Apéry-Fermi pencil of K3-surfaces and 2-isogenies

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Background

History of the family of Laurent polynomials

$$P_k = X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} - k$$

defining the $K3$-surfaces Y_k.

- First studied by Peters and Stienstra (1989)
  "A pencil of $K3$-surfaces related to Apéry's recurrence for $\zeta(3)$ and Fermi surfaces for potential zero"
  (among all they gave its Picard-Fuchs differential equation),
- and used by Bertin to find explicit Mahler measures of the polynomials (2006, 2010, 2018).
- Bertin and Lecacheux found all the 30 elliptic fibrations of Y_2 (2010).

- Recently (April 2018), Bertin and Lecacheux found all the 27 elliptic fibrations with a Weierstrass equation of the generic member Y_k. Moreover when these fibrations are endowed with a 2-torsion section, they studied the corresponding 2-isogenies.
- Almost at the same time (October 2018), Festi and van Straten published
  "Bhabha Scattering and a special pencil of $K3$-surfaces"
  relating this pencil to quantum field theory and Feynman diagrams.

What's a K3-surface?

- A double covering branched along a plane sextic for example defines a K3- surface X.
  In our case
  $$(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4$$
- There is a unique holomorphic 2-form ω on X up to a scalar.
- $H_2(X, \mathbb{Z})$ is a free group of rank 22.

- With the intersection pairing, $H_2(X,\mathbb{Z})$ is a lattice and

$$H_2(X,\mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

  $\mathcal{L}$ is the $K3$-lattice, U_2 the hyperbolic lattice of rank 2, E_8 the unique even positive definite unimodular lattice of rank 8.

- $$Pic(X) \subset H_2(X,\mathbb{Z}) \simeq Hom(H^2(X,\mathbb{Z}),\mathbb{Z})$$

  where $Pic(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles.

- $$Pic(X) \simeq \mathbb{Z}^{\rho(X)}$$

$$\rho(X) := \text{Picard number of } X$$

$$1 \leq \rho(X) \leq 20$$

- $$T(X) := (Pic(X))^{\perp}$$

  is the transcendental lattice of dimension $22 - \rho(X)$
- If $\{\gamma_1, \cdots, \gamma_{22}\}$ is a $\mathbb{Z}$-basis of $H_2(X, \mathbb{Z})$ and ω the holomorphic 2-form,

  $$\int_{\gamma_i} \omega$$

  is called a period of X and

  $$\int_{\gamma} \omega = 0 \text{ for } \gamma \in Pic(X).$$
- If $\{X_z\}$ is a family of $K3$ surfaces, $z \in \mathbb{P}^1$ with generic Picard number ρ and ω_z the corresponding holomorphic 2-form, then the periods of X_z satisfy a Picard-Fuchs differential equation of order $k = 22 - \rho$. For our family $k = 3$.

- In fact, by Morrison, a $\mathcal{M}$-polarized $K3$-surface, with Picard number 19 has a Shioda-Inose s tructure, that means

$$\begin{array}{ccc} X & & A = E \times E/C_N \\ & \searrow \quad \swarrow & \\ & Y = Kum(A/\pm 1) & \end{array}$$

- If the Picard number $\rho = 20$, then the elliptic curve E is CM.

Properties of the Apéry-Fermi pencil

Affine equations:

$$X + \frac{1}{X} + Y + \frac{1}{Y} + Z + \frac{1}{Z} = k, \quad k \in \mathbb{C}$$

Let $k = s + \frac{1}{s}$, the fibers of the Fermi fibration

$$\pi_s : \mathcal{Z} \to \mathbb{P}^1(s)$$

give elliptic $K3$-surfaces Y_k except $s \notin \{0, \infty, \pm 1, 3 \pm 2\sqrt{2}, -3 \pm 2\sqrt{2}\}$.

$$NS(Y_k) = E_8(-1) \oplus E_8(-1) \oplus U \oplus \langle -12 \rangle$$

$$T(Y_k) = U \oplus \langle 12 \rangle$$

(U is the hyperbolic lattice and E_8 the unique positive definite even unimodular lattice of rank 8)

Hence this family appears as a family of M_6-polarized $K3$-surfaces Y_k with period $t \in \mathcal{H}$. And we deduce from a result of Dolgachev the following property. Let $E_t = \mathbb{C}/\left(\mathbb{Z} + t\mathbb{Z}\right)$ and $E'_t = \mathbb{C}/\left(\mathbb{Z} + (-\frac{1}{6t})\mathbb{Z}\right)$ be the corresponding pair of isogenous elliptic curves. Then there exists a canonical involution τ on Y_k such that $Y_k/(\tau)$ is birationally isomorphic to the Kummer surface $E_t \times E'_t/(\pm 1)$.
This result is linked to the Shioda-Inose structure of $K3$-surfaces with Picard number 19 and 20 described first by Shioda and Inose and exten ded by Morrison.
Since the Picard rank of the generic Y_k is 19 and of its singular members is 20, all these $K3$-surfaces are concerned by the following results.

A Shioda's result

Shioda considers an elliptic $K3$-surface S (with a section) with two II*-fibres and proved that every Shioda-Inose structure can be extended to a sandwich, that is, given a $K3$ surface S, if there exists a unique Kummer surface $K = Km(C_1 \times C_2)$ with two rational maps of degree 2, $S \to K$ and $K \to S$ where C_1 and C_2 are elliptic curves.

Then van Geemen-Sarti, Comparin-Garbagnati, Koike and Schütt, realized sandwich Shioda-Inose structures via elliptic fibrations with 2-torsion sections.

Some observations

- The shioda's "Kummer sandwiching" between a $K3$-surface S and its Kummer K is in fact a 2-isogeny between two elliptic fibrations.
- Bertin and Lecacheux have determined all the elliptic fibrations of Y_2 and found many 2-torsion sections.

So our goal was the following:

- Find all the elliptic fibrations of Y_k
- Select those with a 2-torsion section T: they define an involution ι on Y_k called van Geemen-Sarti involution
- Take the minimal resolution of the quotient Y_k/ι and give its caracterization.

The results

Theorem

(B.-Lecacheux (2018))

- *Either* Y_k/ι *is the Kummer surface* K_k *associated to* Y_k *and given by its Shioda-Inose structure (in that case* ι *is called "Morrison-Nikulin" involution)*
- *Or* $Y_k/\iota = S_k$ *where* S_k *is a* $K3$*-surface with transcendental lattice*

$$T(S_k) = \langle -2 \rangle \oplus \langle 2 \rangle \oplus \langle 6 \rangle$$

- *There exist some genus* 1 *fibrations of* K_k *without section whose Jacobian variety is* S_k.

Remark: S_k is not a Kummer surface but is the Hessian $K3$-surface of a general cubic surface with 3 nodes studied by Dardanelli and van Geemen.

Some ingredients in the proof

- The Kneser-Nishiyama technique: in our case all the elliptic fibrations come from all the primitive embeddings of $A_2 \oplus D_5$ into the Niemeier lattices.
- Elkies's 2-neighbor method to derive Weierstrass equations from one of them.

How it specializes to the singular $K3$-surface Y_2

There are more elliptic fibrations with 2-torsion sections on Y_2 than on the generic Y_k.

Theorem

- *The "Morrison-Nikulin" involutions on Y_k remain "Morrison-Nikulin" involutions on Y_2 sending Y_2 to its Kummer surface K_2.*
- $$S_2 = Y_2$$

  That means that the van Geemen-Sarti involutions on Y_2 which are not Morrison-Nikulin, are symplectic automorphisms of order 2 (self-involutions) or exchange two elliptic fibrations of Y_2.

Remark 1

The specialization of the genus 1 fibrations on K_2 provides an example of a Kummer surface K_2 defined by the product of two isogenous elliptic curves (actually the same elliptic curve of j-invariant equal to 8000), having many fibrations of genus one whose Jacobian Y_2 is not a Kummer surface (see a Keum's similar result but with a Kummer surface defined by non isogenous curves).

Remark 2

The specializations to other singular $K3$-surfaces do not provide in general "self-isogenies", thanks to a result of Boissière, Sarti and Veniani.
In fact we get the following result.

Theorem

Among the singular $K3$-surfaces of the Apéry-Fermi family defined for k rational integer, only Y_2 and Y_{10} possess symplectic automorphisms of order 2 that are "self-isogenies".