,1}E_o(x)$ on the left by

$$E_o(s)E_{o\bullet s}(x)^{-1}E_o(s)^{-1} = E_o(s)E_{o\bullet s}(x^{-1})E_{o\bullet s}(s^{-1}) = E_o(s(x^{-1})),$$

and on the right by $E_o(x)^{-1} = E_o(x^{-1})$ to obtain

$$E_o(s)(E_{o\bullet s}(x^{-1}) - E_o(x^{-1})) = -\epsilon_{o\bullet s}(1,s)E_o(s(x^{-1}))B_{o,1},$$

and we use $\epsilon_o(1,s) = -\epsilon_{o\bullet s}(1,s)$. □

Now we relate the Bernstein element at $z \in \Lambda(1)$ to the Bernstein element at z^n for $n \in \mathbb{N}_{>0}$.

Lemma 5.42. *We suppose that $q_s = 1$ for $s \in S^{aff}$. When $s \in (S \cap S_o)(1)$,*

$$E_o(s)(E_{o\bullet s}(z^n) - E_o(z^n)) = \sum_{k=0}^{n-1} E_o(sz^ks^{-1})E_o(s)(E_{o\bullet s}(z) - E_o(z))E_o(z^{n-1-k}).$$

Proof. Using that the orientations o and $o \bullet s$ are fixed by $z \in \Lambda(1)$, we have $E_{o\bullet s}(z^n) - E_o(z^n) = E_{o\bullet s}(z^{n-1})(E_{o\bullet s}(z) - E_o(z)) + (E_{o\bullet s}(z^{n-1}) - E_o(z^{n-1}))E_o(z)$. By induction on n,

$$E_{o\bullet s}(z^n) - E_o(z^n) = \sum_{k=0}^{n-1} E_{o\bullet s}(z^k)(E_{o\bullet s}(z) - E_o(z))E_o(z^{n-1-k}).$$

We multiply this equality on the left by $E_o(s)$, and we observe that $E_o(s)E_{o\bullet s}(z^k) = E_o(sz^k) = E_o(sz^ks^{-1}s) = E_o(sz^ks^{-1})E_o(s)$. □

We end now the proof of Thm. 5.38. For $n \in \mathbb{N}_{>0}$, by the lemma 5.42 applied to $z = x$ and to $z = x^{-1}$, the Bernstein element (84) at x^n is equal to

$$\sum_{k=0}^{n-1} \epsilon_{o\bullet s}(1,s)E_o(s(x^k))B_{o,1}E_o(x)E_o(x^{n-1-k}) = \epsilon_{o\bullet s}(1,s)B_{o,n}E_o(x^n),$$

with $B_{o,n}$ defined in (86), and the Bernstein element (84) at $x^{-n}, n \in \mathbb{N}_{>0}$, is equal to

$$\epsilon_o(1,s)\sum_{k=0}^{n-1} E_o(s(x^{-n+k+1}))E_o(s(x^{-1}))B_{o,1}E_o(x^{-k}) = \epsilon_o(1,s)E_o(s(x^{-n}))B_{o,n}.$$

As $x = \lambda_s, (\beta \circ \nu)(\lambda_s) = -\delta$, an arbitrary $\lambda \in \Lambda(1)$ is equal to

$$\lambda = \begin{cases} x^n y & \text{if } \epsilon_\beta(\lambda) = -1, \\ yx^{-n} & \text{if } \epsilon_\beta(\lambda) = 1 \end{cases}$$

where $y \in \Lambda^s(1)$ and $n \in \mathbb{N}$. We multiply on the right by $E_o(y)$ the Bernstein element at x^n to obtain the Bernstein element at $\lambda = x^n y$,

$$E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda)) = \epsilon_{o\bullet s}(1,s)B_{o,n}E_o(\lambda).$$

We have $E_o(sys^{-1})E_o(s) = E_o(sy) = E_o(s)E_{o\bullet s}(y) = E_o(s)E_o(y)$. We multiply on the left by $E_o(sys^{-1})$ the Bernstein element at x^{-n}, obtain the Bernstein element at $\lambda = yx^{-n}$,

$$E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda)) = E_o(s(y))E_o(s)(E_{o\bullet s}(x^{-n}) - E_o(x^{-n})) = \epsilon_o(1,s)E_o(s(\lambda))B_{o,n}.$$

This ends the proof of Thm. 5.38.

Corollary 5.43. *When $q_s = 1$ for $s \in S^{aff}$, with the notations of* (5.37) *and the hypothesis of Thm. 5.38, the Bernstein element* (84) *is equal to*

$$\epsilon_o(1,s)\epsilon_\beta(\lambda)\sum_{k=0}^{|\beta\circ\nu(\lambda)|-1} c(k,\lambda)E_o(\mu(k,\lambda)) = \epsilon_o(1,s)\epsilon_\beta(\lambda)\sum_{k=0}^{|\beta\circ\nu(\lambda)|-1} E_o(\mu(k,\lambda))c'(k,\lambda)$$

where $c(k,\lambda), c'(k,\lambda), \mu(k,\lambda)$ depend on $c(k), c'(k), \mu(k)$ (Notation 5.37) hence on λ_s, are defined by:
$c(k,\lambda) = c(k),\ c'(k,\lambda) = \lambda^{-1}\bullet c'(k),\ \mu(k,\lambda) = \mu(k)\lambda$ if $\epsilon_\beta(\lambda) = -1$,
$c(k,\lambda) = s(\lambda)\bullet c(k),\ c'(k,\lambda) = c'(k),\ \mu(k,\lambda) = s(\lambda)\mu(k)$ if $\epsilon_\beta(\lambda) = 1$.

Proof. $B_{o,n}E(\lambda) = \sum_{k=0}^{n-1} c(k)E_o(\mu(k)\lambda) = \sum_{k=0}^{n-1} E_o(\mu(k)\lambda)(\lambda^{-1}\bullet c'(k))$,
$E_o(s(\lambda))B_{o,n} = \sum_{k=0}^{n-1} s(\lambda)\bullet c(k)E_o(s(\lambda)\mu(k)) = \sum_{k=0}^{n-1} E_o(s(\lambda)\mu(k))c'(k)$. □

Recalling Notations 5.37, we have

$$\nu(\mu(k)) = k\beta^\vee = \nu(\mu_\beta^{-k}),\quad \beta\circ\nu(\mu(k,\lambda)) = 2k - |\beta\circ\nu(\lambda)|. \tag{87}$$

because
when $\delta = 1$, $\nu(s\lambda_s s^{-1}) = s(\nu(\lambda_s)) = \nu(\lambda_s) + \beta^\vee$ because $\beta\circ\nu(\lambda_s) = -1$,
when $\delta = 2$, $\nu(\mu_\beta) = -\beta^\vee$, $\nu(s\mu_\beta s^{-1}) = \beta^\vee$.
For $k \neq k'$ we have $Z_k\mu(k,\lambda) \neq Z_k\mu(k',\lambda)$. The corollary 5.43 gives the coefficicients of the expansion of (84) in the basis $(E_o(w))_{w\in W(1)}$.

We pass from the case $q_s = 1$ to the general case using the method explained in subsection 4.5. The Bernstein relations in the $R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]$-algebra $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(1,\mathfrak{q}_s^{-1}c_s)$, are given by Cor. 5.43 where c_s and $c_{\tilde{s}}$ are replaced by $\mathfrak{q}_s^{-1}c_s$ and $\mathfrak{q}_{\tilde{s}}^{-1}c_{\tilde{s}}$ in the formula for $c(k)$. A quick inspection shows that this means replacing $c(\delta k)$ by $\mathfrak{q}_s^{-1}c(\delta k)$ and when $\delta = 2$ replacing $c(2k+1)$ by $\mathfrak{q}_{\tilde{s}}^{-1}c(2k+1)$.

Lemma 5.44. *The isomorphism $h \mapsto \tilde{h} : \mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(1,\mathfrak{q}_s^{-1}c_s) \to \mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s,c_s)$* (70) (72) *sends $E_o(w), w\in W(1)$, to*

$$\tilde{E}_o(w) = \mathfrak{q}_w^{-1}E_o(w) \in \mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s,c_s).$$

Proof. Let $s \in S^{aff}(1)$. The image of $T_s^* = T_s - \mathfrak{q}_s^{-1}c_s \in \mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(1,\mathfrak{q}_s^{-1}c_s)$ is $\tilde{T}_s - \mathfrak{q}_s^{-1}c_s = \mathfrak{q}_s^{-1}(T_s - c_s) = \mathfrak{q}_s^{-1}T_s^*$ in $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s,c_s)$. □

We multiply $\tilde{E}_o(s)(\tilde{E}_{o\bullet s}(\lambda) - \tilde{E}_o(\lambda))$ by $\mathfrak{q}_s\mathfrak{q}_\lambda$ and we note that $\mathfrak{q}_{s(\lambda)} = \mathfrak{q}_\lambda$ (Prop. 5.13). Cor. 5.43 implies:

Proposition 5.45. Bernstein relations in $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s,c_s)$
Let $s \in (S\cap S_o)(1),\ \lambda\in\Lambda(1)$. With the notations of (5.37)*, the Bernstein element* (84) *is* 0 *if $\lambda\in\Lambda^s(1)$. Otherwise, it is equal to*

$$\epsilon_\beta(\lambda)\epsilon_o(1,s)\sum_{k=0}^{|\beta\circ\nu(\lambda)|-1}\mathbf{q}(k,\lambda)c(k,\lambda)E_o(\mu(k,\lambda)),$$

where $\mathbf{q}(\delta k,\lambda) = \mathfrak{q}_\lambda\mathfrak{q}_{\mu(\delta k,\lambda)}^{-1}$ and when $\delta = 2$, $\mathbf{q}(2k+1,\lambda) = \mathfrak{q}_\lambda\mathfrak{q}_{\mu(2k+1,\lambda)}^{-1}\mathfrak{q}_s\mathfrak{q}_{\tilde{s}}^{-1}$ depends on s.

They are also the Bernstein relations in the subalgebra $\mathcal{H}_{R[(\mathbf{q_s})]}(\mathbf{q}_s, c_s)$ because this is the expansion of $E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda))$ in the basis $(E_o(w))_{w \in W(1)}$. We deduce:

$$\mathbf{q}(k,\lambda)c(k,\lambda) \in R[(\mathbf{q_s})][Z_k] \quad \text{for } 0 \leq k < |\beta \circ \nu(\lambda)|.$$

This is true for any choice of R and $(c_s)_{s \in S^{aff}(1)}$ satisfying the condition a6 of subsection 4.3. We may choose $R = \mathbb{Z}$ and $c_s \neq 0$ for all s. Then $c(k,\lambda) \in \mathbb{Z}$ is not 0, therefore

$$\mathbf{q}(k,\lambda) = \prod_{s \in S^{aff}/\sim} \mathbf{q}_s^{m_{k,\lambda}(s)} \quad (m_{k,\lambda}(s) \in \mathbb{N}). \tag{88}$$

We will give later (Proposition 5.49) more properties of $\mathbf{q}(k,\lambda)$. The Bernstein relations in the R-algebra $\mathcal{H}_R(q_s, c_s)$ are obtained by specialisation of the Bernstein relations in $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$ by the map $\mathbf{q}_s \to q_s$ for $s \in S^{aff}/\sim$. We denote by $q(k,\lambda)$ the specialisation of $\mathbf{q}(k,\lambda)$.

Theorem 5.46. (Bernstein relations in $\mathcal{H}_R(q_s, c_s)$) *For $s \in (S \cap S_o)(1), \lambda \in \Lambda(1)$, the Bernstein element* (84) *belongs to the subalgebra $\mathcal{A}_o$ of $\mathcal{H}_R(q_s, c_s)$ of basis $(E_o(\lambda))_{\lambda \in \Lambda(1)}$. It vanishes when $\lambda \in \Lambda^s(1)$, otherwise it is equal to*

$$\epsilon_o(1,s)\epsilon_\beta(\lambda) \sum_{k=0}^{|\beta\circ\nu(\lambda)|-1} q(k,\lambda)c(k,\lambda)E_o(\mu(k,\lambda)),$$

with the notations of (5.37) *and of Prop. 5.45.*

For the dominant and antidominant spherical orientation o we have $S = S_o$; for the antidominant orientation o we have also (Example 5.32):

$$E_o(w) = T_w \quad \text{for all } w \in W_0(1).$$

Writing $E(w) = E_o(w)$ where o is the antidominant orientation, we obtain a presentation of the generic algebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$.

Corollary 5.47. (Bernstein presentation of the generic algebra)

The $R[(\mathbf{q}_s)]$-algebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$ is the free $R[(\mathbf{q}_s)]$-module of basis $(E_o(w))_{w \in W(1)}$ endowed with the unique $R[(\mathbf{q}_s)]$-algebra structure satisfying:

- *Braid relations for $w, w' \in W_0(1)$, $E(w)E(w') = E(ww')$ if $\ell(w) + \ell(w') = \ell(ww')$.*
- *Quadratic relations for $s \in S(1)$, $E(s)^2 = \mathbf{q}_s s^2 + c_s E(s)$.*
- *Product formula for $\lambda \in \Lambda(1), w \in W(1)$, $E(\lambda)E(w) = \mathbf{q}_{\lambda,w}E(\lambda w)$.*
- *Bernstein relations for $s \in S(1), \lambda \in \Lambda(1)$,*

$E(s(\lambda))E(s) = E(s)E(\lambda)$ when $\nu(\lambda)$ is fixed by s,

$E(s(\lambda))E(s) - E(s)E(\lambda) = \epsilon_\beta(\lambda)\sum_{k=0}^{|\beta\circ\nu(\lambda)|-1} \mathbf{q}(k,\lambda)c(k,\lambda)E(\mu(k,\lambda))$, when $\nu(\lambda)$ is not fixed by s.

5.5 Variants of the Bernstein relations

This section is motivated by applications to the theory of smooth representations of G over a field C of characteristic p.

In the Bernstein relations in $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$ (Prop. 5.45), we move the term with $k = 0$ from the right hand side to the left hand side which becomes (using (84))

$$E_o(s(\lambda))E_o(s) - E_o(s)E_o(\lambda) - \epsilon_\beta(\lambda)\epsilon_o(1,s)\mathbf{q}(0,\lambda)c(0,\lambda)E_o(\mu(0,\lambda)). \tag{89}$$

Proposition 5.48. (Variant of the Bernstein relations in $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$) *With the hypothesis of Prop. 5.45, (89) is equal to*

$$\begin{aligned}
\text{if } \epsilon_\beta(\lambda) = -1, \quad & E_o(s(\lambda))E_o(s) - (E_o(s) + \epsilon_{o\bullet s}(1,s)c_s)E_o(\lambda) = \mathbf{q}_s(E_o(s\lambda) - E_{o\bullet s}(s\lambda)) \\
& = -\epsilon_o(1,s) \sum_{k=1}^{|\beta\circ\nu(\lambda)|-1} \mathbf{q}(k,\lambda)c(k,\lambda)E_o(\mu(k,\lambda)), \\
\text{if } \epsilon_\beta(\lambda) = 1, \quad & E_o(s(\lambda))(E_o(s) + \epsilon_{o\bullet s}(1,s)c_s) - E_o(s)E_o(\lambda) \\
& = \epsilon_o(1,s) \sum_{k=1}^{|\beta\circ\nu(\lambda)|-1} \mathbf{q}(k,\lambda)c(k,\lambda)E_o(\mu(k,\lambda)).
\end{aligned}$$

The term $\sum_{k=1}^{|\beta\circ\nu(\lambda)|-1}$ *appears only if* $|\beta\circ\nu(\lambda)| > 1$ *and* $\mathbf{q}(k,\lambda)\mathbf{q}_s^{-1} \in \mathbb{Z}[(\mathbf{q}_s)]$ *if* $\epsilon_\beta(\lambda) = -1$.

Proof. The term with $k = 0$ in the Bernstein relation (Thm. 5.46) is

$$\begin{aligned}
\epsilon_\beta(\lambda)\epsilon_o(1,s)q(0,\lambda)c(0,\lambda)E_o(\mu(0,\lambda)) &= -\epsilon_o(1,s)c_sE_o(\lambda) = \epsilon_{o\bullet s}(1,s)c_sE_o(\lambda) \quad \text{if} \quad \epsilon_\beta(\lambda) = -1, \\
&= \epsilon_o(1,s)(s(\lambda) \bullet c_s)E_o(s(\lambda)) = -\epsilon_{o\bullet s}(1,s)E_o(s(\lambda))c_s \quad \text{if} \quad \epsilon_\beta(\lambda) = 1,
\end{aligned}$$

using $\epsilon_{o\bullet s}(1,s) + \epsilon_o(1,s) = 0$, Cor. 5.43, Prop. 5.45, Property a5), Lemma 5.21.

Case $\epsilon_\beta(\lambda) = -1$. Then (89) is equal to

$$\begin{aligned}
& E_o(s(\lambda))E_o(s) - E_o(s)E_o(\lambda) - \epsilon_{o\bullet s}(1,s)c_s)E_o(\lambda) = \\
& E_o(s)E_{o\bullet s}(\lambda) - (E_o(s) + \epsilon_{o\bullet s}(1,s)c_s)E_o(\lambda) = E_o(s)E_{o\bullet s}(\lambda) - E_{o\bullet s}(s)E_o(\lambda) \\
& = \mathbf{q}_s(E_o(s\lambda) - E_{o\bullet s}(s\lambda))
\end{aligned}$$

using $\ell(s\lambda) = \ell(\lambda) - 1, \mathbf{q}_{s,\lambda} = \mathbf{q}_s$ (proof of Prop. 5.35, $E_o(s) + \epsilon_{o\bullet s}(1,s)c_s = E_{o\bullet s}(s)$ (81)). When $R = \mathbb{Z}$, $c(1,\lambda) \in \mathbb{Z}$ is not 0, hence the Bernstein relations imply $\mathbf{q}(k,\lambda)\mathbf{q}_s^{-1} \in \mathbb{Z}[(\mathbf{q}_s)]$.

Case $\epsilon_\beta(\lambda) = 1$. Then (89) is equal to

$$\begin{aligned}
& E_o(s(\lambda))E_o(s) - E_o(s)E_o(\lambda) + \epsilon_{o\bullet s}(1,s)E_o(s(\lambda))c_s = \\
& E_o(s(\lambda))(E_o(s) + \epsilon_{o\bullet s}(1,s)c_s) - E_o(s)E_o(\lambda) = E_o(s(\lambda))E_{o\bullet s}(s) - E_o(s)E_o(\lambda) \\
& = E_o(s\lambda) - E_o(s)E_o(\lambda)
\end{aligned}$$

using $\ell(s\lambda) = 1 + \ell(\lambda)$, $\mathbf{q}_{s,\lambda} = \mathbf{q}_{s(\lambda),s} = 1$ (proof of Prop. 5.35). ☐

By specialisation, the proposition is valid in the algebra $\mathcal{H}_R(q_s, c_s)$. We study now the elements

$$(90) \qquad \mathbf{q}(k,\lambda) = \prod_{s\in S^{aff}/\sim} \mathfrak{q}_s^{2m_{k,\lambda}(s)} = \mathfrak{q}_\lambda \mathfrak{q}_{\mu(k,\lambda)}^{-1} x, \quad m_{k,\lambda}(s) \in \mathbb{N},\ x \in \{1, \mathfrak{q}_s\mathfrak{q}_{\tilde{s}}^{-1}\},$$

by (88) and Proposition 5.45, for $\mathfrak{q}_s^2 = \mathbf{q}_s$ for $s \in S^{aff}(1)$ and $x = 1$ if and only if $\delta = 1$ or k is even.

Proposition 5.49. *Let* $\beta \in \Delta, \lambda \in \Lambda, k \in \mathbb{N}_{>0}$ *with* $k < |\beta \circ \nu(\lambda)|$. *Then,* $\mathbf{q}(k,\lambda) \neq 1$. *If* $\epsilon_\beta\ (\lambda) = -1$, *we have*

a) $\mathbf{q}(k,\lambda)\mathbf{q}_{s_\beta}^{-1} \neq 1$ *for* $1 < k < |\beta \circ \nu(\lambda)| - 1$.

b) $\mathbf{q}(1,\lambda)\mathbf{q}_{s_\beta}^{-1} = 1 \Leftrightarrow \mathbf{q}(|\beta \circ \nu(\lambda)| - 1, \lambda)\mathbf{q}_{s_\beta}^{-1} = 1 \Leftrightarrow \ell(\lambda) - \ell(\mu_\beta^{-1}\lambda) = 2$.

c) If $\lambda \in \Lambda^-$ *then* $\ell(\lambda) - \ell(\mu_\beta^{-1}\lambda) = 2$.

Proof. The proof relies on four claims :

1. $\sum_{s\in S^{aff}/\sim} 2m_{k,\lambda}(s) = \ell(\lambda) - \ell(\mu(k,\lambda))$.

2. $\ell(\mu(k,\lambda)) = \ell(\mu_\beta^k \lambda^{\epsilon_\beta(\lambda)})$ and $\ell(\mu_\beta^k\lambda) = \ell(\mu_\beta^{\beta\circ\nu(\lambda)-k}\lambda)$.

3. If $\epsilon_\beta(\lambda)=1$ then $\ell(\lambda)-\ell(\mu_\beta^k\lambda) \geq 2\,\mathrm{Min}(k, \beta\circ\nu(\lambda)-k) \geq 2$.

4. If $\lambda \in \Lambda^+$ then $\mu_\beta\lambda \in \Lambda^+$.

From (90) and claim 1, $\mathbf{q}(k,\lambda) \neq 1 \Leftrightarrow \ell(\lambda)\neq \ell(\mu(k,\lambda))$. This is always true by claims 2,3.

Supppose now $\epsilon_\beta(\lambda)=-1$. By Prop. 5.48 and claims 1, 2,
$\mathbf{q}(k,\lambda)\mathbf{q}_{s_\beta}^{-1} = 1 \Leftrightarrow \ell(\lambda)-\ell(\mu_\beta^k\lambda^{-1}) = 2$,
$\mathbf{q}(k,\lambda)\mathbf{q}_{s_\beta} \neq 1 \Leftrightarrow \ell(\lambda)-\ell(\mu_\beta^k\lambda^{-1}) > 2$.
We have $\ell(\lambda)=\ell(\lambda^{-1})$ and $\epsilon_\beta(\lambda^{-1})=1$. We deduce from Claim 2 that
$\ell(\lambda)-\ell(\mu_\beta\lambda^{-1}) = \ell(\lambda)-\ell(\mu_\beta^{|\beta\circ\nu(\lambda)|-1}\lambda^{-1})$.
We deduce from claim 3 that $\ell(\lambda)-\ell(\mu_\beta^k\lambda^{-1}) > 2$ if $1<k<|\beta\circ\nu(\lambda)|-1$.

Let ρ be the half-sum of the positive roots of Σ. If $\gamma\in\Sigma$ is positive, $\rho(\gamma^\vee)\geq 1$ with equality if and only if γ is a simple root [Bki, Prop. 29 (iii)]. If $\lambda\in\Lambda^-$, we have $\ell(\lambda) = -2\rho(\nu(\lambda))$ (Cor. 5.10), $\mu_\beta^{-1}\lambda \in \Lambda^-$ by Claim 4, and $-\nu(\mu_\beta)=\beta^\vee$ (87). We obtain $\ell(\lambda)-\ell(\mu_\beta^{-1}\lambda) = -2\rho\circ\nu(\lambda)+2\rho\circ\nu(\mu_\beta^{-1}\lambda) = -2\rho\circ\nu(\mu_\beta) = 2\rho(\beta^\vee)=2$.

It remains to prove the claims. Claim 4 follows from $\nu(\mu_\beta\lambda) = -\beta^\vee+\nu(\lambda)$ and
$\beta(-\beta^\vee+\nu(\lambda)) = -2+\beta\circ\nu(\lambda)\geq 0$ because $0<k<\beta\circ\nu(\lambda)$,
$-\alpha(\beta^\vee)$ and $\alpha\circ\nu(\lambda)$ are ≥ 0 for all simple roots $\alpha\in\Sigma^+ - \{\beta\}$ because $\lambda\in\Lambda^+$.

□

We prove the remaining three claims. Claim 1 is easy and is valid without restriction on $k\in\mathbb{Z}$. Note that $\mu(k,\lambda)$ and $m_{k,\lambda}(s)$ (90) are well defined for $k\in\mathbb{Z}$.

Lemma 5.50. $2\sum_{s\in S^{aff}/\sim} m_{k,\lambda}(s) = \ell(\lambda)-\ell(\mu(k,\lambda))$ *for* $\lambda\in\Lambda, k\in\mathbb{Z}$.

Proof. Choosing reduced decompositions of λ and $\mu(k,\lambda)$, we have

$$\mathfrak{q}_\lambda\mathfrak{q}_{\mu(k,\lambda)}^{-1} = \prod_{s\in S^{aff}/\sim}\mathfrak{q}_s^{n_s(k)}, \quad n_s(k)\in\mathbb{Z}, \qquad \sum_{s\in S^{aff}/\sim} n_s(k) = \ell(\lambda)-\ell(\mu(k,\lambda)).$$

We compare with (90). □

We prove now Claim 2. It is valid without restriction on $k\in\mathbb{Z}$. The second formula is valid for any root $\alpha\in\Sigma, \mu_\alpha = s_{\alpha+1}s_\alpha\in\Sigma^{aff}$ with $\nu(\mu_\alpha) = -\alpha^\vee$.

Lemma 5.51. $\ell(\mu(k,\lambda)) = \ell(\mu_\beta^k\lambda^{\epsilon_\beta(\lambda)})$ *and* $\ell(\mu_\alpha^k\lambda) = \ell(\mu_\alpha^{\alpha\circ(\lambda)-k}\lambda)$ *for* $\lambda\in\Lambda, k\in\mathbb{Z}, \alpha\in\Sigma$.

Proof. Recalling (87) and Cor. 5.43, the value of $\nu(\mu(k,\lambda))$ is
$\nu(\mu(k))+\nu(\lambda) = \nu(\mu_\beta^{-k}\lambda)$ if $\epsilon_\beta(\lambda)=-1$,
$\nu(\mu(k))+\nu(s(\lambda)) = \nu(\mu_\beta^{-k}s(\lambda)) = s(\nu(\mu_\beta^k\lambda))$ if $\epsilon_\beta(\lambda)=1$.
We have

$$s_\alpha(\nu(\mu_\alpha^k\lambda)) = s_\alpha(-k\alpha^\vee+\nu(\lambda)) = (k-\alpha\circ\nu(\lambda))\alpha^\vee+\nu(\lambda) = \nu(\mu_\alpha^{\alpha\circ\nu(\lambda)-k}\lambda).$$

The length of $x\in\Lambda$ depends only on $\nu(x)\in V$ (Cor. 5.10), is constant on the W_0-orbit of x, is stable by taking inverse, and the homomorphism $\nu:\Lambda\to V$ is W_0-equivariant. Hence $\ell(\mu(k,\lambda)) = \ell(\mu_\beta^{-k}\lambda^{-\epsilon_\beta(\lambda)}) = \ell(\mu_\beta^k\lambda^{\epsilon_\beta(\lambda)})$ and $\ell(\mu_\alpha^k\lambda) = \ell(\mu_\alpha^{\alpha\circ\nu(\lambda)-k}\lambda)$. □

We prove Claim 3. It is valid for any root $\alpha\in\Sigma$.

Lemma 5.52. *For* $\alpha\in\Sigma, \lambda\in\Lambda, k\in\mathbb{N}_{>0}$ *such that* $k<\alpha\circ\nu(\lambda)$, *we have*

$$\ell(\lambda)-\ell(\mu_\alpha^k\lambda)\geq 2\min(k,\alpha\circ\nu(\lambda)-k).$$

Proof. a) Reduction to $\nu(\lambda)$ dominant.

There exists $w \in W_0$ such that $w(\lambda) \in \Lambda^+$. By W_0-invariance of ν, we have $\alpha \circ \nu(\lambda) = w(\alpha) \circ \nu(w(\lambda))$. We have $w(\mu_\alpha) = ws_{\alpha+1}s_\alpha w^{-1} = \mu_{w(\alpha)}$. By W_0-invariance of ℓ, $\ell(\mu_\alpha^k \lambda) = \ell(\mu_{w(\alpha)}^k w(\lambda))$. The lemma is true for $(w(\alpha), w(\lambda))$ if and only if it is true for the pair (α, λ). From now on, we suppose that $\lambda \in \Lambda^+$.

b) By [Kumar, 1.3.22 Cor. and 1.4.2 Prop.], $\nu(\lambda) - w(\nu(\lambda)) \in \sum_{\gamma \in \Delta_\Sigma} \mathbb{N}\gamma^\vee$. Hence

$$2\rho(\nu(\lambda) - w(\nu(\lambda)) \geq 0 \quad \text{with equality if and only if } \nu(\lambda) = w(\nu(\lambda)).$$

c) For $w \in W_0$ such that $w(\nu(\mu_\alpha^k \lambda)) = w(-k\alpha^\vee + \nu(\lambda))$ is dominant, we have $\ell(\mu_\alpha^k \lambda) = 2\rho(w(-k\alpha^\vee + \nu(\lambda)))$ as the length is W_0-invariant. Hence

$$\ell(\lambda) - \ell(\mu_\alpha^k \lambda) = 2\rho(\nu(\lambda) - w(\nu(\lambda)) + 2k\rho(w(\alpha^\vee)).$$

We deduce:

1) $\ell(\lambda) - \ell(\mu_\alpha^k \lambda) \geq 2k$ if there exists $w \in W_0$ such that $w(\alpha)$ is positive and $w(-k\alpha^\vee + \nu(\lambda))$ is dominant.

2) $\ell(\lambda) - \ell(\mu_\alpha^k \lambda) = 2k$ if there exists $w \in W_0$ fixing $\nu(\lambda)$ such that $w(\alpha)$ is a simple root and $w(-k\alpha^\vee + \nu(\lambda))$ is dominant.

e) Suppose $2k = \alpha \circ \nu(\lambda)$, or equivalently $-k\alpha^\vee + \nu(\lambda)$ fixed by s_α. For $w \in W_0$, $w(-k\alpha^\vee + \nu(\lambda)) = ws_\alpha(-k\alpha^\vee + \nu(\lambda))$ and either $w(\alpha)$ or $ws_\alpha(\alpha)$ is positive. We deduce from 1) that $\ell(\lambda) - \ell(\mu_\gamma^k \lambda) \geq 2k$. The lemma is proved when $2k = \alpha \circ \nu(\lambda)$.

f) Suppose $2k < \alpha \circ \nu(\lambda)$. For $w \in W_0$ such that $w(-k\alpha^\vee + \nu(\lambda))$ is dominant, $w(\alpha)$ is positive because $w(\alpha)(w(-k\alpha^\vee + \nu(\lambda)) = \alpha(-k\alpha^\vee + \nu(\lambda)) = \alpha \circ \nu(\lambda) - 2k$ is positive by hypothesis. A root which takes a positive value on a point in the dominant Weyl chamber of V is positive. We deduce from 1) that $\ell(\lambda) - \ell(\mu_\alpha^k \lambda) \geq 2k$. The lemma is proved in this case.

g) Suppose $\alpha \circ \nu(\lambda) < 2k < 2\alpha \circ \nu(\lambda)$. We have $\ell(\mu_\alpha^k \lambda) = \ell(\mu_\alpha^{\alpha \circ \nu(\lambda) - k} \lambda)$ by Lemma 5.51. We deduce from f) applied to $k' = \alpha \circ \nu(\lambda) - k$ that $\ell(\lambda) - \ell(\mu_\alpha^k \lambda) \geq 2(\alpha \circ \nu(\lambda) - k)$. This ends the proof of the lemma. $\square$

Corollary 5.53. *We suppose that $q_s = 0$ for all $s \in S^{aff}(1)$. If $\lambda \in \Lambda(1)$, o is a spherical orientation and $\beta \in \Delta$ are such that $s_\beta \in S_o$ and $s \in S^{aff}(1)$ lifts s_β, we have*

$$\text{if} \quad \beta \circ \nu(\lambda) = 0, \quad E_o(s(\lambda))E_o(s) = E_o(s)E_o(\lambda), \tag{91}$$

$$\text{if} \quad \beta \circ \nu(\lambda) < 0, \quad E_o(s(\lambda))E_o(s) = E_{o\bullet s}(s)E_o(\lambda), \tag{92}$$

$$\text{if} \quad \beta \circ \nu(\lambda) > 0, \quad E_o(s(\lambda))E_{o\bullet s}(s) = E_o(s)E_o(\lambda), \tag{93}$$

where the sets $\{E_o(s), E_{o\bullet s}(s)\} = \{T_s, T_s^*\}$ are equal.

Proof. Prop. 5.48, 5.49, (81), and the remark following (84). $\square$

Corollary 5.54. *If o is the dominant or anti-dominant spherical orientation and $w \in W_0(1)$, we have*

$$E_o(w\lambda w^{-1})E_o(\tilde{w}) - E_o(\tilde{w})E_o(\tilde{\lambda}) \in \sum_{v<w} T_{\tilde{v}}\mathcal{A}_o. \tag{94}$$

In particular, $\mathcal{A}_o T_{\tilde{w}} \subset \sum_{v \leq w} T_{\tilde{v}}\mathcal{A}_o$.

Proof. As $S = S_o$, the Bernstein relations (Thm. 5.46) imply $E_o(\tilde{s}(\tilde{\lambda}))E_o(\tilde{s}) - E_o(\tilde{s})E_o(\tilde{\lambda}) \in \mathcal{A}_o$ for any $s \in S$. We prove the lemma by induction on $\ell(w)$. Let $w \in W_0, w \neq 1$, and $s \in S$ such that $\ell(sw) = 1 + \ell(w)$ and (94) is true for s and $v \leq w$. Hence the element

$$E_o(\tilde{s}\tilde{w}(\tilde{\lambda}))E_o(\tilde{s}\tilde{w}) - E_o(\tilde{s}\tilde{w})E_o(\tilde{\lambda}) = E_o(\tilde{s}\tilde{w}(\tilde{\lambda}))E_o(\tilde{s})E_o(\tilde{w}) - E_o(\tilde{s})E_o(\tilde{w})E_o(\tilde{\lambda})$$

lies in

$$(E_o(\tilde{s})E_o(\tilde{w}(\tilde{\lambda})) + \mathcal{A}_o)E_o(\tilde{w}) - E_o(\tilde{s})(E_o(\tilde{w}(\tilde{\lambda}))E_o(\tilde{w}) + (\sum_{v<w} T_{\tilde{v}}\mathcal{A}_o))$$
$$= \mathcal{A}_o E_o(\tilde{w}) + \sum_{v<w} E_o(\tilde{s})T_{\tilde{v}}\mathcal{A}_o.$$

The triangular expression of $E_o(\tilde{w})$ in the Iwahori-Matsumoto basis (Cor. 5.26) and (94) for $v \leq w$ imply $\mathcal{A}_o E_o(\tilde{w}) \subset \sum_{v\leq w} T_{\tilde{v}}\mathcal{A}_o$. The product formula implies that $E_o(\tilde{s})T_{\tilde{v}} \in \sum_{\tilde{x}\leq\tilde{w}} RT_{\tilde{x}}$ if $v < w$. Therefore we have

$$E_o(\tilde{s}\tilde{w}(\tilde{\lambda}))E_o(\tilde{s}\tilde{w}) - E_o(\tilde{s}\tilde{w})E_o(\tilde{\lambda}) \in \sum_{v\leq w} T_{\tilde{v}}\mathcal{A}_o.$$

In particular (94) is true for $\tilde{s}\tilde{w}$. ☐

We give the proof of another variant of the Bernstein relations when $q_s = 1$ for all $s \in S^{aff}/\sim$ using the end of the proof of Thm. 5.38 after the lemma 5.42. When o is the anti-dominant orientation this variant was discovered by Abe.

As in Notation 5.37, for $s \in S(1)$, let $\lambda_s \in \Lambda(1)$, let $\beta \in \Delta_\Sigma$ such that s lifts s_β and let $\delta \in \{1,2\}$ such that $\beta \circ \nu(\Lambda) = \delta\mathbb{Z}$ and $\beta \circ \nu(\lambda_s) = -\delta$, and for $k \in \mathbb{Z}$ let $\mu(k) \in \Lambda(1)$ and $c(k), c'(k) \in R[Z_k]$ depending on s, λ_s, k defined in Notation 5.37.

As in Definitions 5.18, 5.22, for an orientation o, we have $E_o(s) = T_s$ if $\epsilon_{o\bullet s}(1,s) = 1$, meaning that the alcove $\mathfrak{C}$ belongs to the o-negative side of H_s, and $E_o(s) = T_s - c_s$ if $\epsilon_{o\bullet s}(1,s) = -1$.

Proposition 5.55. (Variant of the Bernstein relations) *We suppose that $q_s = 1$ for all $s \in S^{aff}/\sim$. Let $s \in (S \cap S_o)(1)$, let $\lambda^+, \lambda^- \in \Lambda(1)$ such that $\beta \circ \nu(\lambda^+) = -\beta \circ \nu(\lambda^-) = \delta n > 0$. Then,*

$$\begin{aligned}
E_o(s(\lambda^-))E_o(s) - E_o(s)E_o(\lambda^-) &= \epsilon_{o\bullet s}(1,s)\sum_{k=1}^{\delta n} E_o(s(\lambda^-)\mu(-k))c'(-k)\\
&= -\epsilon_{o\bullet s}(1,s)\sum_{k=0}^{\delta n-1} c(k)E_o(\mu(k)\lambda),\\
E_o(s(\lambda^+))E_o(s) - E_o(s)E_o(\lambda^+) &= \epsilon_{o\bullet s}(1,s)\sum_{k=1}^{\delta n} c(-k)E_o(\mu(-k)\lambda^+)\\
&= \epsilon_{o\bullet s}(1,s)\sum_{k=0}^{\delta n-1} E_o(s(\lambda^+)\mu(k))c'(k).
\end{aligned}$$

Proof. Let $x = \lambda_s$ and $y, y' \in \Lambda^s(1)$ such that

$$\lambda^- = x^n y = y' x^n, \quad \lambda^+ = y x^{-n} = x^{-n} y'.$$

The Bernstein element (84) at x^n, computed after the lemma 5.42, multiplied on the left by $E_o(s(y'))$ is the Bernstein element at $\lambda^- = y'x^n$ because $E_o(s(y'))E_o(s) = E_o(s)E_{o\bullet s}(y') = E_o(s)E_o(y')$, and we obtain :

$$\begin{aligned}
E_o(s(\lambda^-))E_o(s) - E_o(s)E_o(\lambda^-) &= E_o(s(y'))(E_o(s(x^n))E_o(s) - E_o(s)E_o(x^n))\\
&= \epsilon_{o\bullet s}(1,s)E_o(s(y'))B_{o,n}E_o(x^n).
\end{aligned}$$

Similarly, the Bernstein element at x^{-n}, computed after the lemma 5.42, multiplied on the right by $E_o(y')$ is the Bernstein element at $\lambda^+ = x^{-n}y'$, and we obtain :

$$E_o(s(\lambda^+))E_o(s) - E_o(s)E_o(\lambda^+) = (E_o(s(x^{-n}))E_o(s) - E_o(s)E_o(x^{-n}))E_o(y')$$
$$= \epsilon_{o\bullet s}(1,s)E_o(s(x^{-n}))B_{o,n}E_o(y').$$

The explicit expansion of $B_{o,n}$ given after (86) gives $B_{o,n}E_o(x^n) = \sum_{k=0}^{\delta n-1} c(k)E_o(\mu(k)x^n)$; we compute the product in the inner terms $c(k)E_o(\mu(k)x^n) = E_o(\mu(k)x^n)(\mu(k)x^n \bullet c(k))$, and we replace k by $\delta n - k$ to obtain

$$B_{o,n}E_o(x^n) = \sum_{k=1}^{\delta n} E_o(\mu(\delta n-k)x^n)((\mu(\delta n-k)x^n)^{-1} \bullet c(\delta n-k)).$$

Multiplying on the left by $\epsilon_{o\bullet s}(1,s)E_o(s(y'))$ we get

$$E_o(s(\lambda^-))E_o(s) - E_o(s)E_o(\lambda^-)$$
$$= \epsilon_{o\bullet s}(1,s)\sum_{k=1}^{\delta n} E_o(s(y')\mu(\delta n-k)x^n)\,((\mu(\delta n-k)x^n)^{-1} \bullet c(\delta n-k)).$$

Analagously, the second explicit expansion of $B_{o,n}$ given after (86) gives

$$E_o(s(x^{-n}))B_{o,n} = \sum_{k=0}^{\delta n-1} E_o(s(x^{-n})\mu(k))c'(k) = \sum_{k=0}^{\delta n-1} (s(x^{-n})\mu(k) \bullet c'(k))\,E_o(s(x^{-n})\mu(k))$$
$$= \sum_{k=1}^{\delta n}(s(x^{-n})\mu(\delta n-k) \bullet c'(\delta n-k))E_o(s(x^{-n})\mu(\delta n-k)).$$

Multiplying on the right by $\epsilon_{o\bullet s}(1,s)E_o(y')$ we get

$$E_o(s(\lambda^+))E_o(s) - E_o(s)E_o(\lambda^+)$$
$$= \epsilon_{o\bullet s}(1,s)\sum_{k=1}^{\delta n}(s(x^{-n})\mu(\delta n-k) \bullet c'(\delta n-k))E_o(s(x^{-n})\mu(\delta n-k)y').$$

Recalling the values of $\mu(k)$ and $c(k)$, for $k \in \mathbb{Z}$, we have:

If $k = \delta r$, then $\mu(\delta n-k)x^n = s(x^{n-r})x^r$ and $s(x)^{-n}\mu(\delta n-k) = s(x^{-r})x^{r-n}$,

$$\begin{aligned}
&s(y')\mu(\delta n-k)x^n = s(y'x^{n-r})x^r = s(\lambda^-)s(x)^{-r}x^r = s(\lambda^-)\mu(-k),\\
&(\mu(\delta n-k)x^n)^{-1} \bullet c(\delta n-k) = x^{-r} \bullet c_s = c'(-k),\\
&s(x^{-n})\mu(\delta n-k)y' = s(x^{-r})x^{r-n}y' = s(x)^{-r}x^r\lambda^+ = \mu(-k)\lambda^+,\\
&s(x)^{-n}\mu(\delta n-k) \bullet c'(n-k) = s(x^{-r}) \bullet c_s = c(-k).
\end{aligned}$$

If $\delta = 2,\ k = 2r-1$, then $\mu(2n-k)x^n = \mu(2(n-r)+1)x^n = s(x^{n-r})x^{r-1}$ and $s(x)^{-n}\mu(\delta n-k) = s(x^{-r})x^{r-n-1}$,

$$\begin{aligned}
&s(y')\mu(2n-k)x^n = s(\lambda^-)s(x)^{-r}x^{r-1} = s(\lambda^-)\mu(-k),\\
&(\mu(2n-k)x^n)^{-1} \bullet c(2n-k) = x^{1-r} \bullet c_\tau = c'(-k),\\
&s(x^{-n})\mu(\delta n-k)y' = s(x^{-r})x^{r-1}\lambda^+ = \mu(-k)\lambda^+,\\
&s(x)^{-n}\mu(\delta n-k) \bullet c'(n-k) = s(x^{-r}) \bullet c_\tau = c(-k).
\end{aligned}$$

□

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