# On number fields without a unit primitive element

T. ZAÏMI, M. J. BERTIN and A. M. AL-JOUIEE

**Abstract**

*We characterize number fields without unit primitive element, and we exhibit some families of such fields, with low degree. Also, we prove that a non-cyclotomic totally complex number field $K$, with degree $2d$, where $d$ is odd, and having a unit primitive element, can be generated by a reciprocal integer, if and only if $K$ is non-CM, and the Galois group of the normal closure of $K$ is contained in the hyperoctahedral group $B_d$.*

2010 *Mathematics Subject Classification:* primary 11R06, secondary 11R32.
*Key words and phrases:* Primitive elements, Pisot numbers, Units, CM-fields, Reciprocal integers.

## 1. Introduction

In its simplest form, the primitive element theorem asserts that any algebraic number field (or simply, any number field) $K$ may be generated, over $\mathbb{Q}$, by a single element, i. e., there is a complex number $\theta$ such that the set $\{1, \theta, ..., \theta^{d-1}\}$ is a base of the $\mathbb{Q}$-vector space $K$. Then, the number $\theta$ is called a primitive element (or a generator) of $K$, and the field $K$ is denoted $\mathbb{Q}(\theta)$. Multiplying such an algebraic number $\theta$ by a certain rational integer we easily obtain a primitive element of $K$ which is an algebraic integer. Five years ago, Miller [11] asked, in a "Mathoverflow.net", the following question.

QUESTION 1.1. *Does there exist a primitive element of $K$ which is a unit?*

In response to 1.1, Brooks exhibited some families of bi-quadratic number fields $K$ without unit generator, and Poonen signaled that there is a positive (resp. a negative) answer to 1.1 when $K$ is not a CM-field (resp. when $K$ belongs to a certain class of CM-fields), without giving details. Recall that the field $K$ is said to be a CM-field (we also say that $K$ is CM) if it is a totally non-real quadratic extension of a totally real number field (which is unique), say $R_K$. These responses of Brooks and Poonen are, respectively, contained in Corollary 1.6, and in Theorem 1.4, below. In fact Theorem 1.4 is a corollary of the next result, which is, itself, a complete answer to the following question, related to some special units generating a given number field.

QUESTION 1.2. *Let $K$ be a real (resp. a non-real) number field. Can we find a primitive element of $K$ which is a Pisot (resp. is a complex Pisot) unit?*

A Pisot number is a real algebraic integer greater than 1, whose other conjugates are of modulus less than one, and a complex Pisot number is a non-real algebraic integer with modulus greater than 1 whose other conjugates, except its complex conjugate, are of modulus less than one. Clearly, a positive answer to 1.2 yields a positive response to 1.1, since a Pisot (resp. a complex Pisot) unit is a Pisot (resp. a complex Pisot) number which is a unit.

Throughout, when we speak about conjugates and the degree of an algebraic number or of a number field, without mentioning the basic field, this is meant over $\mathbb{Q}$. Similarly, if the extension $\mathbb{Q} \subset K$ is normal (resp. is cyclic), then we say that $K$ is normal (resp. $K$ is cyclic). Also, we denote, respectively, by $\Omega_K$, $G_K$, $D$ and $d$ the group of roots of unity in $K$, the Galois group of the normal closure of $K$, i.e., the normal closure of the extension $\mathbb{Q} \subset K$, a negative square-free rational integer and a positive rational integer.

It is clear that we have a negative answer to 1.2 when $K = \mathbb{Q}$, or when $K$ is a non-real quadratic field, since the units of $K$, in these two cases, belong to $\Omega_K$. The following theorem collects some known responses to 1.2; the first one is due to Pisot (see for instance [2], [5], [12], and the other may easily be deduced from the results of [3].

THEOREM 1.3. *(i) A real number field, with degree greater than one, is generated by a Pisot unit.*

*(ii) Let $K$ be a non-real number field $K$ satisfying $K \notin \{\mathbb{Q}(i), \mathbb{Q}(i\sqrt{3})\}$. Then, $K$ is generated by a complex Pisot unit, if and only if $K$ is not CM, or $\Omega_K \neq \{\pm 1\}$, or $K = \mathbb{Q}(i\sqrt{\beta})$ for some totally positive Pisot unit $\beta$, where $i^2 = -1$.*

It is worth noting, when $\beta$ is a totally positive number, that $x^2 + \beta$ is the minimal polynomial of $i\sqrt{\beta}$ over $\mathbb{Q}(\beta)$, the conjugates of $i\sqrt{\beta}$ (over $\mathbb{Q}$) are the numbers $\pm i\sqrt{\beta'}$, where $\beta'$ runs through the set of conjugates of $\beta$, $\mathbb{Q}(i\sqrt{\beta})$ is CM and $R_{\mathbb{Q}(i\sqrt{\beta})} = \mathbb{Q}(\beta)$. As mentioned above, the first consequence of Theorem 1.3 is a characterization of number fields, without unit primitive element.

THEOREM 1.4. *The number field $K$ is generated by a unit, if and only if $K$ is not CM, or $K$ is CM and $\Omega_K \neq \{\pm 1\}$, or $\Omega_K = \{\pm 1\}$ and $K$ is a CM-field of the form $\mathbb{Q}(i\sqrt{\beta})$, where $\beta$ is a totally positive Pisot unit $\beta$ (generating $R_K$).*

The following immediate consequence of Theorem 1.4 would clarify some results presented below.

COROLLARY 1.5. *Let $K$ be CM-field satisfying $\Omega_K = \{\pm 1\}$. If all totally positive units generating $R_K$ are squares in $R_K$, then $K$ has no unit primitive element.*

Recall when $K$ is a quartic CM-field, that the real quadratic field $R_K$ contains a unique (fundamental) unit greater than 1, say $f_K$, such that any unit in $R_K$ is of the form $\varepsilon f_K^n$, where $\varepsilon = \pm 1$ and $n \in \mathbb{Z}$. In this case, we deduce from Theorem 1.4 the following consequence.

COROLLARY 1.6. *Let $K$ be a quartic field. Then, $K$ has no unit primitive element, if and only if $K$ is CM, $\{i,\, i\sqrt{3},\, e^{i2\pi/5}\} \cap K = \phi$ and ($f_K$ has norm $-1$, or $f_K$ has norm $1$ and $K \neq \mathbb{Q}(i\sqrt{f_K})$).*

A simple calculation (see also the proof of Corollary 1.6) shows that the field $K = \mathbb{Q}(\sqrt{2}, \sqrt{D})$, where $D < -3$, is a quartic CM-field, $K \cap \{i,\, i\sqrt{3},\, e^{i2\pi/5}\} = \phi$ and $f_K = 1 + \sqrt{2}$. It follows from Corollary 1.6, that $K$ can not be generated by a unit, since $f_K$ has norm $-1$. For the same reason (see for instance [14]) any quartic CM-field $K$ satisfying the conditions $K \cap \{i,\, i\sqrt{3},\, e^{i2\pi/5}\} = \phi$ and $R_K = \mathbb{Q}(\sqrt{a^2+4})$, where $a$ is a positive rational integer, has no unit generator. Similarly, there is no totally complex quadratic extension $K$ of $\mathbb{Q}(\sqrt{6})$, which is generated by a unit, since $f_K = 5 + 4\sqrt{6}$ and $(1 + i\sqrt{3})/2 \in \mathbb{Q}(i\sqrt{5 + 4\sqrt{6}})$. It is also worth noting that Corollary 1.6 yields that the field $\mathbb{Q}(i\sqrt{2+\sqrt{3}}) = \mathbb{Q}(\sqrt{3}, i\sqrt{2})$ is the unique quartic CM-field $K$, having a unit primitive element, and such that $\sqrt{3} \in K$ and $\Omega_K = \{\pm 1\}$, as the norm of $f_K = 2 + \sqrt{3}$ is 1. From this computation another question arises.

QUESTION 1.7. *Let $R$ be a totally real number field. Does there exists exist a CM-field (resp. Do there exist infinitely many CM-fields) $K$ without unit primitive element and satisfying $R_K = R$?*

Clearly, we have a positive answer to 1.7 when $R = \mathbb{Q}$, since each quadratic field $\mathbb{Q}(\sqrt{D})$, where $D \notin \{-1, -3\}$, has no unit generator. Also we may deduce from Corollary 1.6, a positive answer to 1.7, when $R$ is quadratic.

COROLLARY 1.8. *Let $R$ be a real quadratic field. Then there are infinitely many bi-quadratic CM-fields $K$, without unit primitive element and such that $R_K = R$.*

We are unable to answer 1.7 when the degree of $R$ is greater than 2. Using Corollary 1.5, we can prove a positive response to 1.7, in some special cases. In particular, we have the following result.

THEOREM 1.9. *Let $K$ be a CM-field satisfying $\Omega_K = \{\pm 1\}$. Suppose that there is a fundamental set of units of $R_K$, whose elements $f_1, \ldots, f_{d-1}$, are positive, and such that for each embedding $\sigma$ of $R_K$ into $\mathbb{R}$, other than the identity of $K$, there is one and only one element of the set $\{1, \ldots, d-1\}$, say $j_\sigma$, such that $\sigma(f_{j_\sigma}) < 0$. If the correspondence $\sigma \mapsto j_\sigma$ is one-to-one, then $K$ has no unit primitive element.*

A remarkable class of the set of units is formed by reciprocal integers. An algebraic integer $\alpha$ is said to be reciprocal if $1/\alpha$ is a conjugate of $\alpha$. Then, the inverse of each conjugate of $\alpha$ is also a conjugate of $\alpha$, the degree of $\alpha$ is even, except when $\alpha = \pm 1$, and $G_{\mathbb{Q}(\alpha)}$ is a subgroup of the hyperoctahedral group $B_d = \mathbb{Z}/2 \wr S_d$, where $\mathbb{Z}/2$ is the cyclic group with order 2, $S_d$ is the the symmetric group on $d$ letters, and each $\sigma \in G_{\mathbb{Q}(\alpha)}$ is identified, for an appropriate ordering of the conjugates $\alpha_1, ..., \alpha_{2d}$ of $\alpha$, with an element $\widetilde{\sigma}$ of $S_{2d}$, defined by the equalities $\sigma(\alpha_j) = \alpha_{\widetilde{\sigma}(j)}$ $\forall\ j \in \{1, ..., 2d\}$. The following question has been considered by Lalande [10].

QUESTION 1.10. *Let $K$ be a number field, having a unit primitive element, and such that $G_K \subseteq B_d$, where $2d$ is the degree of $K$. Is $K$ generated by a reciprocal integer?*

Here the notation $G_K \subseteq B_d$, means that $G_{\mathbb{Q}(\theta)} \subseteq B_d$ for a certain generator $\theta$ of $K$, with an appropriate ordering of the conjugates of $\theta$. In fact, Lalande [10] obtained a positive answer to 1.10, when the field $K$ has at least one real conjugate (in this case the condition "*having a unit primitive element*" may be removed). To complete this result it remains to consider the case where $K$ is totally complex, i.e., all conjugates of $K$ are non-real. Clearly, we have a positive answer to 1.10, when $K$ is cyclotomic, that is when $K$ is generated by a root of unity, and in particular, for $K$ being a non-real quadratic field, because in this last case $K = \mathbb{Q}((1 + i\sqrt{3})/2)$ or $K = \mathbb{Q}(i)$. The theorem below gives some partial responses to 1.10.

THEOREM 1.11.

*(i) Let $K$ be a non-cyclotomic totally complex number field, having a unit primitive element, with degree $2d$, where $d$ is odd. Then, $K$ is generated by a reciprocal integer if and only if $G_K \subseteq B_d$ and $K$ is non-CM.*

*(ii) Let $K$ be a quartic totally complex field having a unit primitive element. Then, $K$ is generated by a reciprocal integer if and only if $G_K \subseteq B_2$.*

*(iii) Let $K$ be a CM-field satisfying $\Omega_K = \{\pm 1\}$. Then $K$ is generated by a reciprocal integer if and only if there is a totally positive reciprocal integer $\beta$ such that $\mathbb{Q}(\beta) = R_K$ and $K = \mathbb{Q}(i\sqrt{\beta})$.*

Recall that the problem whether a number field may be generated by a non-reciprocal unit $\alpha$ satisfying certain conditions related to the distribution, in the complex plane, of the conjugates of $\alpha$ has been considered by Dubickas in [6].

The proofs of Theorem 1.11 and some related lemmas, presented in the last section, use Theorem 1.3, Kronecker's theorem, *asserting that an algebraic integer is a root of unity when its conjugates belong to the unit circle*, and a well-known characterization of CM-fields, *which says that a non-real number field $K$ is CM, if and only if $K$ is closed under complex conjugation, and each embedding of $K$ into $\mathbb{C}$ commutes with complex conjugation* (see for instance [4]). In the next section we easily deduce Theorem 1.4 from Theorem 1.3. The corollaries, Theorem 1.9, and two auxiliary results, proved in the third section, allow us to obtain families of number fields with degree at most 10, and without unit generator. All computations are done using the systems PARI [1] and SAGE [13].

## 2. Proof of Theorem 1.4

PROOF. The direct implication in the equivalence: *$K$ is not CM, or $\Omega_K \neq \{\pm 1\}$, or $K = \mathbb{Q}(i\sqrt{\beta})$ for some totally positive Pisot unit $\beta \Leftrightarrow K$ is generated by a unit*, follows trivially from Theorem 1.3, since a real or a complex Pisot unit is a unit, and the fields $\mathbb{Q}$, $\mathbb{Q}(i)$ and $\mathbb{Q}(i\sqrt{3})$ are, respectively, generated by the units 1, $i = e^{i2\pi/4}$, and $(-1+i\sqrt{3})/2 = e^{i2\pi/3}$. To prove the converse, suppose that $K$ is generated by a unit $u$, and $K$ is CM (if $K$ is non-CM, then there is no thing to show). Let $u_1, \overline{u_1}, ..., u_d, \overline{u_d}$, be the conjugates of $u$, and let $\theta$ be a Pisot unit generating the real field $R_K$; such an element $\theta$ exits by Theorem 1.3 (i). If $\theta_1, ..., \theta_d$, designate the corresponding conjugates of $\theta$, then the conjugates of the algebraic integer $u\theta^n$, where $n \in \mathbb{N}$, are $u_1\theta_1^n$, $\overline{u_1}\theta_1^n, ..., u_d\theta_d^n$, $\overline{u_d}\theta_d^n$, and the result follows immediately by Theorem 1.3 (ii), since $u\theta^n$ is a complex Pisot unit generating $K$, when $n$ is sufficiently large.$\square$

## 3. Families of number fields without unit generator

3.1. PROOF OF COROLLARY 1.5

PROOF. Let $K$ be a CM-field with $\Omega_K = \{\pm 1\}$ and such that all totally positive units generating $R_K$ are squares in $R_K$. Assuming on the contrary that $K$ is generated by a unit, we have, by Theorem 1.4, that $K = \mathbb{Q}(i\sqrt{\beta})$ for some totally positive (Pisot) unit $\beta$ generating $R_K$. Then, $\sqrt{\beta} \in R_K$, $i = i\sqrt{\beta}/\sqrt{\beta} \in K$ and this last relation leads to a contradiction, since $K$ does not contain non-real roots of unity.$\square$

3.2 PROOF OF COROLLARY 1.6

PROOF. Let $K$ be a quartic field. If $K \neq \mathbb{Q}(u)$ for all units $u \in K$, then we deduce from Theorem 1.4 that $K$ is CM, and $\Omega_K = \{\pm 1\} \Rightarrow i \notin K$, $(-1 + i\sqrt{3})/2 \notin K$ and $e^{i2\pi/5} \notin K$; thus $K \cap \{i, i\sqrt{3}, e^{i2\pi/5}\} = \phi$. Also, the case where $f_K$ has norm 1 and $K = \mathbb{Q}(i\sqrt{f_K})$ can not hold, because $f_K$ is a totally positive unit, and so $i\sqrt{f_K}$ is a unit generating the CM-field $\mathbb{Q}(i\sqrt{f_K})$.

To prove the converse, notice first when $\xi \in \Omega_K$ and $\xi \neq \pm 1$, that a conjugate of $\xi$ belongs to the set $\{e^{i2\pi/3}, e^{i2\pi/4} = i, e^{i2\pi/6}\}$ if $\xi$ is quadratic, or to the set $\{e^{i2\pi/5}, e^{i2\pi/8}, e^{i2\pi/10}, e^{i2\pi/12}\}$, when $\xi$ is quartic. It follows from the equalities $(e^{i2\pi/6})^2 = e^{i2\pi/3} = (-1 + i\sqrt{3})/2$, $(e^{i2\pi/10})^2 = e^{i2\pi/5}$ and $(e^{i2\pi/8})^2 = (e^{i2\pi/12})^3 = i$, that the relation $\{i, i\sqrt{3}, e^{i2\pi/5}\} \cap K = \phi$ implies $\Omega_K = \{\pm 1\}$. Now, assume on the contrary, that the CM-field $K$, satisfying the two conditions: $\{i, i\sqrt{3}, e^{i2\pi/5}\} \cap K = \phi$ and $f_K$ has norm $-1$ (resp. has norm 1 and $K \neq \mathbb{Q}(i\sqrt{f_K})$), is generated by a unit. It follows, by Corollary 1.5, that there is a totally positive unit $\beta$ such that $R_K = \mathbb{Q}(\beta)$, $K = \mathbb{Q}(i\sqrt{\beta})$ and $\sqrt{\beta} \notin R_K$. Hence, $\beta = f_K^{2(l+1)}$ (resp. $\beta = f_K^{2(l+1)}$, or $\beta = f_K^{2l+1}$), for some non-negative rational integer $l$, $\sqrt{\beta} = f_K^{(l+1)} \in R_K$ (resp. $\sqrt{\beta} = f_K^{(l+1)} \in R_K$, or $i\sqrt{\beta} = if_K^l\sqrt{f_K} \in \mathbb{Q}(i\sqrt{f_K}) \Rightarrow K \subseteq \mathbb{Q}(i\sqrt{f_K})$), and this last relation leads to a contradiction.$\square$

3.3. PROOF OF COROLLARY 1.8

PROOF. Set $R = \mathbb{Q}(\sqrt{N})$, where $N$ is a square-free rational integer greater than one. Then, each bi-quadratic field $K$ of the form $R(\sqrt{D}) = \mathbb{Q}(\sqrt{N}, \sqrt{D})$ is a normal CM-field such that $G_K$ is the Klein group, $R_K = R$, and $e^{i2\pi/5} \notin K$, as $\mathbb{Q}(e^{i2\pi/5})$ is cyclic. Letting (for instance) $|D|$ run through the set of prime numbers greater than $\max\{3, N\}$, we see that the quadratic subfields of $K$ are $\mathbb{Q}(\sqrt{N})$, $\mathbb{Q}(\sqrt{D})$ and $\mathbb{Q}(\sqrt{ND})$, and so $\{i,\ i\sqrt{3}\} \cap K = \phi$. Then, the result follows immediately from Corollary 1.6.$\square$

The following interesting lemma, due to the referee, determines whether a totally real number field $R$ is generated by a totally positive unit which is not a square in $R$, and allow us to simplify the proofs of Theorem 1.9 and Proposition 3.3 below. To state this result, set $f_1 = -1$, $\{f_2, ..., f_d\}$ a fundamental set of units of $R$, $\sigma_1, ..., \sigma_d$ the distinct of embeddings of $K$ into $\mathbb{C}, h$ the usual group isomorphism from the multiplicative group $\{-1, 1\}$ into the additive group $\mathbb{F}_2$ (sending 1 to 0, and $-1$ to 1), and

$$M_R = [h(sg(\sigma_j(f_k)))]_{1 \leq j \leq d}^{1 \leq k \leq d}$$

the $d \times d$ matrix, whose $(j, k)$th entry is the image by $h$ of the signature of $\sigma_j(f_k)$.

LEMMA 3.1. *A totally real number field $R$ is generated by a totally positive unit which is not a square in $R$ if and only if* $\det(M_R) = 0$.

PROOF. It is clear that $\det(M_R)$ belongs to the field $\mathbb{F}_2$. If $\det(M_R) = 0$, then a certain sum of the columns of $M_R$ is the zero vector, so that the product of the corresponding product of distinct $f_k\prime s$ is totally positive. On the other hand, if $\det(M_R) = 1$, then no such product is totally positive and so the only totally positive units in $R$ are squares.$\square$

3.4. PROOF OF THEOREM 1.9

PROOF. With the notation above, let $R = R_K$, where $K$ is a CM-field satisfying $\Omega_K = \{\pm 1\}$. By re-ordering, if necessary, the embedding of $R$ into $\mathbb{R}$, we obtain that $M_R$ is a upper triangular matrix having only ones on its diagonal. Hence, $\det(M_R) = 1$ and the result follows immediately by Lemma 3.1 and Corollary 1.5.$\square$

The following consequence of Theorem 1.9 allows us to answer 1.7, in some particular cases, and also to obtain families of sextic cyclic CM-fields without unit generator.

COROLLARY 3.2. *Let $R$ be a normal real number field with odd prime degree. If there is an element of $R$ whose all conjugates, but one, have the same sign and form a fundamental set of units of $R$, then there are infinitely many cyclic CM-fields $K$, of the form $R(\sqrt{D})$, without unit primitive element, and such that $R_K = R$.*

PROOF. Notice first that if $\theta$ generates the totally real number field $R$, then $\theta + \sqrt{D}$ is a primitive element of the field $K := R(\sqrt{D})$. Hence, $K$ is CM, $R_K = R$, and $K$ is normal, as $R$ is so. By considering the element $\sigma$ of $G_K$, which sends $\sqrt{D}$ to $-\sqrt{D}$, and whose restriction to $R$ is a generator of $G_R$ ($R$ is cyclic because its degree $d$ is prime), we obtain from the equivalences: $\sigma^j(\theta) = \theta \Leftrightarrow j \equiv 0 \bmod d$, and $\sigma^j(\sqrt{D}) = \sqrt{D} \Leftrightarrow j \equiv 0 \bmod 2$, where $j \in \mathbb{Z}$, that the order of $\sigma$ is $2d$ and so $K$ is cyclic.

In order to apply Theorem 1.9 we first prove that the equality $\Omega_K = \{\pm 1\}$ holds, when $D$ is sufficiently small. Indeed, let $\zeta \in \Omega_K$ satisfying $\zeta \neq \pm 1$. Then, the degree of $\zeta$ is 2 or $2d$. Since $K$ is cyclotomic when the degree of $\zeta$ is $2d$, and as there are at most a finite number of cyclotomic fields with a given degree, we immediately see that the degree of $\zeta$ is not $2d$, when $D$ is sufficiently small. Notice also that a calculation similar to the one in the proof of Corollary 1.6, gives that the degree of $\zeta$ can not be equal to 2, when $\{i,\ i\sqrt{3}\} \cap K = \phi$. Because the cyclic field $K$ contains one and only one quadratic field, namely $\mathbb{Q}(\sqrt{D})$, the condition $D \notin \{-1, -3\}$ implies immediately $\{i,\ i\sqrt{3}\} \cap K = \phi$ and so the claim is proved. To conclude consider the conjugates, say $u_1, ...,$ $u_d$ of a unit $u$ of $R$, satisfying the second assumption in Corollary 3.1. By replacing, if necessary, $u$ by $-u$, we may re-order these numbers so that $u_1 > 0, ..., u_{d-1} > 0$, and $u_d < 0$. Also, we may suppose, without loss of generality that $\{u_1, ..., u_{d-1}\}$ is a fundamental set of units of $R$. Now, let $\varphi_j$ be the unique automorphism of $R$ sending $u_d$ to $u_j$, where $j \in \{1, 2, ..., d\}$. Then, $G_R = \{\varphi_1, ..., \varphi_d\} = \{\sigma_1, ..., \sigma_d\}$, where $\sigma_j := \varphi_j^{-1} \ \ \forall\ j \in \{1, ..., d\}$, and so for each $\sigma_j$, with $\sigma_j \neq \sigma_d$ ($\sigma_d$ is the identity of $R$), there is one and only one element $j_\sigma$ ($j_\sigma = j$) of the set $\{1, ..., d-1\}$ such that $\sigma_j(u_{j_\sigma}) < 0$, as $\sigma_j(u_j) = u_d$. Then, the results follows immediately from Theorem 1.9, since the above mentioned correspondence between the group $G_R$ and the set $\{1, ..., d\}$ is trivially one-to-one.$\square$

Using Corollary 3.2, one can easily deduce a positive answer to 1.7 when $R$ runs through the following set of normal cubic fields $C_n := \mathbb{Q}(\theta_n)$, defined by the conditions $\theta_n^3 - n\theta_n + n = 0$, $n$ is a square-free positive rational integer, the residue mod 3 of each prime divisor of $n$ is 1, and $4n - 27$ is the square of a rational integer.

This family of normal cubic fields has been investigated by Francisca [7], who has also shown that each field $C_n$ contains a unit, say $u$, such that the set $\{u, u'\}$, where $u'$ is a conjugate of $u$, is a fundamental set of units of $C_n$ and the conjugates of $u$ do not have the same sign; thus each $C_n$ satisfies the condition of Corollary 3.2, and so there are infinitely many CM-fields of the form $C_n(\sqrt{D})$, without unit primitive element. In fact, using the approach of Gras [9] to show

Godwin's conjecture [8], a fundamental set of units of $C_n$ is explicitly given in [7]. The following proposition exhibits two examples of families of CM-fields, having degrees 8 and 10, without unit generator.

PROPOSITION 3.3. *If* $R = \mathbb{Q}(\theta)$, *where* $\theta^4 + \theta^3 - 3\theta^2 - \theta + 1 = 0$ (*resp. where* $\theta^5 + \theta^4 - 4\theta^3 - 3\theta^2 + 3\theta + 1 = 0$), *then there are infinitely many (resp. many cyclic) CM-fields* $K$ *of the form* $R(\sqrt{D})$, *without unit primitive element, and such that* $R = R_K$.

PROOF. Case $R = \mathbb{Q}(\theta)$ and $\theta^5 + \theta^4 - 4\theta^3 - 3\theta^2 + 3\theta + 1 = 0$

Replacing $(x + 1/x)$ by $x$ in the irreducible cyclotomic polynomial $(x^{11} - 1)/(x - 1)$, we see that $R = R_{\mathbb{Q}(e^{i2\pi/11})}$, $R$ is cyclic and the conjugates of $\theta$ are $2\cos 2\pi/11 = 1.68...$, $2\cos 4\pi/11 = 0.83...$, $2\cos 6\pi/11 = -0.28...$, $2\cos 8\pi/11 = -1.30...$ and $2\cos 10\pi/11 = -1.91....$

With the same argument as in the first part of the proof of Corollary 3.2, we obtain that each field $K$ of the form $R(\sqrt{D})$, where $D \notin \{-1, -3, -11\}$, is a cyclic CM-field such that $R = R_K$ and $\Omega_K = \{\pm 1\}$ (recall that $\mathbb{Q}(\sqrt{-11})$ is the unique quadratic subfield of $\mathbb{Q}(e^{i2\pi/11})$). Using SAGE [13], we find that

$$\{f_1(\theta) = \theta^2 - 2,\ f_2(\theta) = \theta^3 - 2\theta,\ f_3(\theta) = \theta^2 + \theta - 1,\ f_4(\theta) = \theta^4 + \theta^3 - 3\theta^2 - 3\theta\}$$

is a fundamental set of units of $R$. Table 1 gives the approximate values of the elements of the corresponding four fundamental units of $R$.

TABLE 1

| $\theta$ | $f_1(\theta)$ | $f_2(\theta)$ | $f_3(\theta)$ | $f_4(\theta)$ |
|---|---|---|---|---|
| $2\cos 2\pi/11$ | 0.83... | 1.39... | 3.51... | −0.76... |
| $2\cos 4\pi/11$ | −1.30... | −1.08... | 0.52... | −3.51... |
| $2\cos 6\pi/11$ | −1.91... | 0.54... | −1.20... | 0.59... |
| $2\cos 8\pi/11$ | −0.28... | 0.37... | −0.59... | −0.52... |
| $2\cos 10\pi/11$ | 1.68... | −3.22... | 0.76... | 1.20... |

Then, we conclude similarly as in the proof of Theorem 1.9, where the corresponding matrix

$$M_R = \begin{matrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{matrix}$$

satisfies $\det(M_R) = 1$.

Case $R = \mathbb{Q}(\theta)$ and $\theta^4 + \theta^3 - 3\theta^2 - \theta + 1 = 0$

In that case $R$ is a totally real quartic field whose Galois group is the Dihedral group $D_4$. Similarly as in the proofs of Corollary 1.8 and of the case above, we obtain that $K := R(\sqrt{D})$ is a CM-field with $\Omega_K = \{\pm 1\}$, for infinitely many $D$. Using SAGE [13] we find that

$$\{f_1(\theta) = \theta,\ f_2(\theta) = \theta^3 + \theta^2 - 2\theta,\ f_3(\theta) = \theta^3 + 2\theta^2 - \theta - 1\}$$

is a fundamental set of units of $R$. The approximative values of these units, together with their conjugates, are given in Table 2.

TABLE 2

| $\theta = f_1(\theta)$ | $f_2(\theta)$ | $f_3(\theta)$ |
|---|---|---|
| 1.35... | 1.61... | 3.81... |
| 0.47... | −0.61... | −0.91... |
| −0.73... | 1.61... | 0.42... |
| −2.09... | −0.61... | 0.67... |

Then, we conclude as in previous case, since

$$M_R = \begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{matrix}$$

and $\det(M_R) = 1$.□

## 4. Proof of Theorem 1.11

To make clear the proof of Theorem 1.11, let us show some auxiliary results. The first one has already been proved in [10], we prefer to give a different simple proof.

LEMMA 4.1. ([10]) *Suppose that the conjugates $\theta_1, \theta_2, ..., \theta_{2d-1}, \theta_{2d}$ of an algebraic number $\theta$ are ordered so that $G_{\mathbb{Q}(\theta)} \subseteq B_d$. Then, $\theta_{2j} \in \mathbb{Q}(\theta_{2j-1})$ $\forall$ $j \in \{1, ..., d\}$.*

PROOF. Let $\theta_k$ be a conjugate, over the field $\mathbb{Q}(\theta_{2j-1})$, of the number $\theta_{2j}$, for some $j$ and $k$. Then, there is an embedding $\sigma$ of $\mathbb{Q}(\theta_{2j-1}, \theta_{2j})$ into $\mathbb{C}$ sending $\theta_{2j-1}$ to $\theta_{2j-1}$ and $\theta_{2j}$ to $\theta_k$ ($\sigma$ is an extension of the identity of $\mathbb{Q}(\theta_{2j-1})$). By considering an element of $G_{\mathbb{Q}(\theta)} \subseteq B_d$ whose restriction to $\mathbb{Q}(\theta_{2j-1}, \theta_{2j})$ is $\sigma$, we see that $\sigma(\theta_{2j-1}) = \theta_{2j-1}$, $\sigma(\theta_{2j}) = \theta_{2j} \Rightarrow \theta_{2j} = \theta_k$, and so the only conjugate of $\theta_{2j}$ over $\mathbb{Q}(\theta_{2j-1})$ is $\theta_{2j}$; thus $\theta_{2j} \in \mathbb{Q}(\theta_{2j-1})$.□

The following result asserts that the Galois group of the normal closure of a CM-field is always contained in the hyperoctahedral group.

LEMMA 4.2. *If $K$ is a CM with degree $2d$, then $G_K \subseteq B_d$.*

PROOF. Let $\theta$ be a primitive element of the CM-field $K$. Suppose that the conjugates $\theta_1, ..., \theta_{2d}$ of $\theta$ are ordered so that $\theta_{2j} = \overline{\theta_{2j-1}}$ $\forall\, j \in \{1, ..., d\}$, and let

$\sigma \in G_{\mathbb{Q}(\theta)}$ satisfying $\sigma(\theta_{2j}) = \theta_{2l}$ (resp. $\sigma(\theta_{2j}) = \theta_{2l-1}$), for some $j$ and $l$. Since the restriction of $\sigma$ to the CM-field $\mathbb{Q}(\theta_{2j})$ commutes with complex conjugation, i. e. $\sigma(\overline{\theta_{2j}}) = \overline{\sigma(\theta_{2j})}$, we see that $\sigma(\theta_{2j-1}) = \theta_{2l-1}$ (resp. that $\sigma(\theta_{2j-1}) = \theta_{2l}$). In a similar manner we obtain when $\sigma(\theta_{2j-1}) = \theta_{2l}$ (resp. $\sigma(\theta_{2j-1}) = \theta_{2l-1}$), that $\sigma(\theta_{2j}) = \theta_{2l-1}$ (resp. $\sigma(\theta_{2j}) = \theta_{2l}$); thus $G_{\mathbb{Q}(\theta)} \subseteq B_d$ and so $G_K \subseteq B_d$.$\square$

The lemma below may be viewed as a converse of Lemma 4.2.

LEMMA 4.3. *Let $K$ be a totally complex number field with degree $2d$ such that $G_{\mathbb{Q}(\theta)} \subseteq B_d$ for some primitive element $\theta$ of $K$, whose conjugates are ordered so that $\theta_{2j} = \overline{\theta_{2j-1}}$ $\forall\, j \in \{1, ..., d\}$. Then, $K$ is a CM-field.*

PROOF. Notice from Lemma 4.1 that $\theta_{2j} \in \mathbb{Q}(\theta_{2j-1})$, and so $K$ is closed for complex conjugation. Set $\theta := \theta_1$, and let $\varphi$ be an embedding of $\mathbb{Q}(\theta)$ into $\mathbb{C}$, sending $\theta$ to some $\theta_{2j}$ (resp. some $\theta_{2j-1}$). By considering an element of $G_{\mathbb{Q}(\theta)}$, whose restriction to $K$ is $\varphi$, we deduce that $\varphi(\overline{\theta}) = \varphi(\theta_2) = \theta_{2j-1} = \overline{\theta_{2j}}$ (resp. $\varphi(\overline{\theta}) = \varphi(\theta_2) = \theta_{2j} = \overline{\theta_{2j-1}}$), $\varphi(\overline{\theta}) = \overline{\varphi(\theta)}$, and so $\varphi$ commutes with complex conjugation. Hence, $K$ is a CM-field.$\square$

The following result is an analogue of Theorem 1.3 (ii), in terms of reciprocal generators.

LEMMA 4.4. *Let $\alpha$ be a reciprocal integer generating a CM-field $K$. If $\alpha \notin \Omega_K$, then the degree of $\alpha$ is a multiple of 4, and there is a totally positive reciprocal integer $\beta \in R_K$ such that $\alpha^2 = \zeta\beta$ and $\zeta \in \Omega_K \diagdown \{1\}$.*

PROOF. Let $\alpha_1 := \alpha$, ..., $\alpha_{2d}$ be the conjugates of $\alpha$, ordered so that $\alpha_{2j} = \overline{\alpha_{2j-1}}$ $\forall\, j \in \{1, ..., d\}$. Then, similarly as in the proof of Theorem 1.3 (ii) (see [3]), we have that the conjugates of $\alpha$, over $R_K$, are $\alpha$ and $\overline{\alpha}$, and so the conjugates of the algebraic integer $\beta := |\alpha|^2 \in R_K$ (over $\mathbb{Q}$) are the numbers $|\alpha_1|^2$, $|\alpha_3|^2$, ..., $|\alpha_{2d-1}|^2$. It follows that there is no $j \in \{1, ..., d\}$ such that $|\alpha_{2j-1}|^2 = 1$, since otherwise all conjugates of $\alpha$ belong to the unit circle and so, by Kronecker's theorem, $\alpha \in \Omega_K$. Hence, if $\alpha_k$ is a conjugate of $\alpha$, then so are all the distinct numbers $\alpha_k$, $\overline{\alpha_k}$, $1/\alpha_k$ and $1/\overline{\alpha_k}$, and consequently $1/\beta$ is a conjugate of $\beta$; thus $\beta$ is a totally positive reciprocal integer (having even degree), the degree of $R_K$ is even, and so the degree of $K$ is a multiple of 4. Moreover, since $\beta$ is a unit and the conjugates of the algebraic integer $\alpha^2/\beta \in K$ are the numbers $\alpha_1^2/\left|\alpha_1\right|^2$, ..., $\alpha_{2d}^2/\left|\alpha_{2d}\right|^2$, which are all of modulus 1, we have again by Kronecker's theorem, that $\alpha^2/\beta \in \Omega_K$ and so $\alpha^2 = \zeta\beta$, for some $\zeta \in \Omega_K \diagdown \{1\}$.$\square$

The lemma below allow us to obtain a complete answer to 1.10 in the quartic case.

LEMMA 4.5. *A quartic CM-field, having a unit primitive element, is generated by a reciprocal integer.*

PROOF. Let $K$ be a quartic CM-field having a unit generator. It is clear when there is a quartic element $\zeta \in \Omega_K$ that the cyclotomic field $K$ is generated by the reciprocal integer $\zeta$. Notice also, when $\Omega_K = \{\pm 1\}$, that Corollary 1.6 yields $K = \mathbb{Q}(i\sqrt{f_K})$, where the fundamental unit $f_K$ of $R_K$ has norm 1. Hence, $1/\ f_K$ is a conjugate of $f_K$, the conjugates of $i\sqrt{f_K}$ are $i\sqrt{f_K}$, $-i\sqrt{f_K}, i/\sqrt{f_K}, -i/\sqrt{f_K} = 1/i\sqrt{f_K}$, and so $i\sqrt{f_K}$ is a reciprocal integer generating $K$. Finally, suppose that there is a quadratic element $\zeta \in \Omega_K$. Then, the minimal polynomial of $\zeta$ over $R_K$ is $(x-\zeta)(x-\overline{\zeta}) \in \mathbb{Z}[x]$, and since the real quadratic number $f_K^2$ is a reciprocal integer generating $R_K$, we deduce that the conjugates of $\zeta f_K^2$ are the four distinct numbers $\zeta f_K^2, \overline{\zeta} f_K^2, \zeta/f_K^2, \overline{\zeta}/f_K^2 = 1/\zeta f_K^2$, and so $\zeta f_K^2$ is a reciprocal integer generating $K$.$\square$

REMARK. We may also deduce from Lemma 4.5 the following answer to 1.1: *A non-normal quartic CM-field has no unit primitive element.* Indeed, let $K$ be a quartic CM-field, and suppose that $K$ has a unit generator. Then, Lemma 4.5 asserts that $K$ is generated by a reciprocal integer, say $\alpha$. If $\overline{\alpha} \neq 1/\alpha$, then $K$ is a normal extension of $\mathbb{Q}$, because the conjugates of $\alpha$ are $\alpha$, $1/\alpha$, $\overline{\alpha}$, $1/\overline{\alpha}$. Otherwise, $\overline{\alpha} = 1/\alpha$, there is a conjugate $\alpha'$ of $\alpha$ such that $\alpha' \notin \{\alpha, 1/\alpha\}$, and so the conjugates of $\alpha$, are $\alpha$, $1/\alpha$, $\alpha'$, $1/\alpha'$. It follows, in this last case, from the relation $\overline{\alpha'} \notin \{\alpha, 1/\alpha, \alpha'\}$, that $\overline{\alpha'} = 1/\alpha'$, the conjugates of $\alpha$ belong to the unit circle ($\alpha$ is a root of unity by Kronecker's theorem) and so the field $K$ is (cyclotomic) normal.

PROOF OF THEOREM 1.11 (i) Let $K$ be a non-cyclotomic totally complex number field, having a unit generator with degree $2d$, where $d$ is odd. The direct implication, in Theorem 1.11, follows trivially from the introduction and the first assertion in Lemma 4.4, because $K$ is not cyclotomic and $2d$ is not a multiple of 4. Notice also, from Lemma 4.2, that the condition $G_K \subseteq B_d$ is always true, when $K$ is CM.

To prove the converse, suppose that $K$ is not CM and $G_K \subseteq B_d$. Let $\theta$ be a primitive element of $K$ such that $G_{\mathbb{Q}(\theta)} \subseteq B_d$, for an appropriate ordering, say $\theta_1, \theta_2, ..., \theta_{2d-1}, \theta_{2d}$, of the conjugates of $\theta$. It follows by Lemma 4.3 that there is $j \in \{1, ..., d\}$ such that $\theta_{2j} \neq \overline{\theta_{2j-1}}$.

To simplify the notation we may suppose, without loss of generality, that $j = 1$, since any conjugate of a reciprocal integer is reciprocal, and each conjugate of $K$ satisfies the same assumptions as $K$. Let $\alpha_1$ be a complex Pisot unit generating $K$; such an element exists from Theorem 1.3 (ii), since $K$ is not CM (this does not also affect the supposition $j := 1$). Let $\alpha_1, \alpha_2, ..., \alpha_{2d-1}, \alpha_{2d}$, be the corresponding conjugates of $\alpha_1$. Then, $\theta_2 \neq \overline{\theta_1} \Rightarrow \alpha_2 \neq \overline{\alpha_1}$, and we have from Lemma 4.1 that $\alpha_{2j} \in \mathbb{Q}(\alpha_{2j-1})\ \ \forall j \in \{1, ..., d\}$. Let $c$ the unique element of $\{3, ..., n\}$ such that $\overline{\alpha_1} = \alpha_c$, and consider the algebraic integer $\eta_1 := \alpha_1/\alpha_2 \in K$. Then, the corresponding conjugates of $\eta_1$ are

$$\eta_1, \eta_2 := \alpha_2/\alpha_1 = 1/\eta_1, ..., \eta_{2d-1} := \alpha_{2d-1}/\alpha_{2d}, \eta_{2d} := \alpha_{2d}/\alpha_{2d-1},$$

and so $\eta_1$ is a reciprocal integer.

Now, we claim that $\eta_1$ is a primitive element of $K$. Indeed, let $s$ be the degree of $\eta_1$ (over $\mathbb{Q}$) and let $t$ be the degree of $\alpha_1$ over $\mathbb{Q}(\eta_1)$. Then, $st = 2d$ and the claim follows immediately when $t = 1$. Now, assume on the contrary that $t \geq 2$. Then, $\eta_1$ is repeated $t$ times in set $\{\eta_1,\ \eta_2, ...,\ \eta_{2d}\}$, and if $\alpha_1/\alpha_2 =$

$\eta_l = \alpha_l/\alpha_v$ for some $l \in \{2, ..., 2d\}$ and $v \in \{l-1, l+1\}$, then $\alpha_1\alpha_v = \alpha_l\alpha_2$, $1 \leq |\alpha_1\alpha_v| = |\alpha_l\alpha_2|$, because $\alpha_1$ is a complex Pisot number, and so $1 < |\alpha_l|$; thus $\alpha_l = \overline{\alpha_1}$, $l = c$ and $t = 2$. Hence, $s = d$ and this last equality leads to a contradiction, because $d$ is odd and the degree $s$ of the reciprocal integer $\eta_1$ is even ($|\eta_1| > |\alpha_1| > 1 \Rightarrow \eta_1 \neq \pm 1$).

(ii) From the introduction we know that the direct implication is true. To prove the converse, consider a primitive element $\theta$ of the totally complex quartic field $K$ and suppose that the conjugates $\theta = \theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ of $\theta$ are ordered so that $G_{\mathbb{Q}(\theta)} \subseteq B_2$. Recall, by Lemma 4.1, that $\theta_2 \in \mathbb{Q}(\theta_1)$ and $\theta_4 \in \mathbb{Q}(\theta_3)$. Notice also that Lemma 4.5 yields that $K$ may be generated by a reciprocal integer, when it is CM. This happens, in particular when $\theta_2 = \overline{\theta_1}$, because in that case $\theta_4 = \overline{\theta_3}$, and so from Lemma 4.3 $K$ is CM. Suppose, now, that $K$ is non-CM (so $\theta_2 \neq \overline{\theta_1}$), and set for instance $\theta_3 = \overline{\theta_1}$. Then, $\theta_4 = \overline{\theta_2}$, and Theorem 1.3 (ii) gives that there is complex Pisot unit $\alpha$ generating $K$. Let $\alpha_1 = \alpha$, $\alpha_2$, $\alpha_3$, $\alpha_4$ be the corresponding conjugates of $\alpha$. Then, $\alpha_2 \in \mathbb{Q}(\theta_2) = \mathbb{Q}(\theta_1) = \mathbb{Q}(\alpha_1)$, $\alpha_3 = \overline{\alpha_1}$, and $\alpha_4 = \overline{\alpha_2} \in \mathbb{Q}(\theta_3) = \mathbb{Q}(\alpha_3)$. Now, we claim, similarly as in the proof of Theorem 1.11 (i), that the reciprocal integer $\eta := \alpha_1/\alpha_2$, whose conjugates are

$$\alpha_1/\alpha_2, \alpha_2/\alpha_1 = 1/\eta, \alpha_3/\alpha_4 = \overline{\eta}, \alpha_4/\alpha_3 = 1/\overline{\eta},$$

is a generator of $K$. Indeed, assuming on the contrary, we have that $\eta$ is quadratic, since $|\alpha_1| > 1 > |\alpha_2|$ and so $|\eta| > 1 > |1/\eta|$. Moreover, $\eta$ is real, as $\eta = \overline{\eta}$. It follows from the relations $\mathbb{Q} \subset \mathbb{Q}(\eta) \subset \mathbb{Q}(\eta)(\alpha) = \mathbb{Q}(\alpha)$, that $\alpha$ is quadratic over $\mathbb{Q}(\eta)$, the minimal polynomial of $\alpha$, over the real field $\mathbb{Q}(\eta)$, is $(x-\alpha)(x-\overline{\alpha}) = (x-\alpha_1)(x-\alpha_3)$, and so the minimal polynomial of $\alpha_2$, over $\mathbb{Q}(\eta)$, is $(x-\alpha_2)(x-\overline{\alpha_2}) = (x-\alpha_2)(x-\alpha_4)$. Hence, $K$ is a totally non-real quadratic extension of the totally real field $\mathbb{Q}(\eta)$, and this last assertion leads to a contradiction, since $K$ is supposed to be non-CM.

(iii) Let $K$ be a CM-field with $\Omega_K = \{\pm 1\}$. Suppose that there is a reciprocal integer $\alpha$ generating $K$. Then, the second assertion in Lemma 4.4 implies that there is a totally positive reciprocal integer $\beta \in R_K$ such that $\alpha^2 = -\beta$. Hence, the degree of $\beta$ is the half of the degree of $\alpha$, $\beta$ is a primitive element of $R_{\mathbb{Q}(\alpha)}$ and $K = \mathbb{Q}(\alpha) = \mathbb{Q}(i\sqrt{\beta})$. The converse follows trivially from the fact that $i\sqrt{\beta}$ is a reciprocal integer generating the CM-field $\mathbb{Q}(i\sqrt{\beta})$, when $\beta$ is a totally positive reciprocal integer.$\square$

**Acknowledgment.** *We thank the referee for his valuable suggestions about the presentation of this manuscript and Lemma 3.1.*

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T. Zaïmi, Department of Mathematics and Statistics, College of Science, Al-Imam Mohammad Ibn Saud Islamic University, P. O. Box 90950, Riyadh 11623 Saudi Arabia

e-mail: tmzaemi@imamu.edu.sa

M. J. Bertin, Université Pierre et Marie Curie (Paris 6), IMJ, 4 Place Jussieu, 75005 Paris, France

e-mail: marie-jose.bertin@imj-prg.fr

A. M. Al-Jouiee, Department of Mathematics and Statistics, College of Science, Al-Imam Mohammad Ibn Saud Islamic University, P. O. Box 90950, Riyadh 11623 Saudi Arabia

e-mail: amjouiee@imamu.edu.sa