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Elliptic fibrations on the modular surface associated to $\Gamma_1(8)$

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Abstract. We give all the elliptic fibrations of the $K3$ surface associated to the modular group $\Gamma_1(8)$.

1 Introduction

Stienstra and Beukers [27] considered the elliptic pencil

$$xyz + \tau(x+y)(x+z)(y+z) = 0$$

and the associated $K3$ surface $\mathcal{B}$ for $\tau = t^2$, double cover of the modular surface for the modular group $\Gamma_0(6)$. With the help of its L-series, they remarked that this surface should carry an elliptic pencil exhibiting it as the elliptic modular surface for $\Gamma_1(8)$ and deplored it was not visible in the previous model of $\mathcal{B}$.

Later on, studying the link between the logarithmic Mahler measure of some $K3$ surfaces and their L-series, Bertin considered in [3] $K3$ surfaces of the family previously studied by Peters and Stienstra [19]

$$X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = k. \qquad (Y_k)$$

For $k = 2$, Bertin proved that the corresponding $K3$ surface Y_2 is singular (i.e. its Picard rank is 20) with transcendental lattice

$$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

Bertin noticed that Y_2 was nothing else than $\mathcal{B}$, corresponding to the elliptic fibration $X + Y + Z = s$ and $1/\tau = (s-1)^2$. Its singular fibers are of Dynkin type A_{11}, A_5, $2A_1$ and Kodaira type I_{12}, I_6, $2I_2$, $2I_1$. Its Mordell-Weil group is the torsion group $\mathbb{Z}/6\mathbb{Z}$.

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Using an unpublished result of Lecacheux (see also section 7 of this paper), Bertin showed also that Y_2 carries the structure of the modular elliptic surface for $\Gamma_1(8)$. In that case, it corresponds to the elliptic fibration of Y_2 with parameter $Z = s$. Its singular fibers are of Dynkin type $2A_7$, A_3, A_1 and Kodaira type $2I_8$, I_4, I_2, $2I_1$. Its Mordell-Weil group is the torsion group $\mathbb{Z}/8\mathbb{Z}$.

Interested in $K3$ surfaces with Picard rank 20 over $\mathbb{Q}$, Elkies proved in [8] that their transcendental lattices are primitive of class number one. In particular, he gave in [9] a list of 11 negative integers D for which there is a unique $K3$ surface X over $\mathbb{Q}$ with Néron-Severi group of rank 20 and discriminant $-D$ consisting entirely of classes of divisors defined over $\mathbb{Q}$. For $D = -8$, he gave an explicit model of an elliptic fibration with E_8 $(= II^*)$ fibers at $t = 0$ and $t = \infty$ and an A_1 $(= I_2)$ fiber at $t = -1$

$$y^2 = x^3 - 675x + 27(27t - 196 + \frac{27}{t}).$$

For this fibration, the Mordell-Weil group has rank 1 and no torsion.

Independently, Schütt proved in [21] the existence of $K3$ surfaces of Picard rank 20 over $\mathbb{Q}$ and gave for the discriminant $D = -8$ an elliptic fibration with singular fibers A_3, E_7, E_8 (I_4, III^*, II^*) and Mordell-Weil group equal to (0). For such a model, you can refer to [22].

Recall also that Shimada and Zhang gave in [24] a list, without equations but with their Mordell-Weil group, of extremal elliptic $K3$ surfaces. In particular, there are 14 extremal elliptic $K3$ surfaces with transcendental lattice

$$\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}.$$

We mention Beukers and Montanus who worked out the semi-stable, extremal, elliptic fibrations of $K3$ surfaces [4].

As announced in the abstract, the aim of the paper is to determine all the elliptic fibrations with section on the modular surface associated to $\Gamma_1(8)$ and give for each fibration a Weierstrass model. Thus we recover all the extremal fibrations given by Shimada and Zhang and also fibrations of Bertin, Elkies, Schütt and Stienstra-Beukers mentioned above.

The paper is divided in two parts. In the first sections we use Nishiyama's method, as explained in [18] and [23], to determine all the elliptic fibrations of $K3$ surfaces with a given transcendental lattice. The method is based on lattice theoretical ideas. We prove the following theorem

Theorem 1.1 *There are* 30 *elliptic fibrations with section, all distinct up to isomorphism, on the elliptic surface*

$$X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = 2.$$

They are listed in Table 3 with the rank and torsion of their Mordell-Weil group. The list consists of 14 *fibrations of rank* 0*,* 13 *fibrations of rank* 1 *and* 3 *fibrations of rank* 2*.*

In the second part, i.e. sections 7 to 10, we first explain that Y_2 is the modular surface associated to the modular group $\Gamma_1(8)$. From one of its fibrations we deduce that it is the unique $K3$ surface X over $\mathbb{Q}$ with Néron-Severi group of rank 20 and discriminant -8, all of its classes of divisors being defined over $\mathbb{Q}$.

Then, for each fibration, we determine explicitly a Weierstrass model, with generators of the Mordell-Weil group.

We first use the 8-torsion sections of the modular fibration to construct the 16 first fibrations. Their parameters belong to a special group generated by 10 functions on the surface. This construction is similar to the one developed for $\Gamma_1(7)$ by Harrache and Lecacheux in [13]. The next fibrations are obtained by classical methods of gluing and breaking singular fibers. The last ones are constructed by adding a vertex to the graph of the modular fibration.

The construction of some of the fibrations can be done also for the other $K3$ surfaces Y_k of the family. Thus we hope to find for them fibrations of rank 0 and perhaps obtain more easily the discriminant of the transcendental lattice for singular $K3$ members.

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2 Definitions

An integral symmetric bilinear form or a **lattice** of rank r is a free $\mathbb{Z}$-module S of rank r together with a symmetric bilinear form b. If S is a non-degenerate lattice, we write the signature of S, $\mathrm{sign}(S) = (t_+, t_-)$. An indefinite lattice of signature $(1, t_-)$ or $(t_+, 1)$ is called an **hyperbolic** lattice. A lattice S is called **even** if $x^2 := b(x, x)$ is even for all x from S. For any integer n we denote by $\langle \boldsymbol{n} \rangle$ the lattice $\mathbb{Z}e$ where $e^2 = n$. For every integer m we denote by $\boldsymbol{S[m]}$ the lattice obtained from a lattice S by multiplying the values of its bilinear form by m. If $e = (e_1, \ldots, e_r)$ is a $\mathbb{Z}$-basis of a lattice S, then the matrix $G(e) = (b(e_i, e_j))$ is called the **Gram matrix** of S with respect to e.

A homomorphism of lattices $f : S \to S'$ is a homomorphism of the abelian groups such that $b'(f(x), f(y)) = b(x, y)$ for all $x, y \in S$. An injective (resp. bijective) homomorphism of lattices is called an **embedding** (resp. an isometry). The group of isometries of a lattice S into itself is denoted by $O(S)$ and called the orthogonal group of S. Two embeddings $i : S \to S'$ and $i' : S \to S'$ are called isomorphic if there exists an isometry $\sigma \in O(S')$ such that $i' = \sigma \circ i$. An embedding $i : S \to S'$ is called **primitive** if $S'/i(S)$ is a free group. A sublattice is a subgroup equipped with the induced bilinear form. A sublattice S' of a lattice S is called primitive if the identity map $S' \to S$ is a primitive embedding. The **primitive closure** of S inside S' is defined by $\overline{S} = \{x \in S'/mx \in S \text{ for some positive integer } m\}$. A lattice M is an **overlattice** of S if S is a sublattice of M such that the index $[M : S]$ is finite.

By $S_1 \oplus S_2$ we denote the orthogonal sum of two lattices defined in the standard way. We write S^n for the orthogonal sum of n copies of a lattice S. The **orthogonal complement of a sublattice** S of a lattice S' is denoted $(S)^{\perp}_{S'}$ and defined by $(S)^{\perp}_{S'} = \{x \in S'/b(x, y) = 0 \text{ for all } y \in S\}$.

3 Discriminant forms

Let L be a non-degenerate lattice. The **dual lattice** L^* of L is defined by

$$L^* := \mathrm{Hom}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q}/\ b(x,y) \in \mathbb{Z} \text{ for all } y \in L\}.$$

and the **discriminant group** G_L by

$$G_L := L^*/L.$$

This group is finite if and only if L is non-degenerate. In the latter case, its order is equal to the absolute value of the lattice determinant $\mid \det(G(e)) \mid$ for any basis e of L. A lattice L is **unimodular** if G_L is trivial.

Let G_L be the discriminant group of a non-degenerate lattice L. The bilinear form on L extends naturally to a $\mathbb{Q}$-valued symmetric bilinear form on L^* and induces a symmetric bilinear form

$$b_L : G_L \times G_L \to \mathbb{Q}/\mathbb{Z}.$$

If L is even, then b_L is the symmetric bilinear form associated to the quadratic form defined by

$$\begin{array}{rcl} q_L : G_L & \to & \mathbb{Q}/2\mathbb{Z} \\ q_L(x+L) & \mapsto & x^2 + 2\mathbb{Z}. \end{array}$$

The latter means that $q_L(na) = n^2 q_L(a)$ for all $n \in \mathbb{Z}$, $a \in G_L$ and $b_L(a,a') = \frac{1}{2}(q_L(a+a') - q_L(a) - q_L(a'))$, for all $a, a' \in G_L$, where $\frac{1}{2} : \mathbb{Q}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is the natural isomorphism. The pair $(\boldsymbol{G_L, b_L})$ (resp. $(\boldsymbol{G_L, q_L})$) is called the **discriminant bilinear** (resp. **quadratic**) **form** of L.

4 Root lattices

In this section we recall only what is needed for the understanding of the paper. For proofs and details one can refer to Bourbaki [5] or Martinet [14].

Let L be a negative-definite even lattice. We call $e \in L$ a **root** if $q_L(e) = -2$. Put $\Delta(L) := \{e \in L/q_L(e) = -2\}$. Then the sublattice of L spanned by $\Delta(L)$ is called the **root type** of L and is denoted by L_{root}. If $e \in \Delta(L)$, we call **reflection associated with** e the following isometry

$$R_e(x) = x + b(x,e)e.$$

The subgroup of $O(L)$ generated by R_e $(e \in \Delta(L))$ is called the **Weyl group** of L and is denoted by $W(L)$.

The **lattices** $A_n = \langle a_1, a_2, \ldots, a_n \rangle$ $(n \geq 1)$, $D_l = \langle d_1, d_2, \ldots, d_l \rangle$ $(l \geq 4)$, $E_p = \langle e_1, e_2, \ldots, e_p \rangle$ $(p = 6, 7, 8)$ defined by the following **Dynkin diagrams** are called the **root lattices**. All the vertices a_j, d_k, e_l are roots and two vertices a_j and a'_j are joined by a line if and only if $b(a_j, a'_j) = 1$. We use Bourbaki's definitions [5].(Figure 1)

Denote ϵ_i the vectors of the canonical basis of $\mathbb{R}^n$ with the negative usual scalar product.

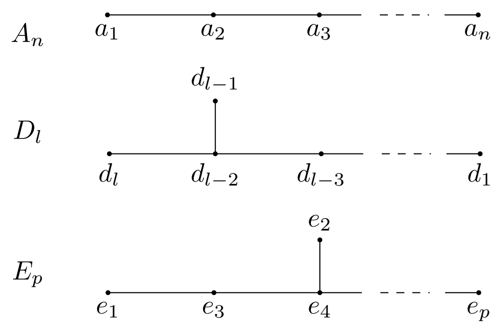


Figure 1: Dynkin diagrams

4.1 A_n^*/A_n**.** We can represent A_n by the set of points in $\mathbb{R}^{n+1}$ with integer coordinates whose sum is zero. Set $a_i = \epsilon_i - \epsilon_{i+1}$ and define

$$\alpha_n = \epsilon_1 - \frac{1}{n+1}\sum_{j=1}^{n+1}\epsilon_j = \frac{1}{n+1}\sum_{j=1}^{n}(n-j+1)a_j.$$

One can show that

$$A_n^* = \langle A_n, \alpha_n\rangle, \quad A_n^*/A_n \simeq \mathbb{Z}/(n+1)\mathbb{Z} \ \text{ and } \ q_{A_n}(\alpha_n) = \left(-\frac{n}{n+1}\right).$$

4.2 D_l^*/D_l**.** We can represent D_l as the set of points of $\mathbb{R}^l$ with integer coordinates of even sum and define

$$\begin{aligned}
\delta_l &= \tfrac{1}{2}(\textstyle\sum_{i=1}^{l}\epsilon_i) = \tfrac{1}{2}\left(\sum_{i=1}^{l-2} id_i + \tfrac{1}{2}(l-2)d_{l-1} + \tfrac{1}{2}ld_l\right)\\
\overline{\delta}_l &= \epsilon_1 = \textstyle\sum_{i=1}^{l-2} d_i + \tfrac{1}{2}(d_{l-1}+d_l)\\
\tilde{\delta}_l &= \delta_l - \epsilon_l = \tfrac{1}{2}\left(\sum_{i=1}^{l-2} id_i + \tfrac{1}{2}ld_{l-1} + \tfrac{1}{2}(l-2)d_l\right).
\end{aligned}$$

One can show that

$$D_l^* = \langle \epsilon_1, \dots, \epsilon_l, \delta_l\rangle.$$

Then, for l odd

$$D_l^*/D_l \simeq \mathbb{Z}/4\mathbb{Z} = \langle\delta_l\rangle, \quad \overline{\delta}_l \equiv 2\delta_l \ \text{ and } \ \tilde{\delta}_l \equiv 3\delta_l \ \text{ mod. } D_l$$

and for l even

$$D_l^*/D_l \simeq \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z},$$

the 3 elements of order 2 being the images of δ_l, $\tilde{\delta}_l$ and $\overline{\delta}_l$. Moreover,

$$q_{D_l}(\delta_l) = \left(-\frac{l}{4}\right), \quad q_{D_l}(\overline{\delta}_l) = (-1), \quad b_{D_l}(\delta_l, \overline{\delta}_l) = -\frac{1}{2}.$$

4.3 E_6^*/E_6**.** We can represent E_6 as a lattice in $\mathbb{R}^8$ generated by the 6 vectors $e_1 = \frac{1}{2}(\epsilon_1+\epsilon_8) - \frac{1}{2}(\sum_{i=2}^{7}\epsilon_i)$, $\quad e_2 = \epsilon_1+\epsilon_2$, $\quad e_i = \epsilon_{i-1} - \epsilon_{i-2}$, $\quad 3 \le i \le 6$.
If $\eta_6 = -\frac{1}{3}(2e_1+3e_2+4e_3+6e_4+5e_5+4e_6)$, then

$$E_6^* = \langle E_6, \eta_6\rangle, \quad E_6^*/E_6 \simeq \mathbb{Z}/3\mathbb{Z}, \quad q_{E_6}(\eta_6) = \left(-\frac{4}{3}\right).$$

4.4 E_7^*/E_7**.** We can represent E_7 as a lattice in $\mathbb{R}^8$ generated by the 6 previous vectors e_i and $e_7 = \epsilon_6 - \epsilon_5$. If $\eta_7 = -\frac{1}{2}(2e_1 + 3e_2 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7)$, then

$$E_7^* = \langle E_7, \eta_7 \rangle, \quad E_7^*/E_7 \simeq \mathbb{Z}/2\mathbb{Z}, \quad q_{E_7}(\eta_7) = \left(-\frac{3}{2}\right).$$

4.5 E_8^*/E_8**.** We can represent E_8 as the subset of points with coordinates ξ_i satisfying

$$2\xi_i \in \mathbb{Z}, \quad \xi_i - \xi_j \in \mathbb{Z}, \quad \sum_{i=1}^{\infty} \xi_i \in 2\mathbb{Z}.$$

Then $E_8^*/E_8 = (0)$.

5 Elliptic fibrations

Before giving a complete classification of the elliptic fibrations on the $K3$ surface Y_2, we recall briefly some useful facts concerning $K3$ surfaces. For more details see [1] [29].

5.1 $K3$ **surfaces and elliptic fibrations.** A $K3$ surface X is a smooth projective complex surface with

$$K_X = \mathcal{O}_X \quad \text{and} \quad H^1(X, \mathcal{O}_X) = 0.$$

If X is a $K3$ **surface**, then $H^2(X, \mathbb{Z})$ is torsion free. With the cup product, $H^2(X, \mathbb{Z})$ has the structure of an even lattice. By the Hodge index theorem it has signature $(3, 19)$ and by Poincaré duality it is unimodular. Moreover, as a lattice, $H^2(X, \mathbb{Z}) = U^3 \oplus E_8(-1)^2$.

The **Néron-Severi group** $NS(X)$ (i.e. the group of line bundles modulo algebraic equivalence), with the intersection pairing, if $\rho(X)$ is the Picard number of X, is a lattice of signature $(1, \rho(X) - 1)$. The natural embedding $NS(X) \hookrightarrow H^2(X, \mathbb{Z})$ is a primitive embedding of lattices. If C is an smooth projective curve over an algebraically closed field K, an **elliptic surface** Σ over C is an smooth surface with a surjective morphism

$$f : \Sigma \to C$$

such that almost all fibers are smooth curves of genus 1 and no fiber contains exceptional curves of the first kind. The morphism f defines an elliptic fibration on Σ. We suppose also that every elliptic fibration has a section and so a Weierstrass form. Thus we can consider the generic fiber as an elliptic curve E on $K(C)$ choosing a section as the zero section $\bar{O}$. In the case of $K3$ surfaces, $C = \mathbb{P}^1$.

The singular fibers were classified by Néron [15] and Kodaira [10]. They are union of irreductible components with multiplicities; each component is a smooth rational curve with self-intersection -2. The singular fibers are classified in the following Kodaira types:

- two infinite series $I_n (n > 1)$ and $I_n^* (n \geq 0)$
- five types $III, IV, II^*, III^*, IV^*$.

The dual graph of these components (a vertex for each component, an edge for each intersection point of two components) is an **extended Dynkin diagram** of type $\tilde{A}_n$, $\tilde{D}_l$, $\tilde{E}_p$. Deleting the zero component (i.e. the component meeting the zero section) gives the Dynkin diagram graph A_n, D_l, E_p. We draw the most

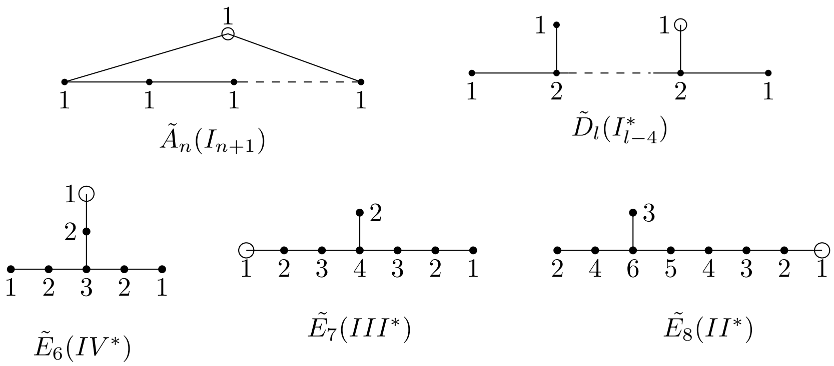


Figure 2: Extended Dynkin diagrams

useful diagrams, with the multiplicity of the components, the zero component being represented by a circle.(Figure 2)

The **trivial lattice** $T(X)$ is the subgroup of the Néron-Severi group generated by the zero section and the fibers components. More precisely, the trivial lattice is the orthogonal sum

$$T(X) =< \bar{O}, F > \oplus_{v \in S} T_v$$

where $\bar{O}$ denotes the zero section, F the general fiber, S the points of C corresponding to the reducible singular fibers and T_v the lattice generated by the fiber components except the zero component. From this formula we can compute the determinant of $T(X)$. From Shioda's results on height pairing [25] we can define a positive-definite lattice structure on the Mordell-Weil lattice $MWL(X) := E(K(C))/E(K(C))_{tor}$ and get the following proposition.

Proposition 5.1 Let X be a $K3$ surface or more generally any elliptic surface with section. We have the relation

$$|disc(NS(X))| = disc(T(X)) disc(MWL(X))/|E(K)_{tor}|^2.$$

Moreover since X is a $K3$ surface, the zero section has self-intersection $\bar{O}^2 = -\chi(X) = -2$. Hence the zero section $\bar{O}$ and the general fiber F generate an even unimodular lattice, called the hyperbolic plane U. The trivial lattice $T(X)$ of an elliptic surface is not always primitive in $NS(X)$. Its primitive closure $\overline{T(X)}$ is obtained by adding the torsion sections. The Néron-Severi lattice $NS(X)$ always contains an even sublattice of corank two, the **frame** $W(X)$

$$W(X) = \langle \bar{O}, F \rangle^{\perp} \subset NS(X).$$

Lemma 5.1 *For any elliptic surface X with section, the frame $W(X)$ is a negative-definite even lattice of rank $\rho(X) - 2$.*

Hence, the Néron-Severi lattice of a $K3$ surface is an even lattice. One can read off the Mordell-Weil lattice, the torsion in the Mordell-Weil group MW and the type of singular fibers from $W(X)$ by

$$MWL(X) = W(X)/\overline{W(X)}_{\text{root}} \quad (MW)_{\text{tors}} = \overline{W(X)}_{\text{root}}/W(X)_{\text{root}}$$
$$T(X) = U \oplus W(X)_{\text{root}}.$$

We can also calculate the heights of points from the Weierstrass equation [11] and test if points generate the Mordell-Weil group, since $disc(NS(X))$ is independent of the fibration.

5.2 Nikulin and Niemeier's results.

Lemma 5.2 (Nikulin [17], Proposition 1.4.1) *Let L be an even lattice. Then, for an even overlattice M of L, we have a subgroup M/L of $G_L = L^*/L$ such that q_L is trivial on M/L. This determines a bijective correspondence between even overlattices of L and subgroups G of G_L such that $q_L\,|_G = 0$.*

Lemma 5.3 (Nikulin [17], Proposition 1.6.1) *Let L be an even unimodular lattice and T a primitive sublattice. Then we have*

$$G_T \simeq G_{T^\perp} \simeq L/(T \oplus T^\perp), \quad q_{T^\perp} = -q_T.$$

In particular, $\det T = \det T^\perp = [L : T \oplus T^\perp]$.

Theorem 5.1 (Nikulin [17] Corollary 1.6.2) *Let L and M be non-degenerate even integral lattices such that*

$$G_L \simeq G_M, \quad q_L = -q_M.$$

Then there exists an unimodular overlattice N of $L \oplus M$ such that

1) the embeddings of L and M in N are primitive

2) $L_N^\perp = M$ and $M_N^\perp = L$.

Theorem 5.2 (Nikulin [17] Theorem 1.12.4) *Let there be given two pairs of nonnegative integers, $(t_{(+)}, t_{(-)})$ and $(l_{(+)}, l_{(-)})$. The following properties are equivalent:*

a) every even lattice of signature $(t_{(+)}, t_{(-)})$ admits a primitive embedding into some even unimodular lattice of signature $(l_{(+)}, l_{(-)})$;

b) $l_{(+)} - l_{(-)} \equiv 0 \ (mod\ 8)$, $t_{(+)} \leq l_{(+)}$, $t_{(-)} \leq l_{(-)}$ and $t_{(+)} + t_{(-)} \leq \frac{1}{2}(l_{(+)} + l_{(-)})$.

Theorem 5.3 (Niemeier [16]) *A negative-definite even unimodular lattice L of rank 24 is determined by its root lattice L_{root} up to isometries. There are 24 possibilities for L and L/L_{root} listed in Table 1.*

The lattices L defined in table 1 are called **Niemeier lattices**.

5.3 Nishiyama's method. Recall that a $K3$ surface may admit more than one elliptic fibration, but up to isomorphism, there is only a finite number of elliptic fibrations [26]. To establish a complete classification of the elliptic fibrations on the $K3$ surface Y_2, we use Nishiyama's method based on lattice theoretic ideas [18]. The technique builds on a converse of Nikulin's results.

Given an elliptic $K3$ surface X, Nishiyama aims at embedding the frames of all elliptic fibrations into a negative-definite lattice, more precisely into a Niemeier lattice of rank 24. For this purpose, he first determines an even negative-definite lattice M such that

$$q_M = -q_{NS(X)}, \quad \text{rank}(M) + \rho(X) = 26.$$

By theorem 5.1, $M \oplus W(X)$ has a Niemeier lattice as an overlattice for each frame $W(X)$ of an elliptic fibration on X. Thus one is bound to determine the (in-equivalent) primitive embeddings of M into Niemeier lattices L. To achieve this, it is essential to consider the root lattices involved. In each case, the orthogonal complement of M into L gives the corresponding frame $W(X)$.

L_{root}	L/L_{root}	L_{root}	L/L_{root}
E_8^3	(0)	$D_5^{\oplus 2} \oplus A_7^{\oplus 2}$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$
$E_8 \oplus D_{16}$	$\mathbb{Z}/2\mathbb{Z}$	$A_8^{\oplus 3}$	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$
$E_7^{\oplus 2} \oplus D_{10}$	$(\mathbb{Z}/2\mathbb{Z})^2$	A_{24}	$\mathbb{Z}/5\mathbb{Z}$
$E_7 \oplus A_{17}$	$\mathbb{Z}/6\mathbb{Z}$	$A_{12}^{\oplus 2}$	$\mathbb{Z}/13\mathbb{Z}$
D_{24}	$\mathbb{Z}/2\mathbb{Z}$	$D_4^{\oplus 6}$	$(\mathbb{Z}/2\mathbb{Z})^6$
$D_{12}^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^2$	$D_4 \oplus A_5^{\oplus 4}$	$\mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/6\mathbb{Z})^2$
$D_8^{\oplus 3}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$A_6^{\oplus 4}$	$(\mathbb{Z}/7\mathbb{Z})^2$
$D_9 \oplus A_{15}$	$\mathbb{Z}/8\mathbb{Z}$	$A_4^{\oplus 6}$	$(\mathbb{Z}/5\mathbb{Z})^3$
$E_6^{\oplus 4}$	$(\mathbb{Z}/3\mathbb{Z})^2$	$A_3^{\oplus 8}$	$(\mathbb{Z}/4\mathbb{Z})^4$
$E_6 \oplus D_7 \oplus A_{11}$	$\mathbb{Z}/12\mathbb{Z}$	$A_2^{\oplus 12}$	$(\mathbb{Z}/3\mathbb{Z})^6$
$D_6^{\oplus 4}$	$(\mathbb{Z}/2\mathbb{Z})^4$	$A_1^{\oplus 24}$	$(\mathbb{Z}/2\mathbb{Z})^{12}$
$D_6 \oplus A_9^{\oplus 2}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$	0	Λ_{24}

Table 1: Niemeier lattices

5.3.1 *The transcendental lattice and argument from Nishiyama paper.* Denote by $\mathbb{T}(X)$ the transcendental lattice of X, i.e. the orthogonal complement of $NS(X)$ in $H^2(X,\mathbb{Z})$ with respect to the cup-product,

$$\mathbb{T}(X) = \text{NS}(X)^{\perp} \subset H^2(X,\mathbb{Z}).$$

In general, $\mathbb{T}(X)$ is an even lattice of rank $t = 22-\rho(X)$ and signature $(2, 20-\rho(X))$. Let $t' = t-2$. By Nikulin's theorem 5.2, $\mathbb{T}(X)[-1]$ admits a primitive embedding into the following indefinite unimodular lattice:

$$\mathbb{T}(X)[-1] \hookrightarrow U^{t'} \oplus E_8.$$

Then define M as the orthogonal complement of $\mathbb{T}(X)[-1]$ in $U^{t'} \oplus E_8$. By construction, M is a negative-definite lattice of rank $2t'+8-t = t+4 = 26-\rho(X)$.

By lemma 5.3 the discriminant form satisfies

$$q_M = -q_{\mathbb{T}(X)[-1]} = q_{\mathbb{T}(X)} = -q_{\text{NS}(X)}.$$

Hence M takes exactly the shape required for Nishiyama's technique.

5.3.2 *Torsion group.* First we classify all the primitive embeddings of M into L_{root}. Let N be the orthogonal complement of M into L_{root} and W the orthogonal complement of M into L. If M satisfies $M_{\text{root}} = M$, we can apply Nishiyama's results [18]. In particular,

- M primitively embedded in $L_{\text{root}} \iff M$ primitively embedded in L,
- N/N_{root} is torsion-free.

Notice that the rank r of the Mordell-Weil group is equal to $\text{rk}(W) - \text{rk}(W_{\text{root}})$ and its torsion part is $\overline{W}_{\text{root}}/W_{\text{root}}$. We need also the following lemma [18] lemma 6.6

Lemma 5.4 1. *If* $\det N = \det M$*, then the Mordell-Weil group is torsion-free.*
2. *If* $r = 0$*, then the Mordell-Weil group is isomorphic to* W/N*.*
3. *In general, there are the following inclusions of groups:*

$$\overline{W}_{root}/W_{root} \subset W/N \subset L/L_{root}.$$

6 Elliptic fibrations of Y_2

We follow Nishiyama's method. Since

$$\mathbb{T}(Y_2) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix},$$

we get, by Nishiyama's computation [18], $M = D_5 \oplus A_1$. Thus we have to determine all the primitive embeddings of M into the root lattices and their orthogonal complements.

6.1 The primitive embeddings of $D_5 \oplus A_1$ into root lattices.

Proposition 6.1 There are primitive embeddings of $D_5 \oplus A_1$ only into the following L_{root}:

$$E_8^{\oplus 3},\ E_8 \oplus D_{16},\ E_7^{\oplus 2} \oplus D_{10},\ E_7 \oplus A_{17},\ D_8^{\oplus 3},\ D_9 \oplus A_{15},$$
$$E_6^{\oplus 4},\ A_{11} \oplus E_6 \oplus D_7,\ D_6^{\oplus 4},\ D_6 \oplus A_9^{\oplus 2},\ D_5^{\oplus 2} \oplus A_7^{\oplus 2}.$$

Proof The assertion comes from Nishiyama's results [18]. The lattice A_1 can be primitively embedded in all A_n, D_l and E_p, $n \geq 1$, $l \geq 2$, $p = 6,7,8$. The lattice D_5 can be primitively embedded only in D_l, $l \geq 5$ and E_p, $p = 6,7,8$. The lattice $D_5 \oplus A_1$ can be primitively embedded only in D_l, $l \geq 7$, E_7 and E_8. The proposition follows from theorem 5.3 and the previous facts. □

Proposition 6.2 Up to the action of the Weyl group, the unique primitive embeddings are given in the following list

- $A_1 = \langle a_n \rangle \subset A_n$
- $A_1 = \langle d_l \rangle \subset D_l,\ l \geq 4$
- $A_1 = \langle e_1 \rangle \subset E_p,\ p = 6,7,8$
- $D_5 = \langle d_{l-1}, d_l, d_{l-2}, d_{l-3}, d_{l-4} \rangle \subset D_l,\ l \geq 6$
- $D_5 = \langle e_2, e_5, e_4, e_3, e_1 \rangle \subset E_n,\ n \geq 6$
- $D_5 \oplus A_1 = \langle d_{l-1}, d_l, d_{l-2}, d_{l-3}, d_{l-4} \rangle \oplus \langle d_{l-6} \rangle \subset D_l,\ l \geq 7$
- $D_5 \oplus A_1 = \langle e_2, e_5, e_4, e_3, e_1 \rangle \oplus \langle e_7 \rangle \subset E_n,\ n \geq 7.$

Proof The first five assertions follow from Nishiyama's computations [18]. Just be careful of the difference of notations between Nishiyama and us. The two last assertions follow from the lemmas below.

Lemma 6.1 *Up to isomorphism by an element of the Weyl group, there is a unique primitive embedding of $D_5 \oplus A_1$ into E_8 given by*

$$\langle e_2, e_5, e_4, e_3, e_1 \rangle \oplus \langle e_7 \rangle.$$

Proof Up to isomorphism in E_8 there is a unique primitive embedding of D_5 into E_8 given by $\langle e_2, e_5, e_4, e_3, e_1 \rangle$ [18]. Moreover

$$(D_5)^{\perp}_{E_8} = \langle e_7, e_8, 3e_2 + 2e_1 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7 + 2e_8 \rangle = A_3$$

Thus we get three primitive embeddings of $D_5 \oplus A_1$ into E_8

1. $\langle e_2, e_5, e_4, e_3, e_1 \rangle \oplus \langle e_7 \rangle$
2. $\langle e_2, e_5, e_4, e_3, e_1 \rangle \oplus \langle e_8 \rangle$
3. $\langle e_2, e_5, e_4, e_3, e_1 \rangle \oplus \langle x = 3e_2 + 2e_1 + 4e_3 + 6e_4 + 5e_5 + 4e_6 + 3e_7 + 2e_8 \rangle$

The primitive embedding (2) is isomorphic to the primitive embedding (3) by the reflection $R = R_{3e_2+2e_1+4e_3+6e_4+5e_5+4e_6+3e_7+e_8}$, since $R(x) = e_8$, $R(e_i) = e_i$, $1 \leq i \leq 6$.

The primitive embedding (1) is isomorphic to the primitive embedding (2) by the isomorphism $R = R_{e_7} \circ R_{e_7+e_8}$, since $R(e_8) = e_7$ and $R(e_i) = e_i$, $1 \leq i \leq 5$.

□

Lemma 6.2 *Up to isomorphism by an element of the Weyl group, there is a unique primitive embedding of $D_5 \oplus A_1$ into E_7 given by*

$$\langle e_2, e_5, e_4, e_3, e_1 \rangle \oplus \langle e_7 \rangle.$$

Proof By Nishiyama [18], up to isomorphism, the unique primitive embedding of D_5 into E_7 is given by $\langle e_2, e_5, e_4, e_3, e_1 \rangle$. And its orthogonal into E_7 is $\langle e_7 \rangle \oplus \langle -4 \rangle$.

□

Lemma 6.3 *Up to isomorphism by an element of the Weyl group, there is a unique primitive embedding of $D_5 \oplus A_1$ into D_l, $l \geq 7$, given by*

$$\langle d_{l-1}, d_l, d_{l-2}, d_{l-3}, d_{l-4} \rangle \oplus \langle d_{l-6} \rangle.$$

Proof The unique primitive embedding, up to isomorphism, of D_5 into D_l, $l \geq 7$ is $\langle d_{l-1}, d_l, d_{l-2}, d_{l-3}, d_{l-4} \rangle$, its orthogonal in D_l being $\langle x, d_{l-6}, d_{l-5}, \ldots, d_1 \rangle$ with $x = d_{l-1} + d_l + 2(d_{l-2} + d_{l-3} + d_{l-4} + d_{l-5}) + d_{l-6}$.

It is sufficient to prove that the primitive embeddings

- $$\langle d_{l-1}, d_l, d_{l-2}, d_{l-3}, d_{l-4} \rangle \oplus \langle x \rangle$$
- $$\langle d_{l-1}, d_l, d_{l-2}, d_{l-3}, d_{l-4} \rangle \oplus \langle d_{l-6} \rangle$$

are isomorphic.

Let

$$R = R_x \circ R_{d_l+d_{l-2}+d_{l-3}+d_{l-4}+d_{l-5}+d_{l-6}} \circ R_{d_{l-1}+d_{l-2}+d_{l-3}+d_{l-4}+d_{l-5}+d_{l-6}}.$$

Now $R(x) = d_{l-6}$, $R(d_{l-1}) = d_l$, $R(d_l) = d_{l-1}$, $R(d_{l-i}) = d_{l-i}$, $2 \leq i \leq 5$. So R gives the isomorphism. □

□

Proposition 6.3 We get the following results about the orthogonal complements of the previous embeddings. For notations we refer to section 4.

1.
$$(A_1)^{\perp}_{A_n} = L^2_{n-2} = \left(\begin{array}{c|cccc} -2\times 3 & 2 & 0 & \ldots & 0 \\ \hline 2 & & & & \\ 0 & & & & \\ \vdots & & A_{n-2} & & \\ 0 & & & & \end{array}\right)$$

with $\det L^2_{n-2} = 2(n+1)$

2.
$$(A_1)^{\perp}_{D_4} = A_1^{\oplus 3}$$
$$(A_1)^{\perp}_{D_n} = A_1 \oplus D_{n-2}, \quad n \geq 5$$

3.
$$(A_1)^{\perp}_{A_7} = (\langle a_7 \rangle)^{\perp}_{A_7} = \langle a_7 + 2a_6, a_5, a_4, a_3, a_2, a_1 \rangle$$
$$\alpha_7 \in (A_1)^{\perp}_{A_7^*} \quad \text{but} \quad k\alpha_7 \notin ((A_1)^{\perp}_{A_7})_{\text{root}} = A_5 \text{ for all } k$$

4.
$$(A_1)^{\perp}_{A_9} = (\langle a_9 \rangle)^{\perp}_{A_9} = \langle a_9 + 2a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1 \rangle$$
$$\alpha_9 \in (A_1)^{\perp}_{A_9^*} \quad \text{but} \quad k\alpha_9 \notin ((A_1)^{\perp}_{A_9})_{\text{root}} = A_7 \text{ for all } k$$

5.
$$(A_1)^{\perp}_{A_{11}} = (\langle a_{11} \rangle)^{\perp}_{A_{11}} = \langle a_{11} + 2a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1 \rangle$$
$$\alpha_{11} \in (A_1)^{\perp}_{A_{11}^*} \quad \text{but} \quad k\alpha_{11} \notin ((A_1)^{\perp}_{A_{11}})_{\text{root}} = A_9 \text{ for all } k$$

6.
$$(A_1)^{\perp}_{D_6} = \langle d_5 \rangle \oplus \langle d_5 + d_6 + 2d_4 + d_3, d_3, d_2, d_1 \rangle = A_1 \oplus D_4$$
$$\overline{\delta}_6 \quad \text{and} \quad \tilde{\delta}_6 \in (A_1)^{\perp}_{D_6^*}, \quad \delta_6 \notin (A_1)^{\perp}_{D_6^*}$$

7.
$$(A_1)^{\perp}_{D_7} = \langle d_7 \rangle^{\perp}_{D_7} = \langle d_6 \rangle \oplus \langle d_6 + d_7 + 2d_5 + d_4, d_4, d_3, d_2, d_1 \rangle = A_1 \oplus D_5$$
$$3\delta_7 \in (A_1)^{\perp}_{D_7^*}$$

8.
$$(A_1)^{\perp}_{D_{10}} = \langle d_{10} \rangle^{\perp}_{D_{10}} = \langle d_9 \rangle \oplus \langle d_9 + d_{10} + 2d_8 + d_7, d_7, d_6, d_5, d_4, d_3, d_2, d_1 \rangle = A_1 \oplus D_8$$
$$2\overline{\delta}_{10} \in A_1 \oplus D_8$$

9.
$$(A_1)^{\perp}_{E_6} = \langle e_1 \rangle^{\perp}_{E_6} = \langle e_1 + e_2 + 2e_3 + 2e_4 + e_5, e_6, e_5, e_4, e_2 \rangle = A_5$$
$$3\eta_6 \in A_5$$

10.
$$(A_1)^{\perp}_{E_7} = \langle e_1 + e_2 + 2e_3 + 2e_4 + e_5, e_7, e_6, e_5, e_4, e_2 \rangle = D_6$$
$$2\eta_7 \in (A_1)^{\perp}_{E_7}$$

11.
$$(A_1)^{\perp}_{E_8} = E_7$$

12.
$$(D_5)^{\perp}_{D_l} = D_{l-5}$$
$$D_1 = (-4) \quad D_2 = A_1^{\oplus 2} \quad D_3 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \simeq A_3$$

13.
$$(D_5)^{\perp}_{D_6} = \langle d_5 + d_6 + 2d_4 + 2d_3 + 2d_2 + 2d_1 \rangle = \langle (-4) \rangle$$
$$\delta_6 \quad \text{and} \quad \tilde{\delta}_6 \notin (D_5)^{\perp}_{D_6^*}, \quad \overline{\delta}_6 \in (D_5)^{\perp}_{D_6^*}$$

14.
$$(D_5)^{\perp}_{E_6} = \langle e_2, e_5, e_4, e_3, e_1 \rangle^{\perp}_{E_6} = \langle 3e_2 + 2e_1 + 4e_3 + 6e_4 + 5e_5 + 4e_6 \rangle = \langle (-12) \rangle$$
$$3\eta_6 = (-12)$$

15.

$$\begin{aligned}(D_5)^{\perp}_{E_7} &= \langle 2e_1+2e_2+3e_3+4e_4+3e_5+2e_6+e_7\rangle \\ &\oplus\langle e_2+e_3+2e_4+2e_5+2e_6+2e_7\rangle \\ &= A_1\oplus(-4)\end{aligned}$$

$$\eta_7\in(D_5)^{\perp}_{E_7^*},\quad 2\eta_7\notin A_1$$

16.

$$(D_5)^{\perp}_{E_8}=A_3$$

17.

$$(D_5\oplus A_1)^{\perp}_{D_7}=A_1=(d_6+d_7+2d_5+2d_4+2d_3+2d_2+d_1)$$

18.

$$\begin{aligned}(D_5\oplus A_1)^{\perp}_{D_8} &= \langle d_7+d_8+2d_6+2d_5+2d_4+2d_3+d_2\rangle \\ &\oplus\langle d_7+d_8+2d_6+2d_5+2d_4+2d_3+2d_2+2d_1\rangle \\ &= A_1\oplus(-4)\end{aligned}$$

$$4\delta_8\notin(D_5\oplus A_1)^{\perp}_{D_8},\quad \overline{\delta}_8\notin A_1$$

19.

$$\begin{aligned}(D_5\oplus A_1)^{\perp}_{D_9} &= A_1\oplus A_1\oplus A_1 \\ &= (d_8+d_9+2d_7+2d_6+2d_5+2d_4+d_3) \\ &\quad\oplus(d_8+d_9+2d_7+2d_6+2d_5+2d_4+2d_3+2d_2+d_1)\oplus(d_1)\end{aligned}$$

$$(D_5\oplus A_1)^{\perp}_{D_{10}}=A_1\oplus A_3\qquad (D_5\oplus A_1)^{\perp}_{D_{12}}=A_1\oplus D_5$$
$$(D_5\oplus A_1)^{\perp}_{D_{16}}=A_1\oplus D_9\qquad (D_5\oplus A_1)^{\perp}_{D_{24}}=A_1\oplus D_{17}$$

20.

$$(D_5\oplus A_1)^{\perp}_{E_7}=\langle 3e_2+2e_1+4e_3+6e_4+5e_5+4e_6+2e_7\rangle=\langle(-4)\rangle$$

$$2\eta_7\notin(D_5\oplus A_1)^{\perp}_{E_7}$$

21.

$$\begin{aligned}(D_5\oplus A_1)^{\perp}_{E_8} &= A_1\oplus(-4) \\ &= (3e_2+2e_1+4e_3+6e_4+5e_5+4e_6+3e_7+2e_8) \\ &\quad\oplus(3e_2+2e_1+4e_3+6e_4+5e_5+4e_6+2e_7)\end{aligned}$$

Proof The orthogonal complements are given in Nishiyama [18] and the rest of the proof follows immediately from the various expressions of η_7, η_6, δ_l, $\overline{\delta}_l$, $\tilde{\delta}$ and α_m given in section 4. □

Once the different types of fibrations are known, we get the rank of the Mordell-Weil group by 5.3.2

To determine the torsion part we need to know appropriate generators of L/L_{root}.

6.2 Generators of L/L_{root}. By lemma 5.2, a set of generators can be described in terms of elements of L^*/L. We list in the Table 2 the generators fitting to the corresponding W. We restrict to relevant L_{root} according to proposition 6.1.

For convenience, the generators are given modulo L_{root}.

L_{root}	L/L_{root}
E_8^3	$\langle (0) \rangle$
$E_8 D_{16}$	$\langle \delta_{16} \rangle \simeq \mathbb{Z}/2\mathbb{Z}$
$E_7^2 D_{10}$	$\langle \eta_7^{(1)} + \delta_{10}, \eta_7^{(2)} + \bar{\delta_{10}} \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$
$E_7 A_{17}$	$\langle \eta_7 + 3\alpha_{17} \rangle \simeq \mathbb{Z}/6\mathbb{Z}$
D_8^3	$\langle \delta_8^{(1)} + \bar{\delta}_8^{(2)} + \bar{\delta}_8^{(3)}, \bar{\delta}_8^{(1)} + \delta_8^{(2)} + \bar{\delta}_8^{(3)}, \bar{\delta}_8^{(1)} + \bar{\delta}_8^{(2)} + \delta_8^{(3)} \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^3$
$D_9 A_{15}$	$\langle \delta_9 + 2\alpha_{15} \rangle \simeq \mathbb{Z}/8\mathbb{Z}$
E_6^4	$\langle \eta_6^{(1)} + \eta_6^{(2)} + \eta_6^{(3)}, 2\eta_6^{(1)} + \eta_6^{(3)} + \eta_6^{(4)} \rangle \simeq (\mathbb{Z}/3\mathbb{Z})^2$
$A_{11} E_6 D_7$	$\langle \alpha_{11} + \eta_6 + \delta_7 \rangle \simeq \mathbb{Z}/12\mathbb{Z}$
D_6^4	$\langle \bar{\delta}_6^{(1)} + \bar{\delta}_6^{(4)}, \bar{\delta}_6^{(2)} + \bar{\delta}_6^{(3)}, \delta_6^{(1)} + \bar{\delta}_6^{(3)} + \delta_6^{(4)}, \bar{\delta}_6^{(1)} + \bar{\delta}_6^{(2)} \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^4$
$D_6 A_9^2$	$\langle \delta_6 + 5\alpha_9^{(2)}, \delta_6 + \alpha_9^{(1)} + 2\alpha_9^{(2)} \rangle \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$
$D_5^2 A_7^2$	$\langle \delta_5^{(1)} + \delta_5^{(2)} + 2\alpha_7^{(1)}, \delta_5^{(1)} + 2\delta_5^{(2)} + \alpha_7^{(1)} + \alpha_7^{(2)} \rangle \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$

Table 2: A set of generators of L/L_{root}

Theorem 6.1 *There are* 30 *elliptic fibrations with section, unique up to isomorphism, on the elliptic surface* Y_2. *They are listed with the rank and torsion of their Mordell-Weil groups on Table 3.*

Proof If the rank is 0, we apply lemma 5.4 (1) and (2) to determine the torsion part of the Mordell-Weil group. Thus we recover the 14 fibrations of rank 0 exhibited by Shimada and Zhang [24].

For the other 16 fibrations we apply proposition 6.3 and lemma 5.4 (3). Recall that $\det W = 8$ and the torsion group is $\overline{W}_{\text{root}}/W_{\text{root}}$.

6.3 $L_{\textbf{root}} = E_8 D_{16}$**.**

$$L/L_{\text{root}} = \langle \delta_{16} + L_{\text{root}} \rangle \simeq \mathbb{Z}/2\mathbb{Z}$$

6.3.1 *Fibration* $A_1 D_{16}$. It is obtained from the primitive embedding $D_5 \oplus A_1 \subset E_8$. Since by proposition 6.3 21 $(D_5 \oplus A_1)^{\perp}_{E_8} = A_1 \oplus (-4)$, $\det N = 8 \times 4$, so $W/N \simeq \mathbb{Z}/2\mathbb{Z} \simeq L/L_{\text{root}} = \langle \delta_{16} + L_{\text{root}} \rangle$. Since $2\delta_{16} \in D_{16} = W_{\text{root}}$, thus $\delta_{16} \in \overline{W}_{\text{root}}$ and

$$\overline{W}_{\text{root}}/W_{\text{root}} = \mathbb{Z}/2\mathbb{Z}.$$

6.4 $L_{\textbf{root}} = E_7^2 D_{10}$**.**

$$L/L_{\text{root}} = \langle \eta_7^{(1)} + \delta_{10}, \eta_7^{(2)} + \bar{\delta}_{10} \text{ mod. } L_{\text{root}} \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^2$$

6.4.1 *Fibration* $A_1 D_6 D_{10}$. It is obtained from the primitive embeddings $A_1 \subset E_7^{(1)}$ and $D_5 \subset E_7^{(2)}$. Since by proposition 6.3 10 and 15 $(A_1)^{\perp}_{E_7^{(1)}} = D_6$ and $(D_5)^{\perp}_{E_7^{(2)}} = A_1 \oplus (-4)$, we get $\det N = 8 \times 4^2$ and $W/N \simeq (\mathbb{Z}/2\mathbb{Z})^2 \simeq L/L_{\text{root}}$.

L_{root}	L/L_{root}			Fibers	R	Tor.
E_8^3	(0)					
		$A_1 \subset E_8$	$D_5 \subset E_8$	$E_7A_3E_8$	0	(0)
		$A_1 \oplus D_5 \subset E_8$		$A_1E_8E_8$	1	(0)
E_8D_{16}	$\mathbb{Z}/2\mathbb{Z}$					
		$A_1 \subset E_8$	$D_5 \subset D_{16}$	E_7D_{11}	0	(0)
		$A_1 \oplus D_5 \subset E_8$		A_1D_{16}	1	$\mathbb{Z}/2\mathbb{Z}$
		$D_5 \subset E_8$	$A_1 \subset D_{16}$	$A_3A_1D_{14}$	0	$\mathbb{Z}/2\mathbb{Z}$
		$A_1 \oplus D_5 \subset D_{16}$		$E_8A_1D_9$	0	(0)
$E_7^2D_{10}$	$(\mathbb{Z}/2\mathbb{Z})^2$					
		$A_1 \subset E_7$	$D_5 \subset D_{10}$	$E_7D_6D_5$	0	$\mathbb{Z}/2\mathbb{Z}$
		$A_1 \subset E_7$	$D_5 \subset E_7$	$D_6A_1D_{10}$	1	(0)
		$A_1 \oplus D_5 \subset E_7$		E_7D_{10}	1	$\mathbb{Z}/2\mathbb{Z}$
		$A_1 \oplus D_5 \subset D_{10}$		$E_7E_7A_1A_3$	0	$\mathbb{Z}/2\mathbb{Z}$
		$D_5 \subset E_7$	$A_1 \subset D_{10}$	$A_1A_1D_8E_7$	1	$\mathbb{Z}/2\mathbb{Z}$
E_7A_{17}	$\mathbb{Z}/6\mathbb{Z}$					
		$A_1 \oplus D_5 \subset E_7$		A_{17}	1	$\mathbb{Z}/3\mathbb{Z}$
		$D_5 \subset E_7$	$A_1 \subset A_{17}$	A_1A_{15}	2	(0)
D_{24}	$\mathbb{Z}/2\mathbb{Z}$					
		$A_1 \oplus D_5 \subset D_{24}$		A_1D_{17}	0	(0)
D_{12}^2	$(\mathbb{Z}/2\mathbb{Z})^2$					
		$A_1 \subset D_{12}$	$D_5 \subset D_{12}$	$A_1D_{10}D_7$	0	$\mathbb{Z}/2\mathbb{Z}$
		$A_1 \oplus D_5 \subset D_{12}$		$A_1D_5D_{12}$	0	$\mathbb{Z}/2\mathbb{Z}$
D_8^3	$(\mathbb{Z}/2\mathbb{Z})^3$					
		$A_1 \subset D_8$	$D_5 \subset D_8$	$A_1D_6A_3D_8$	0	$(\mathbb{Z}/2)^2$
		$A_1 \oplus D_5 \subset D_8$		$A_1D_8D_8$	1	$\mathbb{Z}/2\mathbb{Z}$
D_9A_{15}	$\mathbb{Z}/8\mathbb{Z}$					
		$A_1 \oplus D_5 \subset D_9$		$A_1A_1A_1A_{15}$	0	$\mathbb{Z}/4\mathbb{Z}$
		$D_5 \subset D_9$	$A_1 \subset A_{15}$	D_4A_{13}	1	(0)
E_6^4	$(\mathbb{Z}/3\mathbb{Z})^2$					
		$A_1 \subset E_6$	$D_5 \subset E_6$	$A_5E_6E_6$	1	$\mathbb{Z}/3\mathbb{Z}$
$A_{11}E_6D_7$	$\mathbb{Z}/12\mathbb{Z}$					
		$A_1 \subset E_6$	$D_5 \subset D_7$	$A_5A_1A_1A_{11}$	0	$\mathbb{Z}/6\mathbb{Z}$
		$A_1 \subset A_{11}$	$D_5 \subset D_7$	$A_9A_1A_1E_6$	1	(0)
		$A_1 \oplus D_5 \subset D_7$		$A_{11}E_6A_1$	0	$\mathbb{Z}/3\mathbb{Z}$
		$A_1 \subset A_{11}$	$D_5 \subset E_6$	A_9D_7	2	(0)
		$D_5 \subset E_6$	$A_1 \subset D_7$	$A_{11}A_1D_5$	1	$\mathbb{Z}/4\mathbb{Z}$
D_6^4	$(\mathbb{Z}/2\mathbb{Z})^4$					
		$A_1 \subset D_6$	$D_5 \subset D_6$	$A_1D_4D_6D_6$	1	$(\mathbb{Z}/2)^2$
$D_6A_9^2$	$\mathbb{Z}/2 \times \mathbb{Z}/10$					
		$D_5 \subset D_6$	$A_1 \subset A_9$	A_7A_9	2	(0)
$D_5^2A_7^2$	$\mathbb{Z}/4 \times \mathbb{Z}/8$					
		$D_5 \subset D_5$	$A_1 \subset D_5$	$A_1A_3A_7A_7$	0	$\mathbb{Z}/8\mathbb{Z}$
		$D_5 \subset D_5$	$A_1 \subset A_7$	$D_5A_5A_7$	1	(0)

Table 3: The elliptic fibrations of Y_2

By proposition 6.3 15, $\eta_7^{(1)} \in (D_5)^{\perp}_{E_7^{(1)*}} = A_1 \oplus (-4)$, but $2\eta_7 \notin A_1$ and by proposition 6.3 10 $2\eta_7^{(1)} \in (A_1)^{\perp}_{E_7^{(1)}} = D_6$. So

$$\overline{W}_{\text{root}}/W_{\text{root}} = \langle \eta_7^{(1)} + \delta_{10} + W_{\text{root}} \rangle \simeq \mathbb{Z}/2\mathbb{Z}.$$

6.4.2 *Fibration* E_7D_{10}. It is obtained from $D_5 \oplus A_1 \subset E_7^{(1)}$. Since by proposition 6.3 20 $(D_5 \oplus A_1)^{\perp}_{E_7} = (-4)$, $\det N = 8 \times 4$ so $W/N \simeq \mathbb{Z}/2\mathbb{Z}$. Again by proposition 6.3 20, $2\eta_7^{(1)} \notin (D_5 \oplus A_1)^{\perp}_{E_7^{(1)}}$ and we get $W/N = \langle \eta_7^{(2)} + \bar{\delta}_{10} + N \rangle$. Since $2\eta_7^{(2)} \in E_7$ and $2\bar{\delta}_{10} \in D_{10}$, it follows

$$\overline{W}_{\text{root}}/W_{\text{root}} \simeq \mathbb{Z}/2\mathbb{Z}.$$

6.4.3 *Fibration* $2A_1D_8E_7$. It is obtained from the primitive embeddings $A_1 \subset D_{10}$ and $D_5 \subset E_7^{(1)}$. By proposition 6.3 8 and 15, we get $(A_1)^{\perp}_{D_{10}} = A_1 \oplus D_8$ and $(D_5)^{\perp}_{E_7^{(1)}} = A_1 \oplus (-4)$, so $\det N = 8 \times 4^2$ and $W/N \simeq (\mathbb{Z}/2\mathbb{Z})^2 \simeq L/L_{\text{root}}$.

By proposition 6.3 15, $2\eta_7^{(1)} \notin A_1$ and by proposition 6.3 8 $2\bar{\delta}_{10} \in A_1 \oplus D_8$ so

$$\overline{W}_{\text{root}}/W_{\text{root}} \simeq \mathbb{Z}/2\mathbb{Z}.$$

6.5 $L_{\textbf{root}} = E_7A_{17}$**.**

$$L/L_{\text{root}} = \langle \eta_7 + 3\alpha_{17} + L_{\text{root}} \rangle \simeq \mathbb{Z}/6\mathbb{Z}$$

6.5.1 *Fibration* A_{17}. It is obtained from the primitive embedding $D_5 \oplus A_1 \subset E_7$.

By proposition 6.3 20, $(D_5 \oplus A_1)^{\perp}_{E_7} = (-4)$, so $\det N = 8 \times 9$ and $W/N \simeq \mathbb{Z}/3\mathbb{Z} = \langle 6\alpha_{17} + N \rangle$. Moreover, since $18\alpha_{17} \in A_{17}$, $6\alpha_{17} \in \overline{W}_{\text{root}}$ so

$$\overline{W}_{\text{root}}/W_{\text{root}} \simeq \mathbb{Z}/3\mathbb{Z}.$$

6.5.2 *Fibration* A_1A_{15}. It is obtained from the primitive embeddings $D_5 \subset E_7$ and $A_1 \subset A_{17}$. By proposition 6.3 15 and 1, $(D_5)^{\perp}_{E_7} = A_1 \oplus (-4)$ and $(A_1)^{\perp}_{A_{17}} = L^2_{15}$ with $\det L^2_{15} = 2 \times 18$, so $\det N = 8 \times 6^2$ and $W/N \simeq \mathbb{Z}/6\mathbb{Z} \simeq L/L_{\text{root}}$. But, by lemma 5.2, $W_{\text{root}} = A_1A_{15}$ has no overlattice. Hence

$$\overline{W}_{\text{root}}/W_{\text{root}} \simeq (0).$$

6.6 $L_{\textbf{root}} = D_8^3$**.**

$$\begin{aligned} L/L_{\text{root}} &= \langle \delta_8^{(1)} + \bar{\delta}_8^{(2)} + \bar{\delta}_8^{(3)}, \bar{\delta}_8^{(1)} + \delta_8^{(2)} + \bar{\delta}_8^{(3)}, \bar{\delta}_8^{(1)} + \bar{\delta}_8^{(2)} + \delta_8^{(3)} \rangle \text{ mod. } L_{\text{root}} \\ &\simeq (\mathbb{Z}/2\mathbb{Z})^3 \end{aligned}$$

6.6.1 *Fibration* $A_1D_8D_8$. It comes from the primitive embedding $D_5 \oplus A_1 \subset D_8^{(1)}$. By proposition 6.3 18, $(D_5 \oplus A_1)^{\perp}_{D_8} = A_1 \oplus (-4)$ so $\det N = 8 \times 4^2$ and $W/N \simeq (\mathbb{Z}/2\mathbb{Z})^2$. By proposition 6.3 18, $4\delta_8 \notin (D_5 \oplus A_1)^{\perp}_{D_8}$ so $W/N = \langle \bar{\delta}_8^{(1)} + \delta_8^{(2)} + \bar{\delta}_8^{(3)}, \bar{\delta}_8^{(1)} + \bar{\delta}_8^{(2)} + \delta_8^{(3)} \rangle$. Again by proposition 6.3 18 $\bar{\delta}_8 \notin A_1$, so only $2(\delta_8^{(2)} + \bar{\delta}_8^{(2)} + \bar{\delta}_8^{(3)} + \delta_8^{(3)}) \in W_{\text{root}}$ and

$$\overline{W}_{\text{root}}/W_{\text{root}} \simeq \mathbb{Z}/2\mathbb{Z}.$$

6.7 $L_{\textbf{root}} = D_9A_{15}$**.**

$$L/L_{\text{root}} = \langle \delta_9 + 2\alpha_{15} + L_{\text{root}} \rangle \simeq \mathbb{Z}/8\mathbb{Z}$$

6.7.1 *Fibration* D_4A_{13}. It comes from the primitive embeddings $D_5 \subset D_9$ and $A_1 \subset A_{15}$. By proposition 6.3 12 and 1, $(D_5)^{\perp}_{D_9} = D_4$, $(A_1)^{\perp}_{A_{15}} = L^2_{13}$ with $\det L^2_{13} = 2 \times 16$ so $\det N = 8 \times 16$ and $W/N \simeq \mathbb{Z}/4\mathbb{Z}$. But by lemma 5.2, $W_{\text{root}} = D_4A_{13}$ has no overlattice since $q_{A_{13}}(\alpha_{13}) = \left(-\frac{1}{14}\right)$ and $q_{D_4}(\delta_4) \in \mathbb{Z}$. Hence

$$\overline{W}_{\text{root}}/W_{\text{root}} \simeq (0).$$

6.8 $L_{\text{root}} = E_6^4$.

$$L/L_{\text{root}} = \langle \eta_6^{(1)} + \eta_6^{(2)} + \eta_6^{(3)}, 2\eta_6^{(1)} + \eta_6^{(3)} + \eta_6^{(4)} \rangle \text{ mod.} L_{\text{root}} \simeq (\mathbb{Z}/3\mathbb{Z})^2$$

6.8.1 *Fibration* $A_5E_6E_6$. We can suppose the primitive embeddings $A_1 \subset E_6^{(1)}$ and $D_5 \subset E_6^{(2)}$. By proposition 6.3 9 and 14, $(A_1)^{\perp}_{E_6} = A_5$ and $(D_5)^{\perp}_{E_6} = (-12)$ so $\det N = 8 \times 9^2$ and $W/N = (\mathbb{Z}/3\mathbb{Z})^2$. By proposition 6.3 9, $3\eta_6^{(1)} \in A_5$ so $2\eta_6^{(1)} + \eta_6^{(3)} + \eta_6^{(4)} \in \overline{W}_{\text{root}}$ but $\eta_6^{(1)} + \eta_6^{(2)} + \eta_6^{(3)} \notin \overline{W}_{\text{root}}$ by proposition 6.3 14. Hence

$$\overline{W}_{\text{root}}/W_{\text{root}} \simeq \mathbb{Z}/3\mathbb{Z}.$$

6.9 $L_{\text{root}} = A_{11}E_6D_7$.

$$L/L_{\text{root}} = \langle \alpha_{11} + \eta_6 + \delta_7 + L_{\text{root}} \rangle \simeq \mathbb{Z}/12\mathbb{Z}$$

6.9.1 *Fibration* $A_9A_1A_1E_6$. It follows from the primitive embeddings $A_1 \subset A_{11}$ and $D_5 \subset D_7$. By proposition 6.3 5 and 12, $(A_1)^{\perp}_{A_{11}} = L^2_9$, $\det L^2_9 = 2 \times 12$, $(D_5)^{\perp}_{D_7} = D_2 \simeq A_1^{\oplus 2}$ so $\det N = 8 \times 6^2$ and $W/N \simeq \mathbb{Z}/6\mathbb{Z} = \langle 2\alpha_{11} + 2\eta_6 + 2\delta_7 + N \rangle$. Since $k(2\alpha_{11}) \notin A_9$ by proposition 6.3 5, we get

$$\overline{W}_{\text{root}}/W_{\text{root}} = (0).$$

6.9.2 *Fibration* A_9D_7. By lemma 5.2, it follows that $W_{\text{root}} = A_9D_7$ has no overlattice since $q_{A_9}(\alpha_9) = \left(-\frac{1}{10}\right)$ and $q_{D_7} = \left(-\frac{7}{4}\right)$. Hence

$$\overline{W}_{\text{root}}/W_{\text{root}} = (0).$$

6.9.3 *Fibration* $A_{11}A_1D_5$. It comes from the primitive embeddings $D_5 \subset E_6$ and $A_1 \subset D_7$. By proposition 6.3 14 and 7, $(D_5)^{\perp}_{E_6} = (-12)$ and $(A_1)^{\perp}_{D_7} = A_1 \oplus D_5$, so $\det N = 8 \times 12^2$ and $W/N \simeq \mathbb{Z}/12\mathbb{Z} \simeq L/L_{\text{root}}$. Since $W_{\text{root}} \cap E_6 = \emptyset$, we get also $\overline{W}_{\text{root}} \cap E_6 = \emptyset$. Now $3\alpha_{11} + 3\delta_7 \in \overline{W}_{\text{root}}$ since $3\delta_7 \equiv \tilde{\delta}_7$ and $4(3\alpha_{11} + 3\delta_7) \in W_{\text{root}}$. Hence

$$\overline{W}_{\text{root}}/W_{\text{root}} \simeq \mathbb{Z}/4\mathbb{Z}.$$

6.10 $L_{\text{root}} = D_6^4$.

$$L/L_{\text{root}} = \langle \overline{\delta}_6^{(1)} + \overline{\delta}_6^{(4)}, \overline{\delta}_6^{(2)} + \overline{\delta}_6^{(3)}, \delta_6^{(1)} + \overline{\delta}_6^{(3)} + \delta_6^{(4)}, \overline{\delta}_6^{(1)} + \overline{\delta}_6^{(2)} \rangle \text{ mod. } L_{\text{root}} \simeq (\mathbb{Z}/2\mathbb{Z})^4$$

6.10.1 *Fibration* $A_1D_4D_6D_6$. We assume the primitive embeddings $A_1 \subset D_6^{(1)}$ and $D_5 \subset D_6^{(2)}$. By proposition 6.3 13 and 6, $(A_1)^{\perp}_{D_6} = A_1 \oplus D_4$, $(D_5)^{\perp}_{D_6} = (-4)$, so $\det N = 8 \times 8^2$ and $W/N \simeq (\mathbb{Z}/2\mathbb{Z})^3$. After enumeration of all the elements of L/L_{root}, since by proposition 6.3 13 and 6 $\overline{\delta}_6^{(2)} \in (D_5)^{\perp}_{D_6^*}$ and only $\overline{\delta}_6^{(1)}$ or $\tilde{\delta}_6^{(1)} \in (A_1)^{\perp}_{D_6^*}$, we get

$$W/N = \{\overline{\delta}_6^{(1)} + \overline{\delta}_6^{(4)}, \overline{\delta}_6^{(2)} + \overline{\delta}_6^{(3)}, \overline{\delta}_6^{(1)} + \overline{\delta}_6^{(2)}, \overline{\delta}_6^{(2)} + \overline{\delta}_6^{(4)}, \overline{\delta}_6^{(1)} + \overline{\delta}_6^{(3)}, \overline{\delta}_6^{(3)} + \overline{\delta}_6^{(4)}, \overline{\delta}_6^{(1)} + \overline{\delta}_6^{(2)} + \overline{\delta}_6^{(3)} + \overline{\delta}_6^{(4)}\} \simeq (\mathbb{Z}/2\mathbb{Z})^3$$

As $2\overline{\delta}_6^{(2)} \notin W_{\text{root}}$ and $2\overline{\delta}_6^{(1)} \in A_1 \oplus D_4$, it follows

$$\overline{W}_{\text{root}}/W_{\text{root}} = \{\overline{\delta}_6^{(1)} + \overline{\delta}_6^{(4)}, \overline{\delta}_6^{(1)} + \overline{\delta}_6^{(3)}, \overline{\delta}_6^{(3)} + \overline{\delta}_6^{(4)}, 0\} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

6.11 $L_{\textbf{root}} = D_6 A_9^2$**.**

$$L/L_{\text{root}} = \langle \delta_6 + 5\alpha_9^{(2)}, \delta_6 + \alpha_9^{(1)} + 2\alpha_9^{(2)} \rangle \text{ mod.} L_{\text{root}} \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$$

6.11.1 *Fibration* $A_7 A_9$. We assume the following primitive embeddings $D_5 \subset D_6$ and $A_1 \subset A_9^{(1)}$. By proposition 6.3 13 and 4, $(D_5)^{\perp}_{D_6} = (-4)$, $(A_1)^{\perp}_{A_9} = L_7^2$, $\det L_7^2 = 2 \times 10$ so $\det N = 8 \times 10^2$ and $[W : N] = 10$. Enumerating the elements of L/L_{root} and since $\delta_6 \notin (D_5)^{\perp}_{D_6^*}$ by proposition 6.3 13, we get

$$\begin{aligned} W/N &= \{\alpha_9^{(1)} + 7\alpha_9^{(2)}, 2\alpha_9^{(1)} + 4\alpha_9^{(2)}, 3\alpha_9^{(1)} + \alpha_9^{(2)}, 4\alpha_9^{(1)} + 8\alpha_9^{(2)}, 5\alpha_9^{(1)} + 5\alpha_9^{(2)}, \\ &\quad 6\alpha_9^{(1)} + 2\alpha_9^{(2)}, 7\alpha_9^{(1)} + 9\alpha_9^{(2)}, 8\alpha_9^{(1)} + 6\alpha_9^{(2)}, 9\alpha_9^{(1)} + 3\alpha_9^{(2)}, 0\} \\ &\simeq \mathbb{Z}/10\mathbb{Z} \end{aligned}$$

Since $k\alpha_9^{(1)} \notin A_7$ by proposition 6.3 4, it follows

$$\overline{W}_{\text{root}}/W_{\text{root}} = (0).$$

6.12 $L_{\textbf{root}} = D_5^2 A_7^2$**.**

$$L/L_{\text{root}} = \langle 2\alpha_7^{(1)} + \delta_5^{(1)} + \delta_5^{(2)}, \alpha_7^{(1)} + \alpha_7^{(2)} + \delta_5^{(1)} + 2\delta_5^{(2)} \rangle \text{mod. } L_{\text{root}} \simeq \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$$

6.12.1 *Fibration* $D_5 A_5 A_7$. We assume the primitive embeddings $D_5 \subset D_5^{(2)}$ and $A_1 \subset A_7^{(1)}$. By proposition 6.3 3, $(A_1)^{\perp}_{A_7} = L_5^2$, $\det L_5^2 = 2 \times 8$ so $\det N = 8 \times 8^2$ and $[W : N] = 8$. Now enumerating the elements of L/L_{root} and since $\delta_5^{(2)}$ does not occur in W, we get $W/N = \langle 5\alpha_7^{(1)} + \alpha_7^{(2)} + 3\delta_5^{(1)} + N \rangle$. Since by proposition 6.3 3 $k\alpha_7^{(1)} \notin A_5$, it follows

$$\overline{W}_{\text{root}}/W_{\text{root}} = (0).$$

□

7 Equations of the fibrations

In the next sections we give Weierstrass equations of all the elliptic fibrations. We will use the following proposition ([20] p.559-560 or [23] Prop. 12.10) .

Proposition 7.1 Let X be a $K3$ surface and D an effective divisor on X that has the same type as a singular fiber of an elliptic fibration. Then X admits a unique elliptic fibration with D as a singular fiber. Moreover, any irreducible curve C on X with $D.C = 1$ induces a section of the elliptic fibration.

First we show that one of the fibrations is the modular elliptic surface with base curve the modular curve $X_1(8)$ corresponding to modular group $\Gamma_1(8)$. As we see in the Table 3, it corresponds to the fibration $A_1, A_3, 2A_7$. The Mordell-Weil group is a torsion group of order 8. We draw a graph with the singular fibers $I_2, I_4, 2I_8$ and the 8-torsion sections. Most divisors used in the previous proposition can be drawn on the graph.

From this modular fibration we can easily write a Weierstrass equation of two other fibrations of respective parameter k and v. From the singular fibers of these two fibrations we obtain the divisors of a set of functions on Y_2. These functions generate a group whose horizontal divisors correspond to the 8-torsion sections. These divisors lead to more fibrations.

If X is a $K3$ surface and

$$\pi : X \to C$$

an elliptic fibration, then the curve C is of genus 0 and we define an **elliptic parameter** as a generator of the function field of C. The parameter is not unique but defined up to linear fractional transformations.

From the previous proposition we can obtain equations from the linear system of D. Moreover if we have two effective divisors D_1 and D_2 for the same fibration we can choose an elliptic parameter with divisor $D_1 - D_2$. We give all the details for the fibrations of respective parameter t and ψ.

For each elliptic fibration we will give a Weierstrass model numbered from 1 to 30, generally in the two variables y and x. Parameters are denoted with small latine or greek letters. In most cases we give the change of variables that converts the defining equation into a Weierstrass form. Otherwise we use standard algorithms to obtain a Weierstrass form (see for example [6]). From a Weierstrass equation we get the singular fibers, using [28] for example, thus the corresponding fibration in Table 3; so we know the rank and the torsion of the Mordell-Weil group. If the rank is > 0 we give points and heights of points, which, using the formula of proposition 5.1, generate the Mordell-Weil lattice. Heights are computed with Weierstrass equations as explained in [11]. Alternatively we can compute heights as in [25] and [13].

7.1 Equation of the modular surface associated to the modular group $\Gamma_1(8)$. We start with the elliptic surface

$$X + \frac{1}{X} + Y + \frac{1}{Y} = k.$$

From Beauville's classification [2], we know that it is the modular elliptic surface corresponding to the modular group $\Gamma_1(4) \cap \Gamma_0(8)$. Using the birational transformation

$$X = \frac{-U\,(U-1)}{V} \text{ and } Y = \frac{V}{U-1}$$

with inverse

$$U = -XY \text{ and } V = -Y\,(XY+1)$$

we obtain the Weierstrass equation

$$V^2 - kUV = U(U-1)^2.$$

The point $Q = (U = 1, V = 0)$ is a $4-$torsion point. If we want A with $2A = Q$ to be a rational point, then $k = -s - 1/s + 2$. It follows

$$V^2 + (s + \frac{1}{s} - 2)UV = U(U-1)^2. \qquad (1)$$

and

$$X + \frac{1}{X} + Y + \frac{1}{Y} + s + \frac{1}{s} = 2.$$

The point $A = (U = s, V = -1 + s)$ is of order 8. We obtain easily its multiples

	A	$2A$	$3A$	$4A$	$5A$	$6A$	$7A$
(X,Y)	$(-s,1)$	$(\infty,0)$	$\left(1,\frac{-1}{s}\right)$	$(0,0)$	$\left(\frac{-1}{s},1\right)$	$(0,\infty)$	$(1,-s)$

Thus we get an equation for the modular surface Y_2 associated to the modular group $\Gamma_1(8)$

$$Y_2 : X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} - 2 = 0$$

and the elliptic fibration

$$(X, Y, Z) \mapsto Z = s.$$

Its singular fibers are

$$I_8 \quad (s = 0), \quad I_8 \quad (s = \infty), \quad I_4 \quad (s = 1) \quad I_2 \quad (s = -1) \quad I_1 \quad (s = 3 \pm 2\sqrt{2}).$$

From now on, the expression I_n $(s = 0)$ means a singular fiber of type I_n at $s = 0$.

7.2 Construction of the graph from the modular fibration. At $s = s_0$, we have a singular fiber of type I_{n_0}. We denote $\Theta_{s_0,j}$ with $s_0 \in \{0, \infty, 1, -1\}$, $j \in \{0, \ldots, n_0 - 1\}$ the components of a singular fiber I_{n_0} such that $\Theta_{i,j}.\ \Theta_{k,j} = 0$ if $i \neq k$ and

$$\Theta_{i,j}.\Theta_{i,k} = \begin{cases} 1 \text{ if } |k-j| = 1 \text{ or } |k-j| = n_0 - 1 \\ -2 \text{ if } k = j \\ 0 \text{ otherwise} \end{cases}$$

the dot meaning the intersection product. By definition, the component $\Theta_{k,0}$ intersects the zero section (0). The n_0-gone obtained can be oriented in two ways for $n_0 > 2$. For each s_0 we want to know which component is cut off by the section (A), i.e. the index $j(A, s_0)$ such that $A.\Theta_{s_0,j(A,s_0)} = 1$. For this, we compute the local height for the prime $s - s_0$ with a Weierstrass equation ([11]). Since this height is also equal to $\frac{j(A,s_0)\left(n_{s_0} - j(A,s_0)\right)}{n_{s_0}}$ we can give an orientation to the n_0-gone by choosing $0 \leq j(A, s_0) \leq \frac{n_{s_0}}{2}$. Hence we get the following results: $j(A, 0) = 3$, $j(A, s_0) = 1$ for $s_0 \neq 0$.

For the other torsion-sections (iA) we use the algebraic structure of the Néron model and get $(iA).\Theta_{0,j} = 1$ if $j = 3i \mod 8$, $(iA).\Theta_{0,j} = 0$ if $j \neq 3i$. For $s_0 \in \{\infty, 1, -1\}$ we have $(iA).\Theta_{s_0,j} = 1$ if $i = j \mod n_0$.

Remark 7.1 We can also compute $j(A, s_0)$ explicitly from the Néron model ([15] Theorem 1 and prop. 5 p 96).

Now we can draw the following graph. The vertices are the sections (iA) and the components $\Theta_{s_0,j}$ with $s_0 \in \{0, \infty, 1, -1\}$, $j \in \{0, 1.., n_0\}$. Two vertices B and C are linked by an edge if $B.C = 1$. For simplicity the two vertices $\Theta_{-1,0}$, $\Theta_{-1,1}$ and the edge between them are not represented. The edges joining $\Theta_{-1,0}$ and (jA), j even are suggested by a small segment from (jA), and also edges from $\Theta_{-1,1}$ to (iA), i odd.

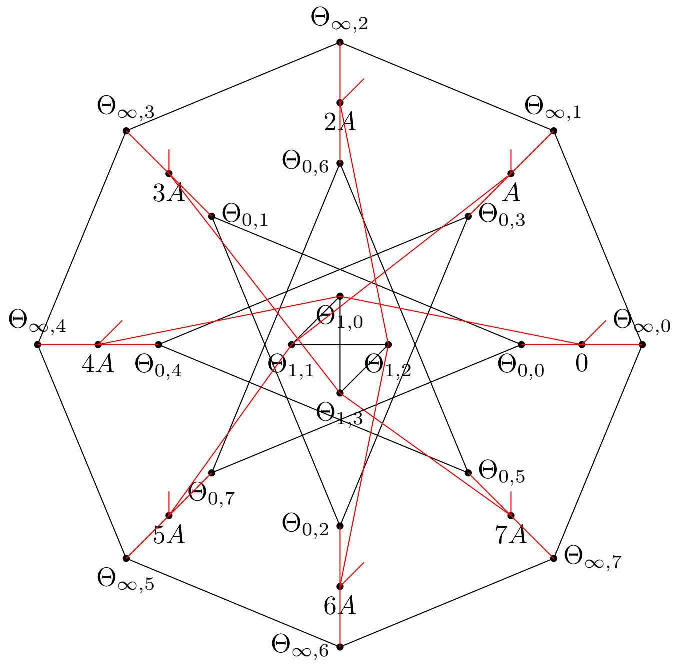


Figure 3: Graph of singular fibers at $s = 0, \infty, 1, -1$ and torsion-sections

7.3 Two fibrations. For the two fibrations to be considered we use the following factorizations of the equation of the surface:

$$(X + Y)(XY + 1)Z + XY(Z - 1)^2 = 0$$

$$(X + ZY)(XZ + Y) + (X + Y)(Y - 1)(X - 1)Z = 0$$

7.3.1 *Fibration of parameter k.* The parameter of the first one is $k = X + Y$. Eliminating for example X, we obtain an equation of degree 2 in Y and Z; easily we have the equation

$$y^2 - x\left(k^2 - 2\,k + 2\right)y = x\,(x - 1)\left(x - k^2\right) \tag{2}$$

with the birational transformation

$$Z = \frac{y}{k\,(x - 1)}, Y = -\frac{yk}{-y + x^2 - x}.$$

The singular fibers of this fibration are

$$I_1^* \quad (k = 0), \quad I_{12} \quad (k = \infty), \quad I_2 \quad (k = 2), \quad I_1 \quad (k = 1), \quad I_1 \quad (k = \pm 2i).$$

The rank of the Mordell-Weil group is one. The point $(x = 1, y = 0)$ is a non-torsion point of height $\frac{4}{3}$; the point $(0, 0)$ is a two-torsion point and (k, k) is of order 4.

7.3.2 *Fibration of parameter v.* This fibration is obtained from the parameter $v = \frac{X+ZY}{Y-1}$. Eliminating Y and using the birational transformation

$$x = \frac{v\,(v+X)\,(-v-1+Z)}{Z}, y = -\frac{v\left(x - v^3 - v^2\right)}{Z}$$

we get the equation

$$y^2 + (v+1)^2 yx - v^2(1+2v)y = (x-v)(x-v^2)(x-v^2-v^3). \qquad (3)$$

The singular fibers of this fibration are

$$I_8 \quad (v=0), \quad I_{10} \quad (v=\infty), \quad I_1 \quad (v=v_0).$$

where v_0 denotes a root of the polynomial $t^6 - 5t^4 + 39t^2 + 2$.

The Mordell-Weil group is of rank two; the two points $(0, v^3), (v, 0)$ are generators of the Mordell-Weil group (the determinant of the heights matrix is $\frac{1}{10}$). The Mordell-Weil torsion-group is 0.

7.4 Divisors. In this section we study the divisors of some functions. Using the elliptic fibration $(X, Y, Z) \mapsto Z = s$ we can compute the horizontal divisor of the following functions. We denote $(f)_h$ the horizontal divisor of f; then we have

$$\begin{array}{llll}
(X)_h & = -(0) - (2A) + (4A) + (6A) & (X+s)_h & = -(0) - (2A) + 2(A) \\
(Y)_h & = -(0) - (6A) + (4A) + (2A) & (Y+s)_h & = -(0) - (6A) + 2(7A) \\
(X-1)_h & = -(0) - (2A) + (3A) + (7A) & (X+\frac{1}{s})_h & = -(0) - (2A) + 2(5A) \\
(Y-1)_h & = -(0) - (6A) + (A) + (5A) & (Y+\frac{1}{s})_h & = -(0) - (6A) + 2(3A) \\
(X+Y)_h & = -(2A) - (6A) + 2(4A) & & \\
(X+sY)_h & = -(0) - (2A) - (6A) + (A) + (3A) + (4A). & &
\end{array}$$

Proposition 7.2 The horizontal divisors of the 7 functions X, Y, $X-1$, $Y-1$, $Y+s$, $X+Y$, $X+sY$ generate the group of principal divisors with support in the 8-torsion sections.

Proof If we write $f_1, f_2, .., f_7$ for these functions, then the determinant of the matrix $(m_{i,j})$ with $m_{i,j} = ord_{iA}(f_j)$, $1 \leq i, j \leq 7$, is equal to 8. □

From this proposition we deduce the corollary used in [3]

Corollary 7.1 *The* 8*-order automorphism* σ_8*, of the surface* Y_2*, leaving invariant the fibration of parameter* Z *and defined by* $M \mapsto M - A$ *on the generic fiber, is given by*

$$\sigma_8 : (X, Y, Z) \mapsto \left(-\frac{Y+XZ}{X+YZ}, \frac{(Y+XZ)\,(1+YZ)}{(X+YZ)\,(X+Y)}, Z\right)$$

Proof The image of X by σ_8 is the unique function of horizontal divisor $(7A) + (5A) - (A) - (3A)$ and equal to 1 at $(4A)$. Using the proposition 7.2 we have $X^{\sigma_8} = \frac{Z(Y-1)(X-1)(X+Y)}{(X+YZ)^2} = -\frac{Y+XZ}{X+YZ}$. A similar argument gives the result for Y. We can notice that

$$\sigma_8^2 : (X, Y, Z) \mapsto (\frac{1}{Y}, X, Z)$$

□

We use the following notations: $Div(f)$ for the divisor of the function f on the surface, $(f)_0$ for the divisor of the zeros of f and $(f)_\infty$ for the divisor of the poles. We get

$$Div(Z)=\sum_{i=0}^7\Theta_{0,i}-\sum_{i=0}^7\Theta_{\infty,i},\quad (Z-1)_0=\sum_{i=0}^3\Theta_{1,i},\quad (Z+1)_0=\sum_{i=0}^1\Theta_{-1,i}.$$

Since X,Y,Z play the same role, the elliptic fibrations $(X,Y,Z)\mapsto X$ and also $(X,Y,Z)\mapsto Y$ have the same property for the singular fibers: two singular fibers of type I_8 for $X=0,\infty$ and $Y=0,\infty$, one singular fiber of type I_4 for $X=1$, $Y=1$. Then we can represent on the graph the divisor of X, drawing two disjoint $8-$gones going throught $(0),(2A)$ and $(4A),(6A)$ and a disjoint $4-$gone throught $(3A),(7A)$. We have

$$\begin{array}{rcl} Div(X) & = & -(0)-\Theta_{\infty,0}-\Theta_{\infty,1}-\Theta_{\infty,2}-(2A)-\Theta_{0,6}-\Theta_{0,7}-\Theta_{0,0} \\ & & +(4A)+\Theta_{\infty,4}+\Theta_{\infty,5}+\Theta_{\infty,6}+(6A)+\Theta_{0,2}+\Theta_{0,3}+\Theta_{0,4} \\ (X-1)_0 & = & (3A)+\Theta_{1,3}+(7A)+\Theta_{-1,1}. \end{array}$$

A similar calculation for Y gives

$$\begin{array}{rcl} Div(Y) & = & -(0)-\Theta_{\infty,0}-\Theta_{\infty,7}-\Theta_{\infty,6}-(6A)-\Theta_{0,2}-\Theta_{0,1}-\Theta_{0,0} \\ & & +(4A)+\Theta_{\infty,4}+\Theta_{\infty,3}+\Theta_{\infty,2}+(2A)+\Theta_{0,6}+\Theta_{0,5}+\Theta_{0,4}\;\; . \\ (Y-1)_0 & = & (A)+\Theta_{1,1}+(5A)+\Theta_{-1,1}. \end{array}$$

The fibration $(X,Y,Z)\mapsto k=X+Y$ has singular fibers of type I_1^*, I_{12} at $k=0$ and $k=\infty$, so we can write the divisor of $X+Y$. By permutation we have also the divisors of $Y+Z$ and $X+Z$

$$\begin{aligned} Div(X+Y) &= -(2A)-\Theta_{\infty,2}-\Theta_{\infty,1}-\Theta_{\infty,0}-\Theta_{\infty,7}-\Theta_{\infty,6} \\ &\quad -(6A)-\Theta_{0,6}-\Theta_{0,7}-\Theta_{0,0}-\Theta_{0,1}-\Theta_{0,2} \\ &\quad +\Theta_{\infty,4}+\Theta_{0,4}+2(4A)+2\Theta_{1,0}+\Theta_{1,1}+\Theta_{1,2} \\ Div(X+Z) &= -(0)-\Theta_{\infty,0}-\Theta_{\infty,7}-\Theta_{\infty,6}-\Theta_{\infty,5}-\Theta_{\infty,4} \\ &\quad -\Theta_{\infty,3}-\Theta_{\infty,2}-(2A)-\Theta_{0,6}-\Theta_{0,7}-\Theta_{0,0} \\ &\quad +\Theta_{1,1}+\Theta_{-1,1}+2(A)+2\Theta_{0,3}+\Theta_{0,2}+\Theta_{0,4} \\ Div(Y+Z) &= -(0)-\Theta_{\infty,0}-\Theta_{\infty,1}-\Theta_{\infty,2}-\Theta_{\infty,3}-\Theta_{\infty,4} \\ &\quad -\Theta_{\infty,5}-\Theta_{\infty,6}-(6A)-\Theta_{0,0}-\Theta_{0,1}-\Theta_{0,2} \\ &\quad +\Theta_{1,3}+\Theta_{-1,1}+2(7A)+2\Theta_{0,5}+\Theta_{0,4}+\Theta_{0,6}. \end{aligned}$$

At last the fibration $(X,Y,Z)\mapsto v=\frac{(X+ZY)}{(Y-1)}$ has two singular fibers of type I_8, I_{12} at $v=0$ and $v=\infty$; thus it follows

$$Div(\frac{X+ZY}{Y-1})=(3A)+\Theta_{1,0}+\Theta_{1,3}+4A+\Theta_{0,4}+\Theta_{0,3}+\Theta_{0,2}+\Theta_{0,1}$$
$$-(2A)-\Theta_{\infty,2}-\Theta_{\infty,1}-\Theta_{\infty,0}-\Theta_{\infty,7}-\Theta_{\infty,6}-\Theta_{\infty,5}-(5A)-\Theta_{0,7}-\Theta_{0,6}.$$

Remark 7.2 We can show that the following twenty elements form a basis of the Néron-Severi group: the eight torsion sections (nA) $0\leq 7$, $\Theta_{\infty,i}$, $1\leq i\leq 7$, $\Theta_{1,j}$ $1\leq j\leq 3$, $\Theta_{-1,1}$, and the fibre. Just compute the Gram matrix using the graph (Figure 3). Its determinant is equal to 8. So, we can recover the divisors of the previous functions by decomposition of others $\Theta_{i,j}$ in this basis.

8 Fibrations from the modular fibration

We give a first set of elliptic fibrations with elliptic parameters belonging to the multiplicative group of functions coming from Proposition 7.2 plus Z and $Z \pm 1$. The first ones come from some easy linear combination of divisors of functions. The others, like t, come from the following remark. We can draw, on Figure 3, two disjoint subgraphs corresponding to singular fibers of the same fibration. We give all the details only in the case of parameter t.

8.0.1 *Fibration of parameter a.* This fibration is obtained with the parameter $a = \frac{Z-1}{X+Y}$. Eliminating Z in the equation and doing the birational transformation

$$\begin{array}{llll} Y & = -\frac{y(1+a)}{x+xa-1} & X & = \frac{x(x+xa-1)}{y(1+a)} \quad \text{with inverse} \\ y & = \frac{Y(XY+XYa+1)}{1+a} & x & = -XY \end{array}$$

we get

$$y^2 - \frac{(x-1)\,y}{(1+a)\,a} = x(x - \frac{1}{1+a})^2. \tag{4}$$

The singular fibers of this fibration are

$$I_8 \quad (a=0), \quad I_1^* \quad (a=\infty), \quad I_6 \quad (a=-1), \quad I_1 \quad (a=a_0),$$

with a_0 root of the ploynomial $16X^3 + 11X^2 - 2X + 1$.

The point $(x = \frac{1}{1+a}, y = 0)$ is of height $\frac{1}{24}$. The torsion-group of the Mordell-Weil group is 0.

8.0.2 *Fibration of parameter d.* This fibration is obtained with parameter $d = XY$ which also is equal to $-x$ in the previous Weierstrass equation. Eliminating X and making the birational transformation

$$y = -(d+1)\,Y\left(d^2 - x\right), x = -ZY d\,(d+1)$$

we get

$$y^2 - 2d\,y\,x = x(x-d^2)(x-d(d+1)^2). \tag{5}$$

The singular fibers are

$$I_2^* \quad (d=0), \quad I_2^* \quad (d=\infty), \quad I_2 \quad (d=1), \quad I_0^* \quad (d=-1).$$

The three points of abscisses $0, d+d^2, d^3+d^2$ are two-torsion points. The point $(d^2, 0)$ is of height 1.

8.0.3 *Fibration of parameter p.* This fibration is obtained with $p = \frac{(XY+1)Z}{X} = \frac{Vs}{U}$ which is also equal to x/d^2 with notation of the previous fibration. We start from the equation in U, V and eliminating V and making the birational transformation

$$s = \frac{xp(p+1)}{y+xp}, U = \frac{x}{p(p+1)}$$

we obtain

$$y^2 = x(x-p)(x-p(p+1)^2). \tag{6}$$

The singular fibers are

$$I_2^* \quad (p=0), \quad I_4^* \quad (p=\infty), \quad I_2 \quad (p=-2), \quad I_4 \quad (p=-1).$$

The Mordell-Weil group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

8.0.4 *Fibration of parameter w.* Using the factorisation of the equation of Y_2

$$(Z+X)(X+Y)(X-1) = X(YZ+X)(X+Y+Z-1)$$

we put $w = X+Y+Z-1 = \frac{(X+Y)(X+Z)(X-1)}{X(YZ+X)}$. Eliminating Z in the equation of Y_2 and doing the birational transformation

$$x = -(1-Y+wY)(1-X+wX), y = -(w-1)Xx$$

we obtain the equation

$$y^2 + w^2(x+1)y = x(x+1)(x+w^2). \tag{7}$$

The singular fibers are

$$I_6 \quad (w=0), \quad I_{12} \quad (w=\infty), \quad I_2 \quad (w=1), \quad I_2 \quad (w=-1), \quad I_1 \quad (w=\pm 2i\sqrt{2}).$$

The Mordell-Weil group is isomorphic to $\mathbb{Z}/6\mathbb{Z}$, generated by $(x=-w^2, y=0)$.

8.0.5 *Fibration of parameter b.* We put $b = \frac{XY}{Z}$. Eliminating Z in the equation of Y_2 and using the birational transformation

$$x = -\frac{b(Y+b)(X+b)}{X}, y = -Yb(x-(b+1)^2)$$

we obtain the equation

$$y^2 + 2(b+1)xy + b^2(b+1)^2y = x(x+b^2)(x-(b+1)^2) \tag{8}$$

or with $z = x+y$

$$z^2 + 2bzx + b^2(b+1)^2z = x^3.$$

The singular fibers are

$$IV^* \quad (b=0), \quad IV^* \quad (b=\infty), \quad I_6 \quad (b=-1), \quad I_1 \quad (b=b_0),$$

with b_0 root of the polynomial $27b^2+46b+27$. The Mordell-Weil group is of rank 1, the point $(x=-b^2, y=0)$ is of height $\frac{4}{3}$ and the torsion-group is of order 3.

8.0.6 *Fibration of parameter r.* Let $r = \frac{(X+Z)(Y+Z)}{ZX}$, r is also equal to x in the fibration of parameter b. Eliminating Y in the equation of Y_2 and doing the birational transformation.

$$x = -\frac{(rX-X-Z)r}{Z}, y = -\frac{X\left(x-r^3\right)x(r-1)}{-r^2+x}$$

we obtain the equation

$$y^2 + 2(r-1)xy = x(x-1)(x-r^3). \tag{9}$$

The singular fibers are

$$I_2^* \quad (r=0), \quad I_6^* \quad (r=\infty), \quad I_2 \quad (r=1), \quad I_1 \quad (r=\pm 2i).$$

The Mordell-Weil group is of rank 1, the point $(1,0)$ is of height 1. The torsion group is of order 2, generated by $(0,0)$.

8.0.7 *Fibration of parameter* e. Let $e=\frac{YX}{(Y+Z)Z}$, e is also equal to $-\frac{x}{r^2}$, where x is from equation (9). Eliminating Y from the equation of Y_2 and doing the birational transformation

$$y=\frac{-\left(2\,e^2+e+x\right)\left(e\left(x-2\,e-1\right)X-x\left(e+1\right)\right)}{\left(2\,e+1\right)\left(e+1\right)}, x=\frac{-e\left(2\,e+1\right)\left(-Ze+X\right)}{X+Z}$$

we obtain the equation

$$y^2=x(x^2-e^2(e-1)x+e^3(2e+1)). \tag{10}$$

The singular fibers are

$$III^* \quad (e=0), \quad I_4^* \quad (e=\infty), \quad I_2 \quad (e=-1), \quad I_2 \quad (e=-\tfrac{1}{2}), \quad I_1 \quad (e=4).$$

The Mordell-Weil group is of rank 1, the point (e^3, e^3+e^4) is of height 1. The torsion group is of order 2, generated by $(0,0)$.

8.0.8 *Fibration of parameter* f. Let $f=\frac{Y(X+Z)^2(Z+Y)}{Z^3X}$, f is also equal to $-x$ where x is from (9). We start from (9) and use the transformation

$$y=\frac{V'}{x(x-1)}, r=-\frac{U'}{x(x-1)}$$

we obtain the equation

$$V'^2-2fV'U'-2f^2(f-1)V'=U'^3+f^4(f-1)^3. \tag{11}$$

The singular fibers are

$$III^* \quad (f=0), \quad II^* \quad (f=\infty), \quad I_4 \quad (f=1), \quad I_1 \quad (f=\tfrac{32}{27}).$$

The Mordell-Weil group is 0.

8.0.9 *Fibration of parameter* g. Let $g=\frac{XY}{Z^2}$. Eliminating Y in the equation of Y_2 and using the birational transformation

$$y=-\frac{\left(g^2-1\right)\left(-gXZ-gZ^2-X^2+gXZ^2\right)g}{Z\left(X+Z\right)^2}, -x=\frac{g\left(g+1\right)\left(Zg+X\right)}{X+Z}$$

we obtain the equation

$$y^2=x^3+4g^2x^2+g^3(g+1)^2x. \tag{12}$$

The singular fibers are

$$III^* \quad (g=0), \quad III^* \quad (g=\infty), \quad I_4 \quad (g=-1), \quad I_2 \quad (g=1).$$

The Mordell-Weil group is of order 2.

8.0.10 *Fibration of parameter* h. Let $h=\frac{(Y+Z)YX^2}{Z^3(X+Z)}$, we can see that $h=\frac{x}{(g+1)}$, with x from (12). We start from (12) and if $y=(g+1)z$ we obtain a quartic equation in z and g with rational points $g=-1, z=\pm 2h$. Using standard transformation we obtain

$$y'^2+(h-\frac{1}{h}-8)x'y'-\frac{96}{h}y'=\left(x'-\frac{1}{4}(h^2+\frac{1}{h^2})+4h+\frac{8}{h}+\frac{1}{2}\right)\left(x'^2-\frac{256}{h}\right)$$

or also

$$y^2=x^3-\frac{25}{3}x-h-\frac{1}{h}-\frac{196}{27}. \tag{13}$$

The singular fibers are

$$II^* \quad (h=0), \quad II^* \quad (h=\infty), \quad I_2 \quad (h=-1), \quad I_1 \quad (h=h_0),$$

where h_0 is a root of the polynomial $27h^2-446h+27$. The Mordell-Weil group is of rank 1 without torsion.

The point $(\frac{1}{16}(h^2+\frac{1}{h^2})+h+\frac{1}{h}+\frac{29}{24}, \frac{1}{64}\,\frac{(h-1)(h+1)\left(h^4+24\,h^3+126\,h^2+24\,h+1\right)}{h^3})$ is of height 4. We recover Elkies' result [9] cited in the introduction.

8.0.11 *Fibration of parameter* t. On the graph (Figure 3), we can see two singular fibers of type I_4^* of a new fibration (Figure 4). They correspond to two divisors D_1 and D_2 with

$$\begin{aligned} D_1 &= \Theta_{\infty,4}+(3A)+2\Theta_{\infty,3}+2\Theta_{\infty,2}+2\Theta_{\infty,1}+2\Theta_{\infty,0}+2(0)+\Theta_{0,0}+\Theta_{1,0} \\ D_2 &= (7A)+\Theta_{0,6}+2\Theta_{0,5}+2\Theta_{0,4}+2\Theta_{0,3}+2\Theta_{0,2}+2(6A)+\Theta_{\infty,6}+\Theta_{1,2}. \end{aligned}$$

We look for a parameter of the new fibration as a function t with divisor D_1-D_2.

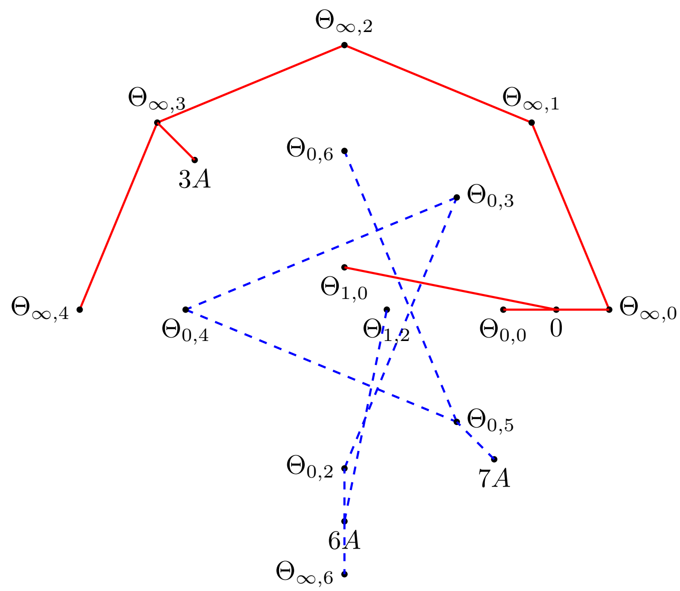


Figure 4: Two singular fiber I_4^*

Let E_s and T_s be the generic fiber and the trivial lattice of the fibration of parameter s. For this fibration we write $D_i=\delta_i+\Delta_i$ with $i=1,2$ and δ_i an horizontal divisor and Δ_i a vertical divisor. More precisely we have $\delta_1=(3A)+2(0)$ and $\delta_2=(7A)+2(6A)$, and the classes of δ_i and D_i are equal $\mathrm{mod}\, T_s$. If $K=\mathbb{C}(s)$ recall the isomorphism: $E_s(K)\sim NS(Y_2)/T_s$. So the class of $\delta_1-\delta_2$ is 0 in $NS(Y_2)/T_s$; thus there is a function $t_0=\frac{X^2(Y+Z)}{(X-1)(X+Y)}$ with divisor $\delta_1-\delta_2$. We can choose for $t=t_0Z^a(Z-1)^b$ where a and b are integer calculated using divisors of section 7.4. We find $a=0, b=1$ and we can take $t=\frac{X^2(Y+Z)(Z-1)}{(X-1)(X+Y)}$. We have also

$$t=-X\frac{Y\left(Z^2-Z+YZ+1\right)Z}{(Z-1)\,(YZ+1)}-\frac{Z^2\,(Y+1)}{(Z-1)\,(YZ+1)}$$

Eliminating X in the equation of Y_2 and then doing the birational transformation

$$Z = \frac{W}{WT+1}, \; Y = \frac{-(WT^2+T+1)}{WT+1}$$

of inverse

$$T = \frac{Y+1}{Z-1}, \; W = \frac{-Z(Z-1)}{YZ+1}$$

we obtain an equation of degree 2 in T. After some classical transformation we get a quartic with a rational point corresponding to $(T=-1+t, W=0)$ and then using a standard transformation we obtain

$$y^2 = x^3 + t(t^2+1+4t)x^2 + t^4 x. \tag{14}$$

The singular fibers are

$$I_4^* \quad (t=0), \quad I_4^* \quad (t=\infty), \quad I_2 \quad (t=-1), \quad I_1 \quad (t=t_0),$$

where t_0 is a root of the polynomial Z^2+6Z+1.

The Mordell-Weil group is of rank one and the point $(-t^3, 2t^4)$ is of height 1. The torsion group is of order 2.

8.0.12 *Fibration of parameter l.* Let $l = \frac{Z(YZ+X)}{X(1+YZ)}$. Eliminating Y in the equation of Y_2 and using the variable $W = \frac{Z+X}{X-1}$, we have an equation of bidegree 2 in W and Z; easily we obtain

$$y^2 - yx - 2l^3 y = (x+l^3)(x+l^2)(x-l+l^3). \tag{15}$$

The singular fibers are

$$I_{10} \quad (l=0), \quad I_3^* \quad (l=\infty), \quad I_1 \quad (l=l_0),$$

with l_0 root of the polynomial $16\,x^5 - 32\,x^4 - 24\,x^3 - 23\,x^2 + 12\,x - 2$. The Mordell-Weil group is of rank 2, without torsion; the two points $(-l^3, 0)$ and $(-l^2, 0)$ are independent and the determinant of the matrix of heights is equal to $\frac{1}{5}$.

9 A second set of fibrations: gluing and breaking

9.1 Classical examples. In the next section we give fibrations obtained using Elkies's methods given in [7] page 11 and explained in [12] Appendix A. If we have two fibrations with fiber F and F' satisfying $F \cdot F' = 2$ the authors explain how can be obtained a parameter from a Weierstrass equation of one fibration. Decomposing F' into vertical and horizontal component, $F' = F'_h + F'_v$ they use F'_h to construct a function on the generic fiber.

9.1.1 *Fibration of parameter o.* Starting with a fibration with two singular fibers of type II^* and the (0) section we obtain a fibration with a singular fiber of type I_{12}^*. Starting from (13) we take x as new parameter. For simplicity, let $o = x + \frac{5}{3}$, we get

$$y^2 = x^3 + (o^3 - 5o^2 + 2)x^2 + x. \tag{16}$$

The singular fibers are

$$I_2 \quad (o=0), \quad I_{12}^* \quad (o=\infty), \quad I_1 \quad (o=1), \quad I_1 \quad (o=5), \quad I_1 \quad (o=o_0),$$

where o_0 is a root of $x^2 - 4x - 4$. The Mordell-Weil group is of rank 1, the torsion group is of order 2.

The point

$$(\frac{1}{16}\,(o-4)^2\,(o-2)^2\,, \frac{1}{64}\,\,(o-4)\,(o-2)\left(o^4 - 4\,o^3 - 20\,o^2 + 96\,o - 80\right))$$

is of height 4.

9.1.2 *Fibration of parameter q.* We start with a fibration with singular fibers of type II^* and III^*, join them with the zero-section and obtain a singular fiber of type I_{10}^* of a new fibration. We transform the equation (11) to obtain

$$y'^2 = x'^3 + (\frac{5}{3} - \frac{2}{f})x' + f + \frac{5}{3f} - \frac{70}{27}.$$

We take x' as the parameter of the new fibration; more precisely, for simplicity, let $q = x' - \frac{1}{3}$. We obtain

$$y^2 = x^3 + (q^3 + q^2 + 2q - 2)x^2 + (1 - 2q)x. \qquad (17)$$

The singular fibers are

$$I_4 \quad (q = 0), \quad I_{10}^* \quad (q = \infty), \quad I_2 \quad (q = 5), \quad I_1 \quad (q = q_0),$$

where q_0 is a root of $X^2 + 2X + 5$.

The Mordell-Weil group is of order 2.

9.2 Fibration with a singular fiber of type I_n, n large. We start with a fibration with two fibers I_n^* and I_m^* and a two-torsion section. Gluing them, we can construct a fibration with a singular fiber of type I_{n+m+8}. The parameter will be $\frac{y}{x}$ in a good model of the first fibration. We can also start from a fibration with two singular fibers of type I_n^* and I_2 and a two-torsion section, join them with the zero-section and obtain a new fibration with a singular fiber of type I_{n+6} or I_{n+7}.

9.2.1 *Fibration of parameter m.* We start from the fibration with parameter t. With the two-torsion section and the two singular fibers of type I_4^*, we can form a singular fiber of of type I_{16} of a new fibration (Figure 5).

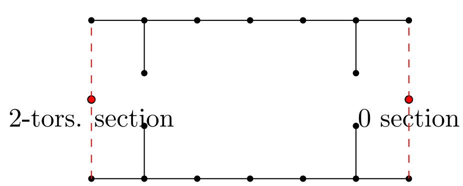


Figure 5: $I_4^*, I_4^* \to I_{16}$

From the equation (14) we get

$$y'^2 = x'^3 + (t + \frac{1}{t} + 4)x'^2 + x'.$$

Let $m = \frac{y'}{x'}$, we obtain

$$y^2 + (m-2)(m+2)yx = x(x-1)^2. \qquad (18)$$

The singular fibers are

$$I_2 \quad (m = 0), \quad I_{16} \quad (m = \infty), \quad I_2 \quad (m = \pm 2), \quad I_1 \quad (m = \pm 2\sqrt{2}).$$

The Mordell-Weil group is cyclic of order 4 generated by $(x = 1, y = 0)$.

9.2.2 *Fibration of parameter n.* With a similar method, we can start from a fibration with two singular fibers of type I_2^* and I_6^*, a two-torsion section and join them to have a fiber of type I_{16}.

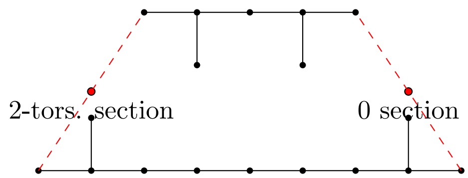


Figure 6: $I_6^*, I_2^* \to I_{16}$

From the equation (9) we obtain

$$y'^2 = x'^3 + (r - 1 + \frac{2}{r})x'^2 - \frac{x'}{r}.$$

Let $n = \frac{y'}{x'}$. The Weierstrass equation is

$$y^2 + (n^2 - 1)yx - y = x^3 - 2x^2. \tag{19}$$

The singular fibers are

$$I_2 \quad (n = 0), \quad I_{16} \quad (n = \infty), \quad I_1 \quad (n = n_0),$$

where n_0 is a root of the polynomial $2x^6 - 9x^4 - 17x^2 + 125$. The Mordell-Weil group is of rank 2. The determinant of the matrix of heights of the two points $(1 \pm t, 0)$ is $\frac{3}{8}$.

9.2.3 *Fibration of parameter j.* Instead of the two-torsion section we can use the section of infinite order $(-t, -2t)$ in the fibration of parameter t.

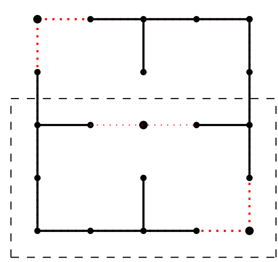


Figure 7: $2I_4^* \to I_{12}$

In (14), let $U' = x + t, V' = y + 2t$ and take the parameter $\frac{V'}{U'}$. For simplification let $j = \frac{V'}{U'} - 2 = \frac{y-2x}{x+t}$; the new fibration obtained has a 3-torsion point which can be put in $(0, 0)$. So it follows

$$y^2 - (j^2 + 4j)x\,y + j^2 y = x^3. \tag{20}$$

The singular fibers are

$$IV^* \quad (j = 0), \quad I_{12} \quad (j = \infty), \quad I_2 \quad (j = -1), \quad I_1 \quad (j = j_0),$$

where j_0 is a root of the polynomial $(x^2 + 10x + 27)$. The Mordell-Weil group is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.

9.2.4 *Fibration of parameter c.* We start from the fibration of parameter o with the equation (16). For $o = 0$ we have a singular fiber of type I_2, the singular point of the bad reduction is $(x = 1, y = 0)$ so we put $x = 1 + u$ and obtain the equation

$$y^2 = u^3 + (-1 - 5o^2 + o^3)u^2 + 2o^2(o - 5)u - o^2(o - 5).$$

The two-torsion section cut the singular fiber I_2 on the zero component, as shown on Figure 8.

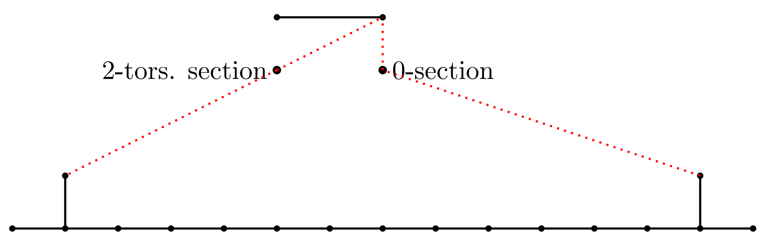


Figure 8: $I_{12}^*, I_2 \to I_{18}$

Let $c = \frac{y}{ox}$ with y, x from the equation (16), we have easily the Weierstrass equation of the fibration. After some translation we can suppose the three-torsion point is $(0, 0)$; we get

$$y^2 + (c^2 + 5)yx + y = x^3. \tag{21}$$

The singular fibers are

$$I_{18} \quad (c = \infty) \quad I_1 \quad (c = c_0),$$

where c_0 is a root of the polynomial $(x^2+2)(x^2+x+7)(x^2-x+7)$. The rank of the Mordell-Weil group is one. The height of $(x = \frac{-1}{4}(c^4 + c^2 + 1), y = \frac{1}{8}(c^2 - c + 1)^3)$ is equal to 4.

9.3 Fibrations with singular fibers of type I_n^*. In this paragraph we obtain new fibrations by gluing two fibers of type I_p^* and I_q^* to obtain, with the zero section, a singular fiber of type I_{p+q+4}^* or I_{p+4}^*.

9.3.1 *Fibration of parameter u.* We start from the fibration with parameter t. With the two singular fibers of type I_4^*, and the 0-section, we can form a singular fiber of type I_8^* of a new fibration (Figure 9).

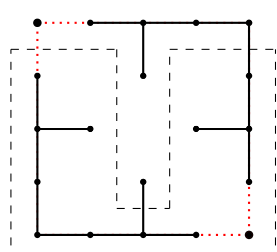


Figure 9: $2I_4^* \to I_8^*$

From the equation (14), it follows

$$\frac{y^2}{x^2t^2} = \frac{x}{t^2} + t + \frac{t^2}{x} + \frac{1}{t} + 4.$$

Taking $u = \frac{x}{t^2} + t$ as the new parameter, we obtain

$$y'^2 = x'^3 + u(u^2 + 4u + 2)x'^2 + u^2x'. \tag{22}$$

The singular fibers are

$$I_1^* \quad (u = 0), \quad I_8^* \quad (u = \infty), \quad I_2 \quad (u = -2), \quad I_1^* \quad (u = -4).$$

The Mordell-Weil group is of order 2.

9.3.2 *Fibration of parameter i.* We start from the fibration of parameter u and from equation (22). With the two singular fibers of type I_8^* and I_1^*, and the 0 section, we can form a singular fiber of type I_{13}^* of a new fibration.We seek for a parameter of the form $\frac{x}{u^2} + ku$, with k chosen to have a quartic equation. We see that $k = 1$ is a good choice so the new parameter is $i = \frac{x}{u^2} + u$ and a Weierstrass equation is

$$y^2 = x^3 + \left(i^3 + 4\,i^2 + 2\,i\right) x^2 + \left(-2\,i^2 - 8\,i - 2\right) x + i + 4. \tag{23}$$

The singular fibers are

$$I_{13}^* \quad (i = \infty), \quad I_2 \quad (i = -\tfrac{5}{2}), \quad I_1^* \quad (i = i_0),$$

where i_0 is a root of the polynomial $4x^3 + 11x^2 - 8x + 16$. The Mordell-Weil group is 0.

9.4 Breaking. In this paragraph we give a fibration obtained by breaking a singular fiber I_{18} and using the three-torsion points as for the fibration of parameter t.

9.4.1 *Fibration of parameter ψ.*

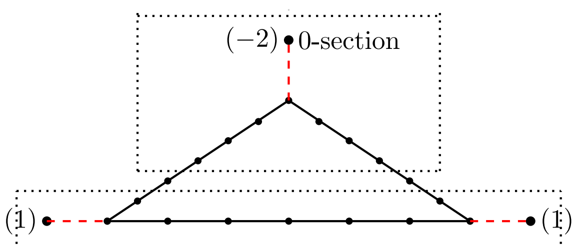


Figure 10: A singular fiber $I_{18} \to III^*, I_6^*$

We start with the fibration (21) of parameter c and 3-torsion sections . We represent the graph of the singular fiber I_{18}, the zero section and two 3-torsion sections (Figure 10). On this graph we can draw two singular fibers III^* and I_6^*. The function x of (21) has the horizontal divisor $-2(0) + P + (-P)$ if P denotes the 3-torsion point and we can take it as the parameter ψ of the new fibration. We get the equation

$$y^2 = x^3 - 5\,x^2\psi^2 - \psi\,x^2 - \psi^5 x. \tag{24}$$

The singular fibers are

$$III^* \quad (\psi = 0), \quad I_6^* \quad (\psi = \infty), \quad I_1 \quad (\psi = -\tfrac{1}{4}), \quad I_1 \quad (\psi = \psi_0),$$

with ψ_0 root of the polynomial $x^2 + 6x + 1$. The Mordell-Weil group is of rank 1. The height of the point

$$(1/4 \left(\psi^2 + 3\,\psi + 1\right)^2, -1/8 \left(\psi^2 + 3\,\psi + 1\right)\left(\psi^4 + 6\,\psi^3 + \psi^2 - 4\,\psi - 1\right))$$

is 4. The torsion-group is of order 2.

10 Last set

From the first set of fibrations we see that not all the components of singular fibers defined on $\mathbb{Q}$ appear on the graph of the Figure 3. We have to introduce some of them to construct easily the last fibrations. For example, we start with the fibration of parameter p (6). Using the Figure 3 we see only 3 components of the singular fiber I_4 for $p = -1$ i.e. $\Theta_{0,1}, \Theta_{0,2}, \Theta_{0,3}$. The fourth is the rational curve named $\Theta_{p,-1,3}$ parametrized by

$$X = 4\,\frac{(w+1)\,w}{1+3\,w^2},\ Y = 1/4\,\frac{1+3\,w^2}{(-1+w)\,w},\ Z = -2\,\frac{(-1+w)\,(w+1)}{1+3\,w^2}.$$

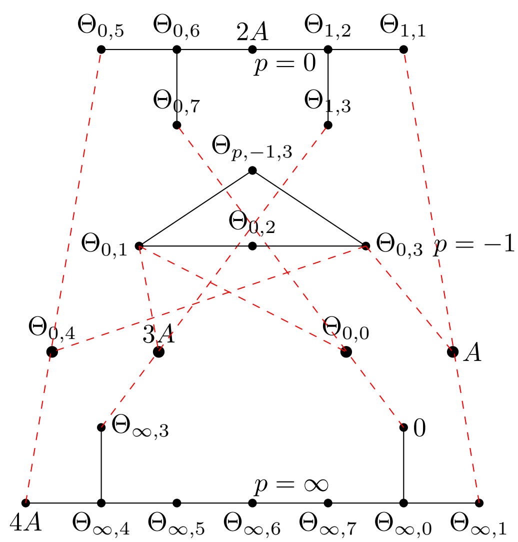


Figure 11: Fibration of parameter p: three singular fibers

The four components $\Theta_{0,0}, \Theta_{0,4}, A, 3A$ are sections of the fibration of parameter p which cut the singular fibers following the previous figure.

10.1 From fibration of parameter p.

10.1.1 *Fibration of parameter δ.* On the Figure 11 we can see the divisor Δ,

$$\Delta = 6\Theta_{\infty,0} + 5\Theta_{\infty,7} + 4\Theta_{\infty,6} + 3\Theta_{\infty,5} + 2\Theta_{\infty,4} + \Theta_{\infty,3} + 3\Theta_{\infty,1} + 4 0 + 2\Theta_{0,0}.$$

The divisor Δ corresponds to a fiber of type II^*. Using the equation (6) and the previous remark, we can calculate the divisors of $p, p+1$ and $Up(p+1)$. The poles of $\delta := Up(p+1)$ give the divisor Δ. Note δ is equal to the x of equation (6). From the zeros of δ we get a fiber of type I_5^*

$$2A + 2\Theta_{1,2} + \Theta_{1,3} + \Theta_{1,1} + 2\sum_{3}^{6}\Theta_{0,i} + \Theta_{0,2} + \Theta_{p,-1,3}.$$

After an easy calculation we get a Weierstrass equation of the fibration

$$y^2 = x^3 + \delta\ (1 + 4\,\delta)\,x^2 + 2\,\delta^4 x + \delta^7. \tag{25}$$

The singular fibers are

$$I_5^* \quad (\delta = 0), \quad II^* \quad (\delta = \infty), \quad I_2 \quad (\delta = -2), \quad I_1 \quad (\delta = -\tfrac{4}{27}).$$

The Mordell-Weil group is equal to 0.

10.1.2 *Fibration of parameter π.* On the previous figure (Figure 11) we can see the singular fiber

$$\Theta_{\infty,1} + \Theta_{\infty,7} + 2\Theta_{\infty,0} + 2(0) + 2\Theta_{0,0} + 2\Theta_{0,7} + 2\Theta_{0,6} + (2A) + 2\Theta_{1,2} + \Theta_{1,1} + \Theta_{1,3}.$$

Using the previous calculation for δ we see that it corresponds to a fibration of parameter $\pi = \frac{U(p+1)}{p}$. The zeros of π correspond to a fiber of type I_3^*. After an easy calculation we have a Weierstrass equation of the fibration

$$y^2 = x^3 + \pi(\pi^2 - 2\pi - 2)x^2 + \pi^2(2\pi + 1)x. \tag{26}$$

The singular fibers are

$$I_3^* \quad (\pi = 0), \quad I_6^* \quad (\pi = \infty), \quad I_2 \quad (\pi = -\tfrac{1}{2}), \quad I_1 \quad (\pi = 4).$$

The Mordell-Weil group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

10.1.3 *Fibration of parameter μ.* From the fibration of parameter p we can also join the I_4^* and I_2^* fibers. Let $\mu = \frac{y}{p(x-p(p+1)^2)}$, with y, x from equation (6). After an easy calculation we obtain a Weierstrass equation of the fibration of parameter μ

$$y^2 + \mu^2(x-1)y = x(x-\mu^2)^2. \tag{27}$$

The singular fibers are

$$IV^* \quad (\mu = 0), \quad I_{10} \quad (\mu = \infty), \quad I_2 \quad (\mu = \pm 1), \quad I_1 \quad (\mu = \mu_0),$$

where μ_0 is a root of the polynomial $2x^2 - 27$. The rank of the Mordell-Weil group is 1, the torsion group is 0. The height of point $(\mu^2, 0)$ is equal to $\frac{1}{15}$.

Remark 10.1 This fibration can also be obtained with the method of the first set and the parameter $\frac{X^2(Y-1)(Z-1)(YZ+X)}{(X-1)(X+Z)(X+Y)}$ or with parameter $\frac{(Us-1)s}{s-1}$.

10.1.4 *Fibration of parameter α.* Let $\alpha = \frac{y}{p(x-p)}$. After an easy calculation we have a Weierstrass equation of the fibration of parameter α

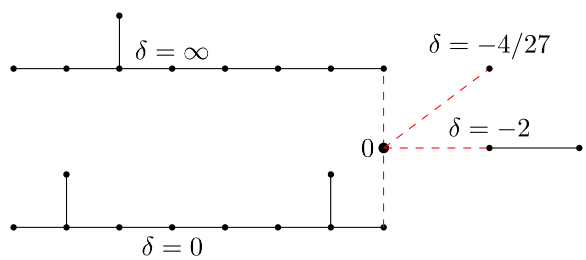
$$y^2 + (\alpha^2 + 2)yx - \alpha^2 y = x^2(x-1). \tag{28}$$

The singular fibers are

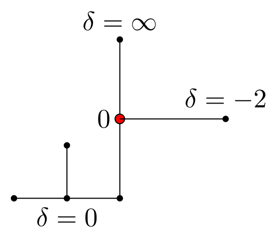
$$I_0^* \quad (\alpha = 0), \quad I_{14} \quad (\alpha = \infty), \quad I_1 \quad (\alpha = \alpha_0),$$

where α_0 is a root of the polynomial $2x^4 + 13x^2 + 64$. The Mordell-Weil group is of rank one, the torsion group is 0. The height of $(0,0)$ is $\frac{1}{7}$.

10.2 From fibration of parameter δ. We redraw the graph of the components of the singular fibers and sections of the fibration of parameter δ, and look for subgraphs.



10.2.1 *Fibration of parameter β.* We can see the subgraph corresponding to a singular fiber of type I_2^*.



To get a parameter β corresponding to this fibration we do the transformation $x = u + \delta^3$ in equation (25) and obtain a new equation

$$y^2 - u^3 - \delta\,(\delta+1)\,(3\,\delta+1)\,u^2 - \delta^4\,(\delta+2)\,(3\,\delta+2)\,u - \delta^7\,(\delta+2)^2\,.$$

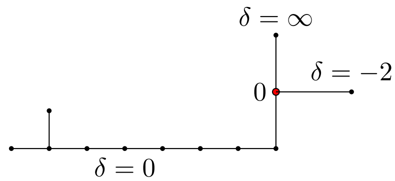
The point $(0,0)$ is singular $\mod \delta$ and $\mod \delta + 2$. By calculation we see that $\beta = \frac{u}{\delta^2(\delta+2)}$ fits. We have a Weierstrass equation

$$y^2 = x^3 + 2\beta^2(\beta-1)x^2 + \beta^3(\beta-1)^2 x. \tag{29}$$

The Mordell-Weil group is of order 2. The singular fibers are

$$III^* \quad (\beta = 0), \quad I_2^* \quad (\beta = \infty), \quad I_1^* \quad (\beta = 1).$$

10.2.2 *Fibration of parameter ϕ.* We can see the subgraph corresponding to a singular fiber of type I_7^*.



As previously, we start with the equation in y, u and seek for a parameter of the form $\phi' = \frac{u}{\delta^2(\delta+2)} + \frac{a'}{\delta}$. We choose a' to get an equation $y^2 = P(u)$ with P of degree ≤ 4; we find $a = \frac{1}{2}$. Let $\phi = \phi' + 1$, a Weierstrass equation is then

$$y^2 = x^3 + 2\,\phi^2\,(4\,\phi - 7)\,x^2 - 4\,\phi^3\left(-3\,\phi + 8\,\phi^2 - 4\right)x + 8\,(3 + 4\,\phi)\,\phi^6. \qquad (30)$$

The singular fibers are

$$III^* \quad (\phi = 0), \quad I_7^* \quad (\phi = \infty), \quad I_1 \quad (\phi = \phi_0),$$

where ϕ_0 is a root of the polynomial $8x^2 - 13x + 16$. The Mordell-Weil group is 0.

The next table gives the correspondence between parameters and elliptic fibrations.

parameter	singular fibers	type of reductible fibers	Rank	Torsion
$1-s$	$2I_8, I_4, I_2, 2I_1$	A_1, A_3, A_7, A_7	0	8
$2-k$	$I_1^*, I_{12}, I_2, 3I_1$	A_{11}, A_1, D_5	1	4
$3-v$	$I_8, I_{10}, 6I_1$	A_7, A_9	2	0
$4-a$	$I_8, I_1^*, I_6, 3I_1$	D_5, A_5, A_7	1	0
$5-d$	$2I_2^*, I_2, I_0^*$	$A_1, D_4, 2D_6$	1	2×2
$6-p$	I_2^*, I_4^*, I_2, I_4	A_1, D_6, A_3, D_8	0	2×2
$7-w$	$I_6, I_{12}, 2I_2, 2I_1$	A_5, A_1, A_1A_{11}	0	6
$8-b$	$2IV^*, I_6, 2I_1$	A_5, E_6, E_6	1	3
$9-r$	$I_6^*, I_2^*, I_2, 2I_1$	D_6, A_1, D_{10}	1	0
$10-e$	$III^*, I_4^*, 2I_2, I_1$	A_1, A_1, D_8, E_7	1	2
$11-f$	III^*, II^*, I_4, I_1	E_7, A_3, E_8	0	0
$12-g$	$2III^*, I_4, I_2$	E_7, E_7, A_1, A_3	0	2
$13-h$	$2II^*, I_2, 2I_1$	A_1, E_8, E_8	1	0
$14-t$	$2I_4^*, I_2, 2I_1$	A_1, D_8, D_8	1	2
$15-l$	$I_{10}, I_3^*, 5I_1$	A_9, D_7	2	0
$16-o$	$I_{12}^*, I_2, 4I_1$	A_1, D_{16}	1	2
$17-q$	$I_{10}^*, I_4, I_2, 2I_1$	A_3, A_1, D_{14}	0	2
$18-m$	$I_{16}, 3I_2, 2I_1$	A_1, A_1, A_1, A_{15}	0	4
$19-n$	$I_{16}, I_2, 6I_1$	A_1, A_{15}	2	0
$20-j$	$IV^*, I_{12}, I_2, 2I_1$	A_{11}, E_6, A_1	0	3
$21-c$	$I_{18}, 6I_1$	A_{17}	1	3
$22-u$	I_8^*, I_1^*, I_2, I_1	A_1, D_5, D_{12}	0	2
$23-i$	$I_{13}^*, I_2, 3I_1$	A_1D_{17}	0	0
$24-\psi$	$III^*, I_6^*, 3I_1$	E_7D_{10}	1	2
$25-\delta$	I_5^*, II^*, I_2, I_1	E_8, A_1D_9	0	0
$26-\pi$	I_3^*, I_6^*, I_2, I_1	A_1, D_{10}, D_7	0	2
$27-\mu$	$IV^*, I_{10}, 2I_2, 2I_1$	A_9, A_1, A_1, E_6	1	0
$28-\alpha$	$I_0^*, I_{14}, 4I_1$	D_4, A_{13}	1	0
$29-\beta$	III^*, I_2^*, I_1^*	E_7, D_6, D_5	0	2
$30-\phi$	$III^*, I_7^*, 2I_1$	E_7, D_{11}	0	0

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