ABOUT ISOGENIES BETWEEN SOME $K3$ SURFACES

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Abstract. We study some 2 and 3-isogenies on the singular $K3$ surface Y_{10} of discriminant 72 and belonging to the Apéry-Fermi pencil (Y_k), and find on it many interesting properties. For example some of its elliptic fibrations with 3-torsion section induce by 3-isogeny either an elliptic fibration of Y_2, the unique $K3$ surface of discriminant 8, or an elliptic fibration of other $K3$ surfaces of discriminant 72.

1. Introduction

In the Apéry-Fermi pencil Y_k defined by the equation

$$(Y_k) \qquad X+\frac{1}{X}+Y+\frac{1}{Y}+Z+\frac{1}{Z}=k,$$

two $K3$ surfaces, namely Y_2 and Y_{10} retain our attention. We observe first the relation between their transcendental lattices $T(Y_2)$ and $T(Y_{10})$ (see [1], [2])

$$T(Y_2)=\begin{pmatrix}2&0\\0&4\end{pmatrix} \qquad T(Y_{10})=\begin{pmatrix}6&0\\0&12\end{pmatrix}=T(Y_2)[3].$$

Even more, from results of Kuwata [10] and Shioda [18] (theorem 2.1), the equality $T(Y_{10})=T(Y_2)[3]$ reveals a relation between Y_2 and Y_{10}. Indeed, starting with an elliptic fibration of Y_2 with two singular fibers II^* and Weierstrass equation

$$(Y_2)_h \qquad y^2=x^3+\alpha x+h+\frac{1}{h}+\beta,$$

a base change $h=u^3$ gives a Weierstrass equation denoted $(Y_2)_h^{(3)}$ of an elliptic fibration of a $K3$ surface with transcendental lattice $T(Y_2)[3]$, which is precisely Y_{10}. If, instead of the previous base change, we use the base change $h=u^2$, we obtain a Weierstrass equation $(Y_2)_h^{(2)}$ of an elliptic fibration of a $K3$ surface with transcendental lattice $T(Y_2)[2]$ which is the Kummer surface K_2. The idea, previously developped in [3] when searching 2-isogenies between some elliptic fibrations of Y_2 and its Kummer K_2, suggests possible 3-isogenies between some elliptic fibrations of Y_2 and Y_{10}. Indeed, in a recent paper, Bertin and Lecacheux [4] obtained Weierstrass equations of two rank 0 elliptic fibrations of Y_{10} by 3-isogenies from Weierstrass equations of rank 0 elliptic fibrations of Y_2. In [3], Bertin and Lecacheux obtained all elliptic fibrations, called generic, of the Apéry-Fermi pencil together with a Weierstrass equation. Some of these fibrations are endowed with a 2 or a 3-torsion section. It was also proved that the quotient $K3$ surface by a 2-torsion section is either the Kummer

Date: March 8, 2022.
2010 *Mathematics Subject Classification.* 11F23, 11G05, 14J28 (Primary); 14J27.
Key words and phrases. Elliptic fibrations of $K3$ surfaces, Isogenies between Elliptic Fibrations, Transcendental lattices .

surface K_k of its Shioda-Inose structure or a non Kummer $K3$ surface S_k with transcendental lattice

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

More precisely the 2-isogenies of Y_k are divided in two classes, the Morrison-Nikulin ones, i.e. 2-isogenies from Y_k to its Kummer K_k, and the others, called van Geemen-Sarti involutions. It was also proved that by specialization $S_2 = Y_2$. But there exist also other non specialized 2-isogenies on Y_2, some of them being Morrison-Nikulin, the others being called "self-isogenies", meaning either they preserve the same elliptic fibration ("PF self-isogenies") or they exchange two elliptic fibrations of Y_2 ("EF self-isogenies").

We shall prove in section 3 a similar result for the specializations K_{10} and S_{10}, that is K_{10} is the Kummer surface with transcendental lattice $\begin{pmatrix} 12 & 0 \\ 0 & 24 \end{pmatrix}$ and $S_{10} = Y_{10}$. We exhibit also a Morrison-Nikulin involution of Y_{10} not coming by specialization and "self isogenies" of rank 0 or positive rank not coming from specialization.

We end this section with the following application that is the determination of the Mordell-Weil group of a certain specialized fibration of Y_{10}. After expliciting the Kummer surface $K_{10} = \mathrm{Kum}(E_1, E_2)$ where E_1 has complex multiplication we exhibit an infinite section on a fibration of K_{10} giving, by a 2-isogeny, a section on the fibration of Y_{10}.

The situation is quite different concerning the 3-isogenies.

We prove in section 4 that the quotient $K3$ surface of a generic member (Y_k) by any 3-torsion section is a $K3$ surface N_k with transcendental lattice $\begin{pmatrix} 0 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ whose specializations satisfy $N_2 = Y_{10}$ while N_{10} is a $K3$ surface of discriminant 72 and transcendental lattice $[4 \quad 0 \quad 18] := \begin{pmatrix} 4 & 0 \\ 0 & 18 \end{pmatrix}$.

We prove also the main result, our first motivation, that is, all the 3-isogenies from elliptic fibrations of Y_2 are 3-isogenies from Y_2 to Y_{10}. It remains a natural question: what about the other 3-isogenies from elliptic fibrations of Y_{10}? Indeed we found 3-isogenies from Y_{10} to two other $K3$ surfaces with respective transcendental lattices $[4 \quad 0 \quad 18]$ and $[2 \quad 0 \quad 36]$.

In the same section we use the elliptic fibration $(Y_2)_h^{(3)}$ to construct elliptic fibrations of Y_{10} of high rank (namely 7 the highest we found) and by the 2-neighbour method, a rank 4 elliptic fibration with a 2-torsion section defining the Morrison-Nikulin involution exhibited in section 3.

In the last section 5, we prove that the L-series of the transcendental lattice of a certain singular $K3$ surface is inchanged by a 2 or a 3-isogeny. This result explains why the isogenous surfaces found in the previous sections have equal discriminants up to square.

Finally we put our results on 2 and 3-isogenies on Y_2 and Y_{10} in the perspective of a result of Bessière, Sarti and Veniani [5].

Computations were performed using partly the computer algebra system PARI [13], partly Sage [14] and mostly the computer algebra system MAPLE and the Maple Library "Elliptic Surface Calculator" written by Kuwata [9].

2. Background

2.1. **Discriminant forms.** Let L be a non-degenerate lattice. The **dual lattice** L^* of L is defined by

$$L^* := \mathrm{Hom}(L, \mathbb{Z}) = \{x \in L \otimes \mathbb{Q} / \ b(x, y) \in \mathbb{Z} \text{ for all } y \in L\}.$$

and the **discriminant group** G_L by

$$G_L := L^*/L.$$

This group is finite if and only if L is non-degenerate. In the latter case, its order is equal to the absolute value of the lattice determinant $\mid \det(G(e)) \mid$ for any basis e of L. A lattice L is **unimodular** if G_L is trivial.

Let G_L be the discriminant group of a non-degenerate lattice L. The bilinear form on L extends naturally to a $\mathbb{Q}$-valued symmetric bilinear form on L^* and induces a symmetric bilinear form

$$b_L : G_L \times G_L \to \mathbb{Q}/\mathbb{Z}.$$

If L is even, then b_L is the symmetric bilinear form associated to the quadratic form defined by

$$\begin{array}{rcl} q_L : G_L & \to & \mathbb{Q}/2\mathbb{Z} \\ q_L(x+L) & \mapsto & x^2 + 2\mathbb{Z}. \end{array}$$

The latter means that $q_L(na) = n^2 q_L(a)$ for all $n \in \mathbb{Z}$, $a \in G_L$ and $b_L(a,a') = \frac{1}{2}(q_L(a+a') - q_L(a) - q_L(a'))$, for all $a, a' \in G_L$, where $\frac{1}{2} : \mathbb{Q}/2\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is the natural isomorphism. The pair $\boldsymbol{(G_L, b_L)}$ (resp. $\boldsymbol{(G_L, q_L)}$) is called the **discriminant bilinear** (resp. **quadratic**) **form** of L.

When the even lattice L is given by its Gram matrix, we can compute its discriminant form using the following lemma as explained in Shimada [16].

Lemma 2.1. *Let A be the Gram matrix of L and U, $V \in Gl_n(\mathbb{Z})$ such that*

$$UAV = D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$$

with $1 = d_1 = \ldots = d_k < d_{k+1} \leq \ldots \leq d_n$. Then

$$G_L \simeq \oplus_{i>k} \mathbb{Z}/(d_i).$$

Moreover the ith row vector of V^{-1}, regarded as an element of L^* with respect to the dual basis e_1^*, ..., e_n^* generate the cyclic group $\mathbb{Z}/(d_i)$.

2.2. **Nikulin's results.**

Lemma 2.2 (Nikulin [12], Proposition 1.6.1)**.** *Let L be an even unimodular lattice and T a primitive sublattice. Then we have*

$$G_T \simeq G_{T^\perp} \simeq L/(T \oplus T^\perp), \quad q_{T^\perp} = -q_T.$$

In particular, $|\det T| = |\det T^\perp| = [L : T \oplus T^\perp]$.

2.3. **Shioda's results.** Let $(S, \Phi, \mathbb{P}^1)$ be an elliptic surface with a section Φ, without exceptional curves of first kind.

Denote by $NS(S)$ the group of algebraic equivalence classes of divisors of S .

Let u be the generic point of $\mathbb{P}^1$ and $\Phi^{-1}(u) = E$ the elliptic curve defined over $K = \mathbb{C}(u)$ with a K-rational point $o = o(u)$. Then, $E(K)$ is an abelian group of finite type provided that $j(E)$ is transcendental over $\mathbb{C}$. Let r be the rank of $E(K)$ and $s_1, ..., s_r$ be generators of $E(K)$ modulo torsion. Besides, the torsion group $E(K)_{tors}$ is generated by at most two elements t_1 of order e_1 and t_2 of order e_2 such that $1 \leq e_2$, $e_2 | e_1$ and $\mid E(K)_{tors} \mid = e_1 e_2$.

The group $E(K)$ of K-rational points of E is canonically identified with the group of sections of S over $\mathbb{P}^1(\mathbb{C})$.

For $s \in E(K)$, we denote by (s) the curve image in S of the section corresponding to s.

Let us define

$$D_\alpha := (s_\alpha) - (o) \quad 1 \leq \alpha \leq r$$

$$D'_\beta := (t_\beta) - (o) \quad \beta = 1,2.$$

Consider now the singular fibers of S over $\mathbb{P}^1$. We set

$$\Sigma := \{v \in \mathbb{P}^1 / C_v = \Phi^{-1}(v) \text{ be a singular fiber}\}$$

and for each $v \in \Sigma$, $\Theta_{v,i}$, $0 \leq i \leq m_v - 1$, the m_v irreducible components of C_v.
Let $\Theta_{v,0}$ be the unique component of C_v passing through $o(v)$.
One gets

$$C_v = \Theta_{v,0} + \sum_{i \geq 1} \mu_{v,i} \Theta_{v,i}, \quad \mu_{v,i} \geq 1.$$

Let A_v be the matrix of order $m_v - 1$ whose entry of index (i,j) is $(\Theta_{v,i}\Theta_{v,j})$, $i,j \geq 1$, where (DD') is the intersection number of the divisors D et D' along S. Finally f will denote a non singular fiber, i.e. $f = C_{u_0}$ for $u_0 \notin \Sigma$.

Theorem 2.1. *The Néron-Severi group $NS(S)$ of the elliptic surface S is generated by the following divisors*

$$f, \Theta_{v,i} \quad (1 \leq i \leq m_v - 1, \quad v \in \Sigma)$$
$$(o), D_\alpha \quad 1 \leq \alpha \leq r, \quad D'_\beta \quad \beta = 1,2.$$

The only relations between these divisors are at most two relations

$$e_\beta D'_\beta \approx e_\beta (D'_\beta(o)) f + \sum_{v \in \Sigma} (\Theta_{v,1}, ..., \Theta_{v,m_v-1}) e_\beta A_v^{-1} \begin{pmatrix} (D'_\beta \Theta_{v,1}) \\ . \\ . \\ . \\ (D'_\beta \Theta_{v,m_v-1}) \end{pmatrix}$$

where $\approx$ stands for the algebraic equivalence.

2.4. **Transcendental lattice.** Let X be an algebraic $K3$ surface; the group $H^2(X,\mathbb{Z})$, with the intersection pairing, has a structure of a lattice and by Poincaré duality is unimodular. The Néron-Severi lattice $NS(X) := H^2(X,\mathbb{Z}) \cap H^{1,1}(X)$ and the transcendental lattice $T(X)$, its orthogonal complement in $H^2(X,\mathbb{Z})$, are primitive sublattices of $H^2(X,\mathbb{Z})$ with respective signatures $(1, \rho - 1)$ and $(2, 20 - \rho)$ where ρ is the rank of the Néron-Severi lattice.
By Nikulin's lemma, their discriminant forms differ just by the sign, that is

$$(G_{T(X)}, q_{T(X)}) \equiv (G_{NS(X)}, -q_{NS(X)}).$$

2.5. **3-isogenous curves.**

2.5.1. *Method.* Let E be an elliptic curve with a 3-torsion point $\omega = (0,0)$

$$E : Y^2 + AYX + BY = X^3$$

and ϕ the isogeny of kernel $< \omega >$.
To determine a Weierstrass equation for the elliptic curve $E/ < \omega >$ we need two functions x_1 of degree 2 and y_1 of degree 3 invariant by $M \to M + \omega$ where $M = (X_M, Y_M)$ is a general point on E. We compute $M + \omega$ and $M + 2\omega$ $(= M - \omega)$ and can choose

$$\begin{aligned} x_1 &= X_M + X_{M+\omega} + X_{M+2\omega} = \frac{X^3 + ABX + B^2}{X^2} \\ y_1 &= Y_M + Y_{M+\omega} + Y_{M+2\omega} \\ &= \frac{Y\left(X^3 - AXB - 2B^2\right) - B\left(X^3 + A^2X^2 + 2AXB + B^2\right)}{X^3}. \end{aligned}$$

The relation between x_1 and y_1 gives a Weierstrass equation for $E/<\omega>$

$$y_1^2+(Ax_1+3B)\,y_1=x_1^3-6ABx_1-B\left(A^3+9B\right).$$

Notice that the points with $x_1=-\frac{A^2}{3}$ are 3-torsion points. Taking one of these points to origin and after some transformation we can obtain a Weierstrass equation $y^2+ayx+by=x^3$ with the following transformations.

2.5.2. *Formulae.* If $j^3=1$ then we define

$$S_1=2\left(j^2-1\right)y+6Ax-2\,(j-1)\left(A^3-27B\right)$$
$$S_2=2\,(j-1)\,y+6Ax-2\left(j^2-1\right)\left(A^3-27B\right)$$

and

$$X=\frac{-1}{324}\frac{S_1S_2}{x^2},\qquad Y=\frac{1}{5832}\frac{S_1^3}{x^3}$$

then we have

$$E/<\omega>:\, y^2+(-3A)\,yx+\left(27B-A^3\right)y=x^3.$$

If $A_1=-3A$, $B_1=27B-A^3$, then we define

$$\sigma_1=2\left(j^2-1\right)3^6Y+6A_13^4X-2\,(j-1)\left(A_1^3-27B_1\right)$$
$$\sigma_2=2\,(j-1)\,3^6Y+6A_13^4X-2\left(j^2-1\right)\left(A_1^3-27B_1\right)$$

and then

$$x=\frac{-1}{324}\frac{\sigma_1\sigma_2}{3^8X^2}=-\frac{3\,X^3+A^2X^2+3\,BAX+3\,B^2}{X^2},\qquad y=\frac{1}{5832}\frac{\sigma_1^3}{3^{12}X^3}.$$

2.5.3. *Other properties of isogenies.* The divisor of the function Y is equal to $-3\,(0)+3\omega$ so $Y=W^3$ where W is a function on the curve $E/<\omega>$. If $X=WZ$, the function field of $E/<\omega>$ is generated by W and Z. So replacing in the equation of E we obtain the relation between Z and W

$$W^3+AZW+B-Z^3=0.$$

This cubic equation, with a rational point at infinity with $W=Z$ can be transformed to obtain a Weierstrass equation in the coordinates X_2 and Y_2:

$$W=\frac{1}{9}\frac{\left(-243B-3X_2A+9A^3-Y_2\right)}{X_2},\quad Z=-\frac{1}{9}\frac{Y_2}{X_2}$$

$$\text{of inverse}\quad X_2=3\frac{A^3-27B}{3\,(W-Z)+A},\quad Y_2=-27Z\frac{A^3-27B}{3\,(W-Z)+A}$$

$$Y_2^2+3AY_2X_2+\left(-9A^3+243B\right)Y_2=$$
$$X_2^3-9X_2^2A^2+27A\left(A^3-27B\right)X_2-27\left(A^3-27B\right)^2.$$

The points of X_2-coordinate equal to 0 are $3-$torsion points and easily we recover the previous formulae.

2.6. **Notation.** The singular fibers of type $I_n, D_m, IV^*, ...$ at $t=t_1,.,t_m$ or at roots of a polynomial $p(t)$ of degree m are denoted $mI_n(t_1,..,t_m)$ or $mI_n(p(t))$. The zero component of a reducible fiber is the component intersecting the zero section and is denoted θ_0 or $\theta_{t_0,0}$. The other components denoted $\theta_{t_0,i}$ satisfy the property $\theta_{t_0,i}\cdot\theta_{t_0,i+1}=1$.

3. 2-isogenies of Y_{10}

In [3], Bertin and Lecacheux classified all the 2-isogenies of Y_2 in two sets, the first defining Morrison-Nikulin involutions, that is from Y_2 to its Kummer surface K_2 and the second giving van Geemen-Sarti involutions that is exchanging two elliptic fibrations (different or the same) of Y_2 named "self-isogenies".
Since we have no exhaustive list of elliptic fibrations of Y_{10} with 2-torsion sections, we cannot give such a classification. However we found on Y_{10} Morrison-Nikulin involutions from Y_{10} to its Kummer surface K_{10}.

3.1. Morrison-Nikulin involutions of Y_{10}.

3.1.1. *Specialized Morrison-Nikulin involutions.* We first recall a result concerning the involutions on the generic member (Y_k) of the Apéry-Fermi pencil.

Theorem 3.1. *(Bertin and Lecacheux [3])*
Suppose Y_k is a generic $K3$ surface of the family with Picard number 19.
Let $\pi : Y_k \rightarrow \mathbb{P}^1$ be an elliptic fibration with a torsion section of order 2 which defines an involution i of Y_k (van Geemen-Sarti involution) then the minimal resolution of the quotient Y_k/i is either the Kummer surface K_k associated to Y_k given by its Shioda-Inose structure or a surface S_k with transcendental lattice $T(S_k) = \langle -2\rangle \oplus \langle 2\rangle \oplus \langle 6\rangle$ and Néron-Severi lattice $NS(S_k) = U \oplus E_8[-1] \oplus E_7[-1] \oplus \langle(-2)\rangle \oplus \langle(-6)\rangle$, which is not a Kummer surface. Thus, π leads to an elliptic fibration either of K_k or of S_k. Moreover there exist some genus 1 fibrations $\theta : K_k \rightarrow \mathbb{P}^1$ without section such that their Jacobian variety satisfies $J_\theta(K_k) = S_k$.
More precisely, among the elliptic fibrations of Y_k (up to automorphisms) 12 of them have a two-torsion section. And only 7 of them possess a Morrison-Nikulin involution i such that $Y_k/i = K_k$.

With the same argument as for specialization to Y_2, Morrison-Nikulin involutions specialized to Y_{10} remain Morrison-Nikulin involutions of Y_{10}. Hence we obtain in the following Table 1 the corresponding Weierstrass equations of such elliptic fibrations of the Kummer surface K_{10} with transcendental lattice $\begin{pmatrix} 12 & 0 \\ 0 & 24 \end{pmatrix}$.

3.1.2. *A non specialized Morrison-Nikulin involution of Y_{10}.*

Theorem 3.2. *The rank 4 elliptic fibration of Y_{10} (4.3), with Weierstrass equation*

$$E_t \qquad y^2 = x^3 - (t^3 + 5t^2 - 2)x^2 + (t^3 + 1)^2 x$$

and singular fibers $I_0^*(\infty)$, $3I_4(t^3+1)$, $I_2(0)$, $4I_1(1, -5/3, t^2-4t-4)$, has a 2-torsion section defining a Morrison-Nikulin involution from Y_{10} to K_{10}, that is $F_t = E_t/\langle(0,0)\rangle$ is a rank 4 elliptic fibration of K_{10} with Weierstrass equation

$$F_t \qquad Y^2 = X^3 + 2(t^3 + 5t^2 - 2)X^2 - (t^2 - 4t - 4)(t - 1)(3t + 5)t^2 X$$

and singular fibers $I_0^*(\infty)$, $I_4(0)$, $7I_2(\pm 1, -5/3, t^2 - t + 1, t^2 - 4t - 4)$.

Proof. Starting from F_t and taking the new parameter $p = \frac{X}{(t^2-4t-4)(t-1)(3t+5)}$, we get a rank 1 elliptic fibration with Weierstrass equation

$$F_p : Y^2 = X^3 + \frac{3}{4}p\,(5p-1)^2 X^2 + \frac{1}{6}p^2\,(2p-1)\,(5p-1)\left(49p^2 - 13p + 1\right) X$$
$$+ \frac{1}{108}p^3\,(2p-1)^2\left(49p^2 - 13p + 1\right)^2$$

No	Weierstrass Equation
#4	$y^2 = x(x - \frac{1}{8}(t-a_1)(t-a_2)(t-a_3))(x - \frac{1}{8}(t-b_1)(t-b_2)(t-b_3))$ $a_1 = -\frac{58}{3} - 4\sqrt{6} + 8\sqrt{3} - 4\sqrt{2}, a_2 = -\frac{58}{3} - 4\sqrt{6} - 8\sqrt{3} + 4\sqrt{2},$ $b_1 = \frac{2}{3} + 4\sqrt{6} + 24\sqrt{2} + 8\sqrt{3}, b_2 = \frac{2}{3} + 4\sqrt{6} - 24\sqrt{2} - 8\sqrt{3}$ $a_3 = \frac{116}{3} + 8\sqrt{6}, b_3 = -\frac{4}{3} - 8\sqrt{6}$ $I_6^*(\infty), 6I_2(a_i, b_i), (r = 2)$
#8	$y^2 = x^3 + 2(t^3 - 25t^2 + 50t - 24)x^2 + (t^2 - 24t + 36)(t^2 - 24t + 16)(t-1)^2 x$ $I_3(0), I_4(1), 4I_2(t^2 - 24t + 16, t^2 - 24t + 36), I_3^*(\infty), (r = 2)$
#16	$y^2 = x(x + t(t^2 + (238 - 96\sqrt{6})t + 1)(x + t(t^2 + (42 - 16\sqrt{6})t + 1))$ $2I_2^*(0, \infty), 4I_2(t^2 + (-16\sqrt{6} + 42)t + 1, t^2 + (-96\sqrt{6} + 238)t + 1), (r = 2)$
#17	$y^2 = x(x - (t^2 - 48)(t^2 - 8))(x - (t^2 + 4t - 20)(t^2 - 4t - 20))$ $I_8(\infty), 8I_2(t^2 - 4t - 20, t^2 + 4t - 20, t^2 - 8, t^2 - 48), (r = 3)$
#23	$y^2 = x(x + \frac{1}{4}(4t^2 - 12t + 1)(8t - 1))(x + (12t - 1)(4t^2 - 8t + 1))$ $I_0^*(\infty), I_6(0), 6I_2(\frac{1}{8}, \frac{1}{12}, 4t^2 - 8t + 1, 4t^2 - 12t + 1), (r = 3)$
#24	$y^2 = x^3 + 2t(t^2 - 22t + 1)x^2 + t^2(t^2 - 14t + 1)(t^2 - 34t + 1)x$ $2I_1^*(0, \infty), 5I_2(-1, t^2 - 14t + 1, t^2 - 34t + 1), (r = 3)$
#26	$y^2 = x(x - \frac{1}{4}(t^2 - 14t + 1)(t^2 - 6t + 1))(x - \frac{1}{4}(t^2 - 10t + 1)^2)$ $4I_4(0, \infty, t^2 - 10t + 1), 4I_2(t^2 - 14t + 1, t^2 - 6t + 1), (r = 2)$

TABLE 1. Specialized fibrations of the Kummer K_{10}

and singular fibers $I_0^*(\infty)$, $I_3^*(0)$, $3I_3(\frac{1}{2}, 49p^2 - 13p + 1)$. The infinite section $P = (-\frac{1}{12}p(49p^2 - 13p+1), \frac{1}{8}p^2(49p^2 - 13p + 1))$ is of height $h(P) = \frac{2}{3}$, is not equal to $2Q$ or $3Q$, hence the discriminant of the Néron-Severi lattice satisfies $\Delta = 4 \times 4 \times 3^3 \times \frac{2}{3} = 72 \times 4$.
Now we are going to compute its Gram matrix and deduce its transcendental lattice.
To compute the Néron-Severi lattice we order the generators as (0) the zero section, (f) the generic fiber, $\theta_i, 1 \leq i \leq 4$, $\eta_i, 1 \leq i \leq 7$, γ_i, δ_i, ϵ_i, $1 \leq i \leq 2$ the rational components of respectively D_4, D_7 and the three A_2 and finally the infinite section (P).

$$NS = \begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -2
\end{pmatrix}.$$

Fom lemma 2.1 we deduce that $G_{NS} = \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z}$ and we get generators f_1 and f_2 with respective norms $q(f_1) = -\frac{41}{12}$, $q(f_2) = -\frac{59}{24}$ modulo 2 and scalar product $f_1.f_2 = \frac{7}{4}$ modulo 1.
In order to prove that the transcendental lattice corresponds to the Gram matrix $\begin{pmatrix} 12 & 0 \\ 0 & 24 \end{pmatrix}$ we must find for the corresponding quadratic form generators g_1 and g_2 satisfying $q(g_1) = \frac{41}{12}$, $q(g_2) = \frac{59}{24}$

No	Weierstrass Equation
#7	$E7: y^2=x^3+2x^2t\,(11t+1)-t^2\,(t-1)^3\,x$ $III^*(\infty), I_1^*(0), I_6(1), 2I_1(t^2+118t+25)$ $EE7:=Y^2=X^3-4X^2t\,(11t+1)+4t^3\left(118t+25+t^2\right)X$ $III^*(\infty), I_2^*(0), I_3(1), 2I_2(t^2+118t+25)$
#9	$E9: y^2=x^3+28t^2x^2+t^3\left(t^2+98t+1\right)x$ $2III^*(0,\infty), 2I_2(t^2+98t+1), 2I_1(t^2-98t+1)$ $EE9: Y^2=X^3-56t^2X^2-4t^3\left(t^2-98t+1\right)X$ $2III^*(0,\infty), 2I_2(t^2-98t+1), 2I_1(t^2+98t+1)$
#14	$E14: y^2=x^3+t\left(98t^2+28t+1\right)x^2+t^6x$ $I_8^*(0), I_0^*(\infty), I_1(4t+1), I_1(24t+1), 2I_1(100t^2+28t+1)$ $EE14: Y^2=X\left(X-96t^3-28t^2-t\right)\left(X-100t^3-28t^2-t\right)$ $I_4^*(0), I_0^*(\infty), I_2(4t+1), I_2(24t+1), 2I_2(100t^2+28t+1)$
#15	$E15: y^2=x^3-t\left(2+t^2-22t\right)x^2+t^2\,(t+1)^2\,x$ $I_1^*(0), I_4^*(\infty), I_4(-1), I_1(24), 2I_1(t^2-20t+4)$ $EE15: Y^2=X\left(X+t^3-24t^2\right)\left(X+t^3-20t^2+4t\right)$ $I_2^*(0), I_2^*(\infty), I_2(-1), I_2(24), 2I_2(t^2-20t+4)$
#20	$E20: y^2=x^3+\left(\frac{1}{4}t^4-5t^3+\frac{53}{2}t^2-15t-\frac{3}{4}\right)x^2-t\,(t-10)\,x$ $I_2(0), I_{12}(\infty), 2I_3(t^2-10t+1), I_2(10), I_1(1), I_1(9)$ $EE20: Y^2=X^3+\left(-\frac{1}{2}t^4+10t^3-53t^2+30t+\frac{3}{2}\right)X^2$ $+\frac{1}{16}\,(t-1)\,(t-9)\left(t^2-10t+1\right)^3 X$ $I_1(0), I_6(\infty), 2I_6(t^2-10t+1), I_2(1), I_2(9), I_1(10)$

TABLE 2. Fibrations $E\#i$ of Y_{10} and $EE\#i$ of S_{10}

modulo 2 and scalar product $g_1.g_2=-\frac{7}{4}$ modulo 1. This is obtained with $g_1=\begin{pmatrix}\frac{1}{4}\\\frac{1}{3}\end{pmatrix}$ and $g_2=\begin{pmatrix}\frac{5}{12}\\\frac{1}{8}\end{pmatrix}$. Thus F_p, hence F_t, are elliptic fibrations of the Kummer surface K_{10}.

□

In a previous paper, Bertin and Lecacheux [3] explained that, in the Apéry-Fermi family, only Y_2 and Y_{10} may have "self-isogenies". A "self-isogeny" is a van Geemen-Sarti involution which either preserve an elliptic fibration (called "PF self-isogeny" for more precision) or exchanges two elliptic fibrations "EF self-isogeny".

Moreover, in the same paper, all the "self-isogenies" of Y_2 were listed. Since it is quite difficult to get all the elliptic fibrations of Y_{10} with 2-torsion sections, we shall give "self-isogenies" of Y_{10} obtained either as specializations or from rank 0 fibrations or from non specialized positive rank elliptic fibrations.

3.2. **Specialized "self-isogenies".** In [3] we characterized the surface S_k obtained by 2-isogeny deduced from van Geemen-Sarti involutions of Y_k which are not Morrison-Nikulin. Let us recall the specialized Weierstrass equations of S_{10}. We denote $E\#n$ (resp. $EE\#n$) a Weierstrass equation of a fibration of Y_{10} (resp. S_{10}).

The specialization of S_k for $k=10$ has the following five elliptic fibrations given on Table 2. The first Weierstrass equation concerns Y_{10} and the second S_{10} obtained as its 2-isogenous curve.

Theorem 3.3. *The previous 2-isogenies are in fact "self-isogenies", the surface S_{10} being equal to Y_{10}.*

Proof. We observe that $E9$ and $EE9$ have the same singular fibers. In fact these two fibrations are isomorphic, the isomorphism being defined by $t = -T$, $x = -\frac{X}{2}$, $y = \frac{Y}{2\sqrt{-2}}$. This property is sufficient to identify S_{10} to Y_{10}.
Among these "self-isogenies" only the number #9 is "PF".

□

3.3. **Rank** 0 **"self-isogenies".**

Theorem 3.4. *There are four* 2*-isogenies from* Y_{10} *to* Y_{10} *defined by extremal elliptic fibrations with* 2*-torsion sections, denoted number* 8, 87, 153, 262 *as in Shimada and Zhang's paper* [17]*. They are all "PF self-isogenies".*

Proof. We write below Weierstrass equation E_n, its 2-isogenous EE_n and the corresponding isomorphism.

E_{262} $\quad y^2 = x^3 + x^2(9(t+5)(t+3) + (t+9)^2) - t^3(t+5)^2 x$
$III^*(\infty)$, $I_6(0)$, $I_4(-5)$, $I_3(-9)$, $I_2(-4)$
EE_{262} $\quad Y^2 = X^3 - 2(9(T+5)(T+3) + (T+9)^2)X^2 + 4(T+4)^2(T+9)^3 X$
$III^*(\infty)$, $I_6(-9)$, $I_4(-4)$, $I_3(0)$, $I_2(-5)$
Isomorphism: $t = -T - 9$, $x = -\frac{X}{2}$, $y = \frac{Y}{2\sqrt{-2}}$.

E_{153} $\quad y^2 = x^3 + t(t^2 + 10t - 2)x^2 + (2t+1)^3 t^2 x$
$I_3(4)$, $I_6(-1/2)$, $I_1^*(0)$, $I_2^*(\infty)$
EE_{153} $\quad Y^2 = X^3 - 2T(T^2 + 10T - 2)X^2 + T^3(T-4)^3 X$
$I_6(4)$, $I_3(-1/2)$, $I_2^*(0)$, $I_1^*(\infty)$
Isomorphism: $t = -\frac{2}{T}$, $x = -\frac{2X}{T^4}$, $y = -\frac{2\sqrt{-2}Y}{T^6}$.

E_{87} $\quad y^2 = x^3 - (9t^4 + 9t^3 + 6t^2 - 6t + 4)x^2 + (21t^2 - 12t + 4)x$
$I_{12}(\infty)$, $I_6(0)$, $2I_2(21t^2 - 12t + 4)$, $2I_1(3t^2 + 6t + 7)$
EE_{87} $\quad Y^2 = X^3 - (9T^4 + 9T^3 + 6T^2 - 6T + 4)X^2 + (21T^2 - 12T + 4)X$
$I_{12}(0)$, $I_6(\infty)$, $2I_2(3t^2 + 6t + 7)$, $2I_1(21t^2 - 12t + 4)$
Isomorphism: $t = -\frac{2}{T}$, $x = -\frac{2X}{9T^4}$, $y = \frac{2\sqrt{-2}}{27T^6}$.

E_8 $\quad y^2 = x^3 - (3t^4 - 60t^2 - 24)x^2 - 144(t^2 - 1)^3 x$
$I_2(0)$, $2I_3(t^2 + 8)$, $I_4(\infty)$, $2I_6(t^2 - 1)$
EE_8 $\quad y^2 = x^3 + 2(3t^4 - 60t^2 - 24)x^2 + 9t^2(t^2 + 8)^3 x$
$I_2(\infty)$, $2I_3(t^2 - 1)$, $I_4(0)$, $2I_6(t^2 + 8)$
Isomorphism: $t = \frac{2\sqrt{-2}}{T}$, $x = \frac{4X}{T^4}$, $y = \frac{8Y}{T^6}$.

□

3.4. **Positive rank non specialized "EF" and "PF self-isogenies".** Using the 2-neighbor method we found many examples of 2-torsion elliptic fibrations of Y_{10}.
Denote E, E_1, E_2, E_3, E_4 the following elliptic fibrations of Y_{10} obtained in the following way. Starting from E_{153} and new parameter $\frac{x}{t(2t+1)^2}$ we get E. Starting from $EE15$ we get successively E_2, E_3, E_4 with the successive parameters $\frac{x}{t^2(t-24)}$, $\frac{x}{t^2(t+1)}$, $\frac{x}{t(t-4)(t-24)}$. And from $EE_2 = E_2/\langle(0,0)\rangle$ we get E_1 with the new parameter $\frac{x}{t(t-1)(t-4)}$.

Theorem 3.5. *(1) The* 2*-isogenies, from* E_3 *to* EE_3*, from* E_4 *to* EE_4*, from* E_1 *to* EE_1 *are "PF self-isogenies".*
(2) The 2*-isogenies from* E *to* EE*, from* $EE14$ *to* $EE14/\langle(100t^2+28t+1,0)\rangle$ *and from* E_2 *to* EE_2 *are "EF self-isogenies".*

Proof. (1) We only need to give the respective Weierstrass equations, singular fibers and isomorphisms concerning the 2-isogenies from E_i to EE_i.

E_3 $\quad y^2 = x^3 - 2t(t^2 - 14t - 2)x^2 + t^4(t-4)(t-24)x$

$I_4^*(0)$, $2I_2(4, 24)$, $2I_1(-1/2, -1/12)$, $I_2^*(\infty)$

$EE_3 = E_3/\langle(0,0)\rangle$

$Y^2 = X^3 + T(T^2 - 14T - 2)X^2 + T^2(2T+1)(12T+1)X$

$I_2^*(0)$, $2I_1(4, 24)$, $2I_2(-1/2, -1/12)$, $I_4^*(\infty)$

Isomorphism: $t = -\frac{2}{T}$, $\quad x = -\frac{8X}{T^4}$ $\quad y = \frac{16\sqrt{-2}Y}{T^6}$

E_4 $\quad y^2 = x^3 - 28t^2(t-1)x^2 + 4t^3(t-1)^2(24t+1)x$

$III^*(0)$, $I_0^*(1)$, $I_2(-1/24)$, $I_1(1/25)$, $I_0^*(\infty)$

$EE_4 = E_4/\langle(0,0)\rangle$

$Y^2 = X^3 + 56T^2(T-1)X^2 + 16T^3(T-1)^2(25T-1)X$

$III^*(0)$, $I_0^*(1)$, $I_2(1/25)$, $I_1(-1/24)$, $I_0^*(\infty)$

Isomorphism: $t = \frac{T}{T-1}$, $\quad x = -\frac{X}{2(T-1)^4}$ $\quad y = -\frac{\sqrt{-2}Y}{4(T-1)^6}$

E_1 $\quad y^2 = x^3 - t(5t^2 + 56t + 160)x^2 + 4t^2(t+6)^2(t+4)^2x$

$I_0^*(0)$, $2I_4(-4, -6)$, $2I_2(-8, -16/3)$, $I_0^*(\infty)$

$EE_1 = E_1/\langle(0,0)\rangle$

$Y^2 = X^3 + 2T(5T^2 + 56T + 160)X^2 + T^2(T+8)^2(3T+16)^2X$

$I_0^*(0)$, $2I_4(-8, -16/3)$, $2I_2(-6, -4)$, $I_0^*(\infty)$

Isomorphism: $t = \frac{32}{T}$, $\quad x = -\frac{2^9X}{T^4}$ $\quad y = \frac{2^{13}\sqrt{-2}Y}{T^6}$

(2) Let us give Weierstrass equations and singular fibers of E and EE.

E $\quad y^2 = x^3 + 2t(2t^2 + 5t + 1)x^2 + t^3(4t+1)(t-1)^2x$

$I_2^*(0)$, $I_4(1)$, $I_3(-1/3)$, $I_2(-1/4)$, $I_1^*(\infty)$

$EE = E/\langle(0,0)\rangle$

$Y^2 = X^3 - 4T(2T^2 + 5T + 1)X^2 + 4T^2(3T+1)^3X$

$I_1^*(0)$, $I_6(-1/3)$, $I_2(1)$, $I_1(-1/4)$, $I_2^*(\infty)$

The fibration EE is a fibration of Y_{10}, since with the new parameter $\frac{X}{(3T+1)^3}$, we get the rank 0 elliptic fibration E_{252}, that is the extremal elliptic fibration numbered 252 in Shimada and Zhang's paper [17].

We also obtain

E_2 $\quad y^2 = x^3 - 4t(t+1)(6t+5)x^2 + 4t^2(t+1)^3x$

$I_0^*(0)$, $I_2^*(-1)$, $2I_1(-8/9, -3/4)$

$EE_2 = E_2/\langle(0,0)\rangle$

$Y^2 = X^3 + 8T(T+1)(6T+5)X^2$
$+16T^2(T+1)^2(9T+8)(4T+3)X$

$I_1^*(-1)$, $I_0^*(0)$, $2I_2(-3/4, -8/9)$, $I_1^*(\infty)$

To prove that EE_2 is an elliptic fibration of Y_{10} first we change the parameter $T = 1/u - 1$ to get the new equation $EE_2(1)$

$$EE_2(1) \qquad y^2 = x^3 + 8u(u-1)(u-6)x^2 + 16u^2(u-4)(u-9)(u-1)^2x.$$

Now with the new parameter $\frac{x}{u(u-1)(u-4)(u-9)}$, we obtain

$$EE_2(2) \qquad y^2 = x^3 - t(59t^2 - 88t + 32)x^2 + 32t^2(t-1))(3t-2)^3x.$$

Again, from $EE_2(2)$, the parameter $\frac{x}{t^2(t-1)}$ leads to the rank 0 fibration E_{252} of Y_{10}.

Finally, the fibration $EE14/\langle(100t^2+28t+1,0)\rangle$ with Weierstrass equation

$$y^2 = x^3 - 2t(104t^2+28t+1)x^2 + t^2(4t+1)^2(24t+1)^2x$$

is a fibration of Y_{10}, since with the new parameter $\frac{x}{t(4t+1)^2}$ we obtain EE_2.

□

In the previous theorem we gave "self-isogenies" of elliptic fibrations with rank less than 2. However we found in section 4 an interesting 2-torsion rank 4 fibration. We present it in the following theorem.

Theorem 3.6. *The rank* 4 *elliptic fibration of* Y_{10} *(4.2) with singular fibers* $3I_4$*,* $3I_2$*,* $2III$*, Weierstrass equation*

$$F \qquad y^2 = x^3 + 4t^2x^2 + t(t^3+1)^2x$$

and its 2*-isogenous* $F/\langle(0,0)\rangle$ *are "PF self-isogenous".*

Proof. We get

$$F/\langle(0,0)\rangle \qquad Y^2 = X^3 - 8T^2X^2 - 4T(T^3-1)^2X$$

with the same type of singular fibers. The isomorphism is given by

$$T = -t, \qquad Y = -2\sqrt{-2}y, \qquad X = -2x.$$

□

3.5. **Generators for specialization of #16 fibration on** Y_{10}**.** The rank of the specializations for $k = 10$ of generic elliptic fibrations increases by one ([4], Theorem 4.1), so we have to determine one more generator for the Mordell-Weil group. We give an example where the computation is easy using a 2-isogeny between an elliptic fibration of Y_{10} and an elliptic fibration of the Kummer surface $K_{10} = \text{Kum}(E_1, E_2)$ associated to Y_{10}, where E_1, E_2 are elliptic curves with complex multiplication. Then using the method developped in [20] and [11], we determine a section on an elliptic fibration of K_{10}.

From [3] Corollary 4.1, the two elliptic curves E_1 and E_2 have respective invariants $j_1 = 8000$ and $j_2 = 188837384000 - 77092288000\sqrt{6}$. Take

$$E_1 : Y^2 = X\left(X^2+4X+2\right)$$

as a model of the first curve. The 2-torsion sections have X-coordinates 0, $-2\pm\sqrt{2}$, the $3-$ torsion sections have X-coordinates $\frac{1}{3}\left(1\pm i\sqrt{2}\right)$ and $-1\pm\sqrt{6}$ that are roots of $(3X^2-2X+1)(X^2+2X-5)$. The elliptic curve E_1 has complex multiplication by $m_2 = \sqrt{-2}$ defined by

$$(X,Y) \overset{m_2}{\mapsto} \left(-\frac{1}{2}\frac{X^2+4X+2}{X}, \frac{i\sqrt{2}}{4}\frac{Y\left(X^2-2\right)}{X^2}\right).$$

Let C_3 and $\widetilde{C_3}$ the two groups of order 3 generated by the points of respective X-coordinates $\frac{1}{3}\left(-2+i\sqrt{2}\right)$ and $\frac{1}{3}\left(-2-i\sqrt{2}\right)$. These groups are fixed by m_2 while the two order 3 groups Γ_3 and $\widetilde{\Gamma_3}$ generated by the points of respective X-coordinates $-2+\sqrt{6}$ and $-2-\sqrt{6}$ are exchanged by m_2.

If $M = (X,Y)$ is a general point on E_1, the 3-isogenous curve by the isogeny w_3 of kernel Γ_3 is thus obtained with $X_2 = \sum_{S\in\Gamma_3} X_{M+S} + c$ and $Y_2 = \sum_{S\in\Gamma_3} Y_{M+S}$ where c can be chosen so that the image of $(0,0)$ is $X_2 = 0$. It follows the 3-isogeny w_3

$$w_3 : X_2 = \frac{X\left(X-2-\sqrt{6}\right)^2}{\left(X+2-\sqrt{6}\right)^2}, \quad Y_2 = -\frac{Y\left(X^2+\left(8-2\sqrt{6}\right)X+2\right)\left(X-2-\sqrt{6}\right)}{\left(X+2-\sqrt{6}\right)^3}$$

and its 3-isogenous curve E_2

$$E_2 : Y_2^2 = X_2^3 + 28X_2^2 + \left(98 + 40\sqrt{6}\right) X_2$$
$$j\left(E_2\right) = 188837384000 - 77092288000\sqrt{6}.$$

An equation for the Kummer surface K_{10} is therefore

$$K_{10} : X\left(X^2 + 4X + 2\right) = y^2 X_2 \left(X_2^2 + 28X_2 + 98 + 40\sqrt{6}\right).$$

3.5.1. *Elliptic fibrations of* K_{10} *and* Y_{10}. We use the following units of $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$

$$r_1 = 1 + \sqrt{2} + \sqrt{6}, \quad r_1' = 1 - \sqrt{2} + \sqrt{6}, \quad r_1 r_1' = s = \left(\sqrt{2} + \sqrt{3}\right)^2$$
$$r_2 = 1 - 2\sqrt{3} - \sqrt{6}, \quad r_2' = 1 + 2\sqrt{3} - \sqrt{6}.$$

In this paragraph we construct an elliptic fibration of K_{10} giving after a two-isogeny the specialization of the elliptic fibration #16 on Y_{10}.
We consider the fibration

$$K_{10} \to \mathbb{P}^1$$
$$(X, X_2, y) \mapsto t = \frac{X_2}{X}.$$

Notice that $X_2 = tX$ and a Weierstrass equation $(K_{10})_t$ for this fibration is obtained with the following transformation

$$X = -\sqrt{2}\left(1 + \sqrt{2}\right) \frac{X_1 - 2t\left(t - r_1^2\right)\left(t - r_2^2\right)}{X_1\left(3 + 2\sqrt{2}\right) - 2t\left(t - r_1^2\right)\left(t - r_2^2\right)}, \qquad y = 2\sqrt{2}\frac{X_1}{Y_1}$$

$$(K_{10})_t : Y_1^2 = X_1\left(X_1 - 2t\left(t - r_1^2\right)\left(t - r_1'^2\right)\right)\left(X_1 - 2t\left(t - r_2^2\right)\left(t - r_2'^2\right)\right). \tag{3.1}$$

The singular fibers are in $t = 0$ and ∞ of type I_2^* and at $t = r_1^2, r_2^2, r_1'^2, r_2'^2$ of type I_2. The rank of the Mordell-Weil group is 2.

Remark 3.1. We can show that $(K_{10})_t$ is the fibration of line #16 of Table 1 . More precisely if $t = -st_0$ and $X = b^2 x$, $Y = b^3 y$ with $b = \sqrt{2}(\sqrt{2} + \sqrt{3})^3$ we get exactly the fibration #16 with parameter t_0.

3.5.2. *Sections on the elliptic fibration* $(K_{10})_t$. In many papers ([18] [19] [20] Th 1.2. and [8], [11]) results on the Mordell-Weil lattice of the Inose fibration are given. We follow the same idea here, with the previous fibration $(K_{10})_t$ of parameter t.
We find a section on this fibration using $w_3 \in \mathrm{Hom}\left(E_1, E_2\right)$. The graph of w_3 on $E_1 \times E_2$ and the image on $K_{10} = E_1 \times E_2 / \pm 1$ correspond to $X_2 = \frac{X\left(X-2-\sqrt{6}\right)^2}{\left(X+2-\sqrt{6}\right)^2}$ or $t = \frac{\left(X-2-\sqrt{6}\right)^2}{\left(X+2-\sqrt{6}\right)^2}$. If we consider the base-change of the fibration $u^2 = t$ we obtain a section defined by $u = \frac{\left(X-2-\sqrt{6}\right)}{\left(X+2-\sqrt{6}\right)}$ or $X = \frac{\left(-2+\sqrt{6}\right)(u-s)}{u-1}$, that is $P_u = \left(X_1\left(u\right), Y_1\left(u\right)\right)$ on the Weierstrass equation $(K_{10})_{u^2}$

$$X_1\left(u\right) = \frac{2}{s_2^2 s_3^2} u^2 \left(u^2 - r_2^2\right)\left(u + r_1\right)\left(u - r_1'\right)$$
$$Y_1\left(u\right) = 2uX_1\left(u\right)\left(\left(\sqrt{3} - \sqrt{2}\right)u^2 + \left(-2\sqrt{3} + \sqrt{2}\right)u + \sqrt{3} + \sqrt{2}\right),$$

where $s_2 = \frac{\sqrt{2}}{2}\left(-\sqrt{3}+1\right), s_3 = \sqrt{2}+1$. If $\widetilde{P_u} = (X_1(-u), Y_1(-u))$, then $\widetilde{P_u} \in (K_{10})_{u^2}$ and $P = \widetilde{P_u} + P_u \in (K_{10})_t$, thus

$$P = (x_P, y_P)$$
$$x_P = \frac{1}{s}(t+s)^2\left(t-r_1^2\right)\left(t-r_1'^2\right), y_P = x_P\frac{2-\sqrt{6}}{2}\left(\frac{t-s}{t+s}\right)\left(t^2-14t-4\sqrt{6}t+s^2\right)$$

so we recover an infinite section P on the fibration $(K_{10})_t$ of the Kummer surface K_{10}.

3.5.3. *Sections on the fibration #16 of* Y_{10}. The 2-isogenous elliptic curve to (3.1) in the isogeny of kernel $(0,0)$ has a Weierstrass equation

$$Y_3^2 = X_3\left(X_3^2 + 8\,t\left(t^2-28\,t+s^2\right)X_3 + 64\,\frac{t^4}{s^2}\right) \tag{3.2}$$
$$X_3 = \left(\frac{Y_1}{X_1}\right)^2, \qquad Y_3 = \frac{Y_1(B-X_1^2)}{X_1}$$

where B is the coefficient of X_1 in (3.1). Singular fibers are in $t=0$ and ∞ of type I_4^* and of type I_1 at $t = r_1^2, r_2^2, r_1'^2, r_2'^2$.

Using the remark 3.1 this is fibration #16 of Y_{10}. The image of P by this isogeny, in the Weierstrass equation (3.2), is $Q = (\xi_Q, \eta_Q)$ with

$$\xi_Q = \frac{1}{2s}\frac{\left(t^2-14t-4\sqrt{6}t+s^2\right)^2(t-s)^2}{(t+s)^2}$$
$$\eta_Q = -\frac{\left(-2+\sqrt{6}\right)}{4s}\frac{\left(t^2-14t-4\sqrt{6}t+s^2\right)(t-s)\,L_t}{(t+s)^3}$$
$$\text{where}\quad L_t = t^6 + 2\left(1+2\sqrt{6}\right)t^5 - \left(993+404\sqrt{6}\right)t^4 + \left(17820+7272\sqrt{6}\right)t^3$$
$$-\left(97137+39656\sqrt{6}\right)t^2 + \left(56642+23124\sqrt{6}\right)t + s^6.$$

Recall that by specialization of the generic case [3] we have also a point $P' = (\xi', \eta')$ of X_3 -coordinate

$$\xi' = -8\frac{t^3(t-1)^2}{\left(t-s^2\right)^2}$$
$$\eta' = \frac{i32}{19}\frac{\left(5\sqrt{2}+2\sqrt{3}\right)t^4\left(19t^2-(326+140\sqrt{6})t+931+380\sqrt{6}\right)}{\left(t-s^2\right)^3}.$$

We verify using definitions that $< P', Q >= 0$ and $h(P').h(Q) = 18$. So by Shioda-Tate formula ([21] [22]) P' and Q and $(0,0)$ generate the Mordell-Weil group.

4. 3-isogenies from Y_2 and from Y_{10}

4.1. **Generic 3-isogenies.** In [3], Bertin and Lecacheux exhibited all the elliptic fibrations of a generic member of the Apéry-Fermi pencil, called generic elliptic fibrations and found two 3-torsion elliptic fibrations defined by a Weierstrass equation, namely $E_{\#19}$ with rank 1 and $E_{\#20}$ with rank 0. We are giving their 3-isogenous $K3$ surface.

Theorem 4.1. *The* 3*-isogenous elliptic fibrations of fibration* #19 *(resp.* #20*) defined by Weierstrass equations* $H_{\#19}(k)$ *(resp.* $H_{\#20}(k)$*) are elliptic fibrations of the same* $K3$ *surface* N_k *with transcendental lattice* $\left(\begin{smallmatrix}0&0&3\\0&4&0\\3&0&0\end{smallmatrix}\right)$ *and discriminant form of its Néron-Severi lattice* $G_{NS} = \mathbb{Z}/3\mathbb{Z}(-\frac{2}{3}) \oplus \mathbb{Z}/12\mathbb{Z}(\frac{5}{12})$.

Proof. The 6-torsion elliptic fibration #20 has a Weierstrass equation

$$E_{\#20}(k) \qquad y^2-(t^2-tk+3)xy-(t^2-tk+1)y=x^3$$

with singular fibers $I_{12}(\infty)$, $2I_3(t^2-kt+1)$, $2I_2(0,k)$, $2I_1(t^2-kt+9)$ and 3-torsion point $(0,0)$. Using 2.5.2, it follows the Weierstrass equation of its 3-isogenous fibration

$$H_{\#20}(k)=E_{\#20}(k)/\langle(0,0)\rangle \qquad Y^2+3(t^2-tk+3)XY+t^2(t^2-tk+9)(t-k)^2Y=X^3$$

with singular fibers $2I_6(0,k)$, $I_4(\infty)$, $2I_3(t^2-kt+9)$, $2I_1(t^2-kt+1)$. Thus it is a rank 0 and 6-torsion elliptic fibration of a $K3$-surface with Picard number 19 and discriminant $\frac{6\times6\times3\times3\times4}{6\times6}=12\times3$.
Now we shall compute the Gram matrix $NS(20)$ of the Néron-Severi lattice of the $K3$ surface with elliptic fibration $H_{\#20}(k)$ in order to deduce its discriminant form.
Applying Shioda's result 2.3, we order the following elements as, s_0, F, $\theta_{0,i}$, $1\leq i\leq 4$, s_3, $\theta_{k,i}$, $1\leq i\leq 5$, $\theta_{\infty,i}$, $1\leq i\leq 3$, $\theta_{t_0,i}$, $1\leq i\leq 2$, $\theta_{t_1,i}$, $1\leq i\leq 2$, where s_0 and s_3 denotes respectively the zero and 3-torsion section, F the generic section, $\theta_{k,i}$ the components of reducible singular fiber s, t_0 and t_1 being roots of t^2-kt+9. We obtain

$$NS(20)=\begin{pmatrix}
-2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-2&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&-2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&-2&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&-2&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&-2&1&0&0&0&0&0&1&0&0&1&0&1\\
0&0&0&0&0&0&1&-2&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&-2&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&-2&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&-2&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&-2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-2&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&1&-2&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&-2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&1&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&-2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&-2
\end{pmatrix}.$$

We get $\det(NS(20))=12\times3$ and applying Shimada's lemma 2.1, the discriminant form $G_{NS(20)}\simeq\mathbb{Z}/3\oplus\mathbb{Z}/12$ is generated by vectors L_1 and L_2 satisfying $q_{L_1}=0$, $q_{L_2}=-\frac{11}{12}$ and $b(L_1,L_2)=\frac{1}{3}$.
Denoting $M(20)$ the following Gram matrix of the lattice $U(3)\oplus\langle4\rangle$,

$$M(20)=\begin{pmatrix}0&0&3\\0&4&0\\3&0&0\end{pmatrix},$$

we find for generators of its discriminant form the vectors

$$g_1=\begin{pmatrix}0\\0\\\frac{1}{3}\end{pmatrix} \qquad g_2=\begin{pmatrix}\frac{1}{3}\\\frac{1}{4}\\\frac{1}{3}\end{pmatrix}$$

satisfying $q_{g_1}=0$, $q_{g_2}=\frac{11}{12}$ and $b(g_1,g_2)=\frac{1}{3}$. We deduce that $M(20)$ is the transcendental lattice of the $K3$ surface with elliptic fibration $H_{\#20}(k)$.
A Weierstrass equation of the 3-torsion, rank 1, elliptic fibration #19 can be written as

$$E_{\#19}(k) \qquad y^2+ktxy+t^2(t^2+kt+1)y=x^3$$

with singular fibers $2IV^*(0,\infty)$, $2I_3(t^2+kt+1)$, $2I_1(k^3t-27kt-27t^2-27)$ and infinite order point $P=(-t^2,-t^2)$ of height $h(P)=\frac{4}{3}$. Using 2.5.2, its 3-isogenous elliptic fibration $E_{\#19}(k)/\langle(0,0)\rangle$ has a Weierstrass equation

$$H_{\#19}(k) \qquad Y^2-3ktXY-Yt^2(27t^2-k(k^2-27)t+27)=X^3.$$

It is a 3-torsion, rank 1, elliptic fibration of a $K3$-surface with Picard number 19 and singular fibers $2IV^*(0,\infty)$, $2I_3(27t^2-k(k^2-27)t+27)$, $2I_1(t^2+kt+1)$ and infinite order point Q with X-coordinate $X_Q=-3-3kt-(k^2+3)t^2-3kt^3-3t^4$ and height $h(Q)=4$. This point Q is the image of the point P in the 3-isogeny and non 3-divisible, hence generator of the non torsion part of the Mordell-Weil lattice. We deduce the discriminant of this $K3$-surface $\frac{3\times3\times3\times3\times4}{3\times3}=12\times3$.
Applying Shioda's result 2.3, we order the components of the singular fibers as, s_0, F, $\theta_{0,i}$, $1\leq i\leq 6$, $\theta_{\infty,i}$, $1\leq i\leq 6$, $\theta_{t_0,i}$, $1\leq i\leq 2$, s_3, $\theta_{t_1,2}$, s_∞, where s_0, s_3, s_∞ denotes respectively the zero, 3-torsion and infinite section, F the generic section, t_0 and t_1 being roots of $27t^2-k(k^2-27)t+27$. The numbering of components of IV^* is done using Bourbaki's notations [6]. It follows the Gram matrix $NS(19)$ of the corresponding $K3$ surface

$$NS(19)=\begin{pmatrix}
-2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&-2&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-2&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&1&-2&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&-2&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&-2&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-2&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-2&1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&-2&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&-2&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&-2&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&-2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-2&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-2&1&0&0\\
0&1&0&1&0&0&0&0&0&1&0&0&0&0&0&1&-2&1&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-2&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&-2
\end{pmatrix}.$$

Its determinant satisfies $\det(NS(19))=12\times3$ and according to Shimada's lemma 2.1, $G_{NS(19)}\simeq\mathbb{Z}/3\oplus\mathbb{Z}/12$ is generated by vectors M_1 and M_2 satisfying $q_{M_1}=-\frac{2}{3}$, $q_{M_2}=\frac{5}{12}$ and $b(M_1,M_2)=0$. We find also generators for the transcendental discriminant form $M(20)$

$$h_1=\begin{pmatrix}\frac{1}{3}\\0\\\frac{1}{3}\end{pmatrix}\qquad h_2=\begin{pmatrix}\frac{1}{3}\\\frac{1}{4}\\-\frac{1}{3}\end{pmatrix}$$

satisfying $q_{h_1}=\frac{2}{3}$, $q_{h_2}=-\frac{5}{12}$ and $b(h_1,h_2)=0$. We deduce that $M(20)$ is also the transcendental lattice of the $K3$ surface with elliptic fibration $H_{\#19}(k)$.
It follows the discriminant form of its Néron-Severi lattice,
$G_{NS(19)}=\mathbb{Z}/3\mathbb{Z}(-\frac{2}{3})\oplus\mathbb{Z}/12\mathbb{Z}(\frac{5}{12})$, which is also $G_{NS(20)}$ since the generators $L_1'=15L_1+4L_2$, $L_2'=L_1-L_2$ satisfy $q_{L_1'}=-\frac{2}{3}$, $q_{L_2'}=\frac{5}{12}$, $b(L_1',L_2')=0$.

□

We can prove the following specializations of N_k for $k=2$ and $k=10$.

Theorem 4.2. *For $k=2$, the $K3$ surface N_2 is Y_{10} with transcendental lattice* $[6\quad 0\quad 12]=T(Y_2)[3]$.
For $k=10$, the $K3$ surface N_{10} is the $K3$-surface with discriminant 72 *and transcendental lattice* $[4\quad 0\quad 18\ ]$.

Proof. To prove that $N_2=Y_{10}$ it is sufficient to prove that $H_{\#20}(2)$ is an elliptic fibration of Y_{10} since, by the previous theorem, $H_{\#19}(2)$ is another fibration of the same $K3$-surface.
But we see easily that $H_{\#20}(2)$ is the 6-torsion extremal fibration of Y_{10} numbered 8 in Shimada and Zhang [17].
Similarly, N_{10} is the $K3$ surface with transcendental lattice $[4\quad 0\quad 18]$ since $H_{\#20}(10)$ is a fibration of that surface according to the following proof.

A Weierstrass equation for $H_{\#20}\,(10)$ has a Weierstrass equation

$$Y^2+3\left(t^2-22\right)YX+\left(t^2-25\right)^2\left(t^2-16\right)Y=X^3$$

with singular fibers $I_4\,(\infty)\,,2I_6\,(\pm5)\,,2I_3\,(\pm4)\,,2I_1\left(t^2-24\right)$. We have a 2-torsion section $s_2=\left(-\left(t^2-25\right)^2,\left(t^2-25\right)^3\right)$ and a 6-torsion section
$s_6=\left(-\left(t^2-25\right)\left(t^2-16\right),-\left(t^2-25\right)\left(t^2-16\right)^2\right)$.
The section $P_w=\left(4\,(t+5)^2\,(t-4)\,,-\,(t+5)^4\,(t-4)^2\right)$ is of infinite order and is a generator of the Mordell Weil lattice. Moreover we have $\theta_{\pm5,1}.s_6=1,\theta_{\infty,2}.s_6=1,\theta_{\pm3,1}.s_6=1$. So the following divisor is 6 divisible

$$\sum_{i=1}^{5}\left(6-i\right)\theta_{\pm5,i}+3\theta_{\infty,1}+3\theta_{\infty,3}+4\theta_{\pm,4,1}+2\theta_{\pm4,2}\approx6s_6$$

So we can replace $\theta_{5,5}$ by s_6.
Moreover we can compute $s_6.P_w=1$ (for $t=-3$). We have also $\theta_{5,0}.P_w=1,\theta_{-5,4}.P_w=1$ and $\theta_{\infty,0}.P_w=1,\theta_{4,1}.P_w=1,\theta_{-4,0}.P_w=1$. All these computations give the Gram matrix of the Néron-Severi lattice of discriminant -72

$$\begin{pmatrix}
-2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1\\
0&0&-2&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&1&-2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&-2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&-2&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&-2&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-2&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&-2&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&-2&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&-2&0&0&0&0&0&0&0&0&1\\
0&1&1&0&0&0&0&1&0&0&0&-2&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&-2&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&-2&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&-2&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&-2&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&-2&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-2&0\\
0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&1&0&0&0&-2
\end{pmatrix}.$$

According to Shimada's lemma 2.1, $G_{NS}=\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/36$ is generated by the vectors L_1 and L_2 satisfying $q_{L_1}=\frac{-1}{2},q_{L_2}=\frac{-35}{36}$ and $b(L_1,L_2)=\frac{-1}{2}$.
Moreover the following generators of the discriminant group of the lattice with Gram matrix $M_{18}=\left(\begin{smallmatrix}4&0\\0&18\end{smallmatrix}\right)$, namely $f_1=(0,\frac{1}{2})$,$f_2=(\frac{1}{4},\frac{7}{18})$ verify $q_{f_1}=\frac{1}{2}$, $q_{f_2}=\frac{35}{36}$ and $b(f_1,f_2)=\frac{1}{2}$.
So the Gram matrix of the transcendental lattice is $\begin{pmatrix}4&0\\0&18\end{pmatrix}$.

Remark 4.1. Instead of proving that $H_{\#20}(10)$ is a fibration of the $K3$ surface with transcendental lattice $[4\quad0\quad18]$, we may prove that $H_{\#19}(10)$ is a fibration of the same $K3$ surface.
We have the Weierstrass equation

$$H_{\#19}\,(10):Y^2-30tYX+t^2\,(t-27)\,(27t-1)\,Y=X^3$$

with the two sections

$$P'=(-100t^2-3-3t(t+10)(1+t^2),\frac{1}{72}i\sqrt{3}(6t^2+10i\sqrt{3}t+30t+3i\sqrt{3}+3)^3)$$

image of the point $\left(-t^2,-t^2\right)$ on $E_{\#19}\,(10)$, and

$$P''=\left(-\,(t+1)\,(27t-1)\,,-\,(27t-1)\,(t+1)^3\right).$$

The Néron-Severi lattice, with the following basis ($s_0, F, \theta_{\infty,i}$ $1 \leq i \leq 6$, $\theta_0, i, 1 \leq i \leq 6$, $\theta_{t_0,1}, \theta_{t_0,2}$, $s_3, \theta_{t_1,2}, P', P''$) has for Gram matrix

$$\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 2 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & -2
\end{pmatrix}.$$

According to Shimada's lemma 2.1, $G_{NS} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/36$ is generated by the vectors L_1 and L_2 satisfying $q_{L_1} = \frac{-1}{2}, q_{L_2} = \frac{13}{36}$ and $b(L_1, L_2) = \frac{-1}{2}$.
Moreover the following generators of the discriminant group of the lattice with Gram matrix $M_{18} = \left(\begin{smallmatrix} 4 & 0 \\ 0 & 18 \end{smallmatrix}\right)$ namely $f_1 = (0, \frac{1}{2})$,$f_2 = (\frac{1}{4}, \frac{5}{18})$ verify $q_{f_1} = \frac{1}{2}$, $q_{f_2} = -\frac{13}{36}$ and $b(f_1, f_2) = \frac{1}{2}$.
So the transcendental lattice is $M_{18} = \begin{pmatrix} 4 & 0 \\ 0 & 18 \end{pmatrix}$.
An alternative proof: From the equation $H_{\#19}\,(10)$ and with the parameter $m = \frac{Y}{(t-27)^2}$ we obtain another elliptic fibration defined by the following cubic equation in W and t with $X = W\,(t-27)$

$$W^3 + 30Wtm - m\left(t^2\,(27t-1) + m\,(t-27)\right) = 0$$

and the rational point

$$W = 676\frac{m\,(m-1)}{m^2 - 648m + 27}, t = \frac{27m^2 - 648 + 1}{m^2 - 648m + 27}.$$

This fibration has a Weierstrass equation of the form

$$y^2 = x^3 - 3ax + \left(m + \frac{1}{m} - 2b\right)$$

with $a = 38425/9, b = -7521598/27$. So the Kummer surface associated is the product of two elliptic curves with $J, J' = 2950584125/27 \pm 1204567000/27\sqrt{6}$ that is $j, j' = 188837384000 \pm 77092288000\sqrt{6}$. So the fibration $H_{\#19}\,(10)$ corresponds to a surface with transcendental lattice of Gram matrix $\begin{pmatrix} 4 & 0 \\ 0 & 18 \end{pmatrix}$.

□

Theorem 4.3. *Define $Y_k^{(3)}$ the elliptic surface obtained by the base change τ of the elliptic fibration of Y_k with two singular fibers of type II^*, where τ is the morphism given by $u \mapsto h = u^3$. Then the $K3$ surface $Y_k^{(3)}$ has a genus one fibration without section such that its Jacobian variety satisfies $J\left(Y_k^{(3)}\right) = N_k$.*

Proof. Recall a Weierstrass equation for fibration #19 (see [3], Table 4)

$$(*) \qquad Y^2 + tkYX + t^2\,(t+s)\,(t+1/s)\,Y = X^3$$

where $k=s+\frac{1}{s}$. The fibration of Y_k with two singular fibers II^* can be obtained from (*) with the parameter $h_k=\frac{Y}{(t+s)^2}$ ([3], Table 3). The surface $Y_k^{(3)}$ is defined by $h_k=u^3$ and has then the following equation

$$u^3s\,(t+s)+tkuWs+t^2\,(ts-1)-W^3s=0,$$

where $X=(t+s)\,uW$.
We consider the fibration

$$\begin{aligned} Y_k^{(3)} &\to \mathbb{P}^1 \\ (u,t,W) &\mapsto t; \end{aligned}$$

this is a genus one fibration since we have a cubic equation in u, W.
However, this fibration seems to have no section. Nevertheless, taking its Jacobian fibration produces an elliptic fibration with section and the same fiber type.
If we make a base change of this fibration: $(t+s)=m^3$ then we obtain the following elliptic fibration with $U=um$.

$$U^3sm-\left(s-m^3\right)ksWU-W^3sm-\left(s-m^3\right)^2\left(s^2-sm^3-1\right)m=0.$$

The transformation

$$W=\frac{-24y+12(s^2+1)(s-m^3)x+\left(s-m^3\right)^2Q}{18sm(4x-m^2(s^2+1)^2)}$$

$$U=\frac{-24y-12(s^2+1)(s-m^3)x+\left(s-m^3\right)^2Q}{18sm(4x-m^2(s^2+1)^2)}$$

where $Q=\left(108m^6s^3+(s^6+105s^2-111s^4-1)m^3+s(s^2+1)^3\right)$ gives a Weierstrass equation, the point π_3 of x coordinate $\frac{1}{4}m^2(s^2+1)^2$ is a 3-torsion point. Taking again $m^3=(t+s)$, and $\pi_3=(X=0,Y=0)$, we recover a Weierstrass equation for the 3-isogenous fibration #19

$$Y^2-3tkYX-t^2\left(27t^2-k\left(k^2-27\right)t+27\right)Y=X^3$$

hence a fibration of N_k. Recall that the transcendental lattice of $Y_k^{(3)}$ is $T(Y_k)[3]$ [18]. □

4.2. **3-isogenies of Y_2.** Recall the results, given in [2], about the 4 elliptic fibrations of Y_2 with 3-torsion.

	Weierstrass Equation Singular Fibers	Rank
$\#20\,(7-w)$	$Y^2-\left(w^2+2\right)YX-w^2Y=X^3$ $I_{12}\,(\infty),\quad I_6\,(0),\quad 2I_2\,(\pm1),\quad 2I_1$	0
$\#19\,(8-b)$	$Y^2+2bYX+b^2\,(b+1)^2\,Y=X^3$ $2IV^*\,(\infty,0),\quad I_6\,(-1),\quad 2I_1$	1
$20-j$	$Y^2-4\left(j^2-1\right)YX+4\,(j+1)^2\,Y=X^3$ $I_{12}\,(\infty),\quad IV^*\,(-1),\quad I_2\left(-\frac{1}{2}\right),\quad 2I_1$	0
$21-c$	$Y^2+\left(c^2+5\right)YX+Y=X^3$ $I_{18}\,(\infty),\quad 6I_1$	1

Theorem 4.4. *The $K3$ surface Y_2 has 4 elliptic fibrations with 3-torsion, two of them being specializations. The 3-isogenies induce elliptic fibrations of Y_{10}.*

Proof. Using 2.5.2 we compute the 3-isogenous elliptic fibrations named H_w, H_b , H_j and H_c and given in the next table. To simplify we denote H_w (resp. H_b) the specialized elliptic fibration $H_{\#20}(2)$ (resp. $H_{\#19}(2)$).
We know from Theorem 4.2 that H_w and H_b are elliptic fibrations of Y_{10}; we present here a proof for H_j and H_c together with remarks using ideas from [10], [8], [18],[19].

	Weierstrass Equation Singular Fibers
H_w	$Y^2+3\left(w^2+2\right)YX+\left(w^2+8\right)\left(w^2-1\right)^2Y=X^3$ $I_4(\infty),\quad 2I_6(\pm 1),\quad I_2(0),\quad 2I_3$
H_b	$Y^2-6bYX+b^2\left(27b^2+46b+27\right)Y=X^3$ $2IV^*(\infty,0),\quad 2I_3,\quad I_2(-1)$
H_j	$Y^2+12\left(j^2-1\right)YX+4\left(4j^2-12j+11\right)(j+1)^2(2j+1)^2Y=X^3$ $I_4(\infty),\quad IV^*(-1),\quad I_6\left(\frac{-1}{2}\right),\quad 2I_3$
H_c	$Y^2-3\left(c^2+5\right)YX-\left(c^2+2\right)\left(c^2+c+7\right)\left(c^2-c+7\right)Y=X^3$ $I_6(\infty),\quad 6I_3$

Recall that the transcendental lattice of Y_{10} is $T(Y_2)[3]=T(Y_{10})$. Notice that the surface Y_2 has an elliptic fibration with singular fibers $2II^*(\infty,0), I_2, 2I_1$ and Weierstrass equation

$$E_h \qquad y^2=x^3-\frac{25}{3}x+h+\frac{1}{h}-\frac{196}{27} \quad \text{or} \quad y^2=z^3-5z^2+\frac{(h+1)^2}{h} \quad \text{with} \quad x=z-\frac{5}{3}.$$

The base change of degree 3, $h=u^3$ ramified at the two fibers II^* induces an elliptic fibration of the resulting $K3$ surface named $Y_2^{(3)}$ in [10]. As the transcendental lattice of the surface $Y_2^{(3)}$ is $T(Y_2)[3]$ [18], this surface $Y_2^{(3)}$ is Y_{10}. Moreover we can precise the fibration obtained: a Weierstrass equation is

$$E_u: Y^2=X^3-5u^2X^2+u^3(u^3+1)^2 \tag{4.1}$$

$$2I_0^*(\infty,0), 3I_2\left(u^3+1\right), 6I_1 \text{ rank } 7.$$

Now we are going to show that every elliptic fibration of Y_2 with 3-torsion is linked to the elliptic fibration of Y_2 with $2II^*(\infty,0), I_2, 2I_1$.
For the fibration $20-j$ we can obtain the elliptic fibration $2II^*(\infty,0), I_2, 2I_1$ from the Weierstrass equation given in the table and the elliptic parameter $h=Y$. So the fibration $20-j$ induces a fibration on Y_{10} with parameter j and an equation obtained after substitution of Y by u^3. So, with the previous computations 2.5.3, this is the 3-isogenous to $20-j$.
The same proof can be done for fibration $21-c$.

Remark 4.2. Moreover we can remark using 2.5.3 that the 3-isogenous to $21-c$ fibration has an equation

$$W^3+\left(c^2+5\right)ZW+1-Z^3=0.$$

Since the general elliptic surface with $(\mathbb{Z}/3\mathbb{Z})^2$ torsion is $x^3+y^3+t^3+3kxyt=0$, we deduce that the torsion on the fibration H_c induced on Y_{10} is $(\mathbb{Z}/3\mathbb{Z})^2$.

For the fibration #19 $(8-b)$ we obtain the elliptic fibration $2II^*(\infty,0), I_2, 2I_1$ from the Weierstrass equation given in the table and the elliptic parameter $h=\frac{Y}{(b+1)^2}$. Substituing h by u^3 and defining W as $X=(b+1)^2uW$ we obtain a cubic equation in u and W with a rational point $u=1, W=1$, so an elliptic fibration of Y_{10}

$$Y_{10}\to\mathbb{P}_1$$
$$(u,b,W)\mapsto b.$$

Computation gives the 3-isogenous elliptic curve to #19 $(8-b)$.
For the last fibration #20 $(7-w)$ the relation with the fibration $2II^*, I_2, 2I_1$ is less direct. □

Remark 4.3. With the previous method we can construct two elliptic fibrations of Y_{10} of rank 4. First from Weierstrass equation #20 $(7-w)$ and with the parameter $m=Y$ we obtain the fibration #1 $(11-f)$ of Y_2 ([3] last table) with singular fibers $II^*(\infty), III^*(0), I_4(1), I_1\left(\frac{32}{27}\right)$. A Weierstrass equation

$$E_m : Y_1^2 = X_1^3 - m(2m-3)X_1^2 + 3m^2\left(m-1\right)^2 X_1 + m^3\left(m-1\right)^4$$

is obtained with the following transformations

$$\begin{aligned} m &= Y, \quad X_1 = -\frac{Y\left(Y-1\right)^2}{X+1}, \quad Y_1 = w\frac{Y^2\left(Y-1\right)^2}{X+1} \\ w &= \frac{-Y_1}{X_1 m}, \quad X = -\frac{X_1 + m\left(m-1\right)^2}{X_1}, \quad Y = m. \end{aligned}$$

The base change $m = u'^3$ gives an elliptic fibration of Y_{10} with singular fibers $I_0^*(\infty)$, $III(0)$, $3I_4\left(1,j,j^2\right)$, $3I_1$, rank 4, a Weierstrass equation and sections

$$\begin{aligned} y'^2 &= x'^3 + u'^2x'^2 + 2u'\left(u'^3-1\right)x' + u'^3\left(u'^3-1\right)^2 \\ P &= \left(x_P\left(u'\right), y_P\left(u'\right)\right) = \left(-\left(u'^3-1\right), \left(u'-1\right)^2\left(u'^2+u'+1\right)\right) \\ Q &= \left(x_Q\left(u'\right), y_Q\left(u'\right)\right) = \left(-\left(u'+2\right)\left(u'^2+u'+1\right), 2i\sqrt{2}\left(u'^2+u'+1\right)^2\right). \end{aligned}$$

Also we have the points P' with $x_{P'} = jx_P\left(ju'\right)$ and Q' with $x_{Q'} = jx_Q\left(ju'\right)$ (with $j^3=1$). As explained in the next paragraph we can compute the height matrix and show that the Mordell-Weil lattice is generated by P, P', Q, Q' and is equal to $A_2\left(\frac{1}{4}\right)\oplus A_2\left(\frac{1}{2}\right)$.
The second example is obtained from #20 $(7-w)$ with the parameter $n=\frac{Y}{t^2}$ we obtain the fibration #9 $(12-g)$ of Y_2 ([3] last table)

$$E_n : y^2 = x^3 + 4x^2n^2 + n^3(n+1)^2x$$

with the following transformation

$$\begin{aligned} X &= \frac{x^2\left(x+2n^2\right)\left(n-1\right)}{y^2}, Y = \frac{1}{n}\frac{x^2\left(x+2n^2\right)^2}{y^2}, t = \frac{x\left(x+2n^2\right)}{ny} \\ x &= \frac{Y^2\left(Y-2X-t^2\right)}{Xt^4}, y = \frac{Y^3\left(Y-t^2\right)\left(Y-2X-t^2\right)}{X^2t^7}, n = \frac{Y}{t^2}. \end{aligned}$$

Notice that if $n=\frac{Y}{t^2}=v^3$ in E_w then we have the equation of H_w. More precisely if $X=tQv$, the equation becomes

$$-tv^3 + t^2Qv + 2Qv + Q^3 + t = 0,$$

a cubic equation in Q and v with a rational point $v=1, Q=0$. Easily we obtain H_w. So in the Weierstrass equation E_n, if we replace the parameter n by g^3 we obtain the following fibration of Y_{10}

$$y^2 = x^3 + 4g^2x^2 + g\left(g+1\right)^2\left(g^2-g+1\right)^2 x \tag{4.2}$$

with singular fibers $2III\left(0,\infty\right), 3I_4\left(-1, g^2-g+1\right), 3I_1\left(1, g^2+g+1\right)$ and rank 4. Notice the two infinite sections with x coordinates $(t+1)^2(t^2-t+1)$ and $-\frac{1}{3}(t-1)^2(t^2-t+1)$.

4.3. **Mordell-Weil group of E_u.** The aim of this paragraph is to construct generators of the Mordell-Weil lattice of the previous fibration of rank 7 with Weierstrass equation

$$E_u : Y^2 = X^3 - 5u^2X^2 + u^3(u^3+1)^2$$
$$2I_0^*\left(\infty,0\right), 3I_2\left(u^3+1\right), 6I_1.$$

Notice that the j-invariant of E_u is invariant by the two transformations $u \mapsto \frac{1}{u}$ and $u \mapsto ju$. These automorphisms of the base $\mathbb{P}^1$ of the fibration E_u can be extended to the sections as explained below.
Let $S_3 =< \gamma, \tau; \gamma^3 = 1, \tau^2 = 1 >$ be the non abelian group of order 6 and define an action of S_3 on the sections of E_u by

$$(X(u), Y(u)) \overset{\tau}{\mapsto} \left(u^4 X\left(\frac{1}{u}\right), u^6 Y\left(\frac{1}{u}\right)\right)$$
$$(X\left(u\right), Y\left(u\right)) \overset{\gamma}{\mapsto} \left(jX\left(ju\right), Y\left(ju\right)\right).$$

To obtain generators of E_u following Shioda [18] we use the rational elliptic surface $X^{(3)+}$ with $\sigma = u + \frac{1}{u}$ and a Weierstrass equation

$$E_\sigma : y^2 = x^3 - 5x^2 + \left(\sigma-1\right)^2\left(\sigma+2\right)$$
$$I_0^*\left(\infty\right), I_2\left(-1\right), 4I_1$$

of rank 3.
The Mordell-Weil lattice of a rational elliptic surface is generated by sections of the form $(a + b\sigma + c\sigma^2, d + e\sigma + f\sigma^2 + g\sigma^3)$. Moreover since we have a singular fiber of type I_0^* at ∞ the coefficients c and f, g are 0 [8]. So after an easy computation we find the 3 sections (with $j^3 = 1, i^2 = -1$).

$$q_1 = \left(-\left(\sigma-1\right), i\sqrt{2}\left(\sigma-1\right)\right)$$
$$q_2 = \left(-j\left(\sigma-1\right), \left(3+j\right)\left(\sigma-1\right)\right) \quad q_3 = \left(-j^2\left(\sigma-1\right), \left(3+j^2\right)\left(\sigma-1\right)\right).$$

These sections give the sections π_1, π_2, π_3 on E_u which are fixed by τ.

$$\pi_1 = \left(-u\left(u^2-u+1\right), i\sqrt{2}u^2\left(u^2-u+1\right)\right)$$
$$\pi_2 = \left(-ju\left(u^2-u+1\right), \left(3+j\right)u^2\left(u^2-u+1\right)\right)$$
$$\pi_3 = \left(-j^2u\left(u^2-u+1\right), \left(3+j^2\right)u^2\left(u^2-u+1\right)\right).$$

We notice $\rho_i = \gamma\left(\pi_i\right)$ and $\mu_i = \gamma^2\left(\pi_i\right)$ for $1 \leq i \leq 3$ which give 9 rational sections with some relations.
Moreover we have another section from the fibration E_h of rank 1.
The point of x coordinate $\frac{1}{16}\left(h^2 + \frac{1}{h^2}\right) - h - \frac{1}{h} + \frac{29}{24}$ is defined on $\mathbb{Q}\left(h\right)$. Passing to E_u we obtain $\omega =$

$$\left(\frac{1}{16}\frac{(1-16u^3+46u^6-16u^9+u^{12})}{u^4}, -\frac{1}{64}\frac{(u^6-1)(1-24u^3+126u^6-24u^9+1)}{u^6}\right).$$

We hope to get a generator system with π_i, ρ_i and ω so we have to compute the height matrix. The absolute value of its determinant is $\frac{81}{16}$. Since the discriminant of the surface is 72, we obtain a subgroup of index a with $\frac{81}{16} \times \frac{1}{a^2} \times 2^3 4^2 = 72$ so $a = 3$.

After some specializations of $u \in \mathbb{Z}$ (for example if $u = 11$, E_u has rank 3 on $\mathbb{Q}$) we find other sections with x coordinate of the shape $(au+b)\left(u^2-u+1\right)$

$$\begin{aligned}
\mu &= \left(-(u-1)\left(u^2-u+1\right), -\left(u^2-u+1\right)\left(u^2+2u-1\right)\right)\\
\mu_1 &= \left(-(u-9)\left(u^2-u+1\right), \left(u^2-u+1\right)\left(5u^2-18u+27\right)\right)\\
\mu_2 &= \left(-\left(u+\frac{1}{3}\right)\left(u^2-u+1\right), \frac{i\sqrt{3}}{9}\left(u^2-u+1\right)\left(9u^2+4u+1\right)\right).
\end{aligned}$$

We deduce the relations

$$\begin{aligned}
3\mu &= \omega + \pi_2 - \gamma(\pi_3) + \pi_3 - \gamma^2(\pi_2)\\
&= \omega + 2\pi_2 + \gamma(\pi_2) + \pi_3 - \gamma(\pi_3)
\end{aligned}$$

so, the Mordell-Weil lattice is generated by $\pi_j, \rho_j = \gamma(\pi_j)$ for $1 \leq j \leq 3$ and μ with Gram matrix

$$\begin{pmatrix}
1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0\\
-\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2}\\
0 & 0 & -\frac{1}{2} & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2}\\
0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & \frac{-1}{2}\\
0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{-1}{2} & 2
\end{pmatrix}.$$

4.4. **A fibration for Theorem 3.2.** From the previous fibration E_u we construct by a 2-neigbour method a fibration with a 2-torsion section used in Theorem 3.2.
We start from the Weierstrass equation (4.1)

$$Y^2 = X^3 - 5u^2X^2 + u^3\left(u^3+1\right)^2$$

and obtain another elliptic fibration with the parameter $m = \frac{X}{u(u^2-u+1)}$, which gives the Weierstrass equation

$$E_m : y^2 = x^3 - \left(m^3+5m^2-2\right)x^2 + \left(m^3+1\right)^2 x \tag{4.3}$$

with singular fibers $I_0^*(\infty)$, $3I_4\left(m^3+1\right)$, $I_2(0)$, $4I_1\left(1, -\frac{5}{3}, m^2-4m-4\right)$ and rank 4.

Remark 4.4. From this fibration with the parameter $q = \frac{y}{xm}$ we recover the fibration H_c.

4.5. **3-isogenies from** Y_{10}**.**

Theorem 4.5. *Consider the two* $K3$ *surfaces of discriminant* 72 *and of transcendental lattice* $[4\quad 0\quad 18]$ *or* $[2\quad 0\quad 36]$. *There exist elliptic fibrations of* Y_{10} *with a* 3*-torsion section inducing by* 3*-isogeny elliptic fibrations of one or the other previous surface.*

Proof. In Bertin and Lecacheux [4] we observe that the fibration numbered 89 of rank 0 has a 3-torsion section. A Weierstrass equation for the 3-isogenous fibration is

$$Y^2 + \left(-27t^2 - 18t + 27\right)YX + 27(4t+3)(5t-3)^2 Y = X^3$$

with singular fibers $I_9(\infty), I_6\left(\frac{3}{5}\right), I_4, (0), I_3\left(-\frac{3}{4}\right), 2I_1$.
From singular fibers, torsion and rank we see in Shimada and Zhang table [17] that it is the $n°48$ case. So the transcendental lattice of the surface is $\left(\begin{smallmatrix}4 & 0\\ 0 & 18\end{smallmatrix}\right)$.
In Bertin and Lecacheux [4] is given also a rank 2 elliptic fibration of Y_{10} numbered (11) with a 3-torsion section. We shall prove that this 3-torsion section induces by 3-isogeny an elliptic fibration of the $K3$ surface with transcendental lattice $[4\quad 0\quad 18]$.

Starting with the Weierstrass equation given in [4], after a translation to put the 3-torsion section in $(0,0)$ we obtain the following Weierstrass equation and p_1 and p_2 generators of the Mordell-Weil lattice

$$
\begin{aligned}
E_{11} : Y^2 + \left(t^2-4\right)YX + t^2\left(2t^2-3\right)Y = X^3 \\
p_1 = (6t^2, 27t^2), \qquad p_2 = (6i\sqrt{3}t - 3t^2, 27t^2) \\
2I_6\,(\infty,0)\,, \quad 2I_3\left(2t^2-3\right), \quad 2I_2\,(\pm1)\,, \quad 2I_1\,(\pm 8)\,.
\end{aligned} \tag{4.4}
$$

We see that the 3-isogenous elliptic fibration has a Weierstrass equation, generators of Mordell-Weil lattice and singular fibers

$$
\begin{aligned}
H_{11} : & Y^2 - 3\left(t^2-4\right)YX - \left(t^2-1\right)^2\left(t^2-64\right)Y = X^3 \\
& 2I_6\,(\pm1)\,, \quad 2I_3\,(\pm8)\,, \quad 2I_2\,(\infty,0)\,, \quad 2I_1\left(2t^2-3\right) \text{ of rank 2.}
\end{aligned}
$$

Notice the two sections

$$
\begin{aligned}
\pi_1 &= \left(-\frac{1}{4}\left(t^2-1\right)\left(t^2-64\right), \frac{1}{8}\left(t-8\right)\left(t-1\right)\left(t+1\right)^2\left(t+8\right)^2\right) \\
\omega &= \left(-7\left(t^2-1\right)^2, 49\alpha\left(t^2-1\right)^3\right) \quad \text{where } 49\alpha^2+20\alpha+7=0.
\end{aligned}
$$

So, computing the height matrix of π_1 and ω, we see the discriminant is 72.
For each reducible fiber at $t=i$ we denote (X_i, Y_i) the singular point of H_{11}

$$
\begin{array}{ccc}
t=\pm1 \quad t=\pm 8 & t=0 & t=\infty \\
(X_{\pm1}=0, Y_{\pm1}=0) & (X_0=-16, Y_0=64) & (x_\infty=-1, y_\infty=-1) \\
(X_{\pm8}=0, Y\pm_8=0) & &
\end{array}
$$

where if $t=\infty$ we substitute $t=\frac{1}{T}$, $x=T^4X, y=T^6Y$. We notice also $\theta_{i,j}$ the j-th component of the reducible fiber at $t=i$. A section $M=(X_M,Y_M)$ intersects the component $\theta_{i,0}$ if and only if $(X_M,Y_M) \not\equiv (X_i,Y_i) \bmod (t-i)$. Using the additivity on the component, we deduce that ω does not intersect $\theta_{i,0}, 2\omega$ intersects $\theta_{i,0}$ and so ω intersects $\theta_{i,3}$ for $i=\pm1$. Also ω intersects $\theta_{i,0}$ for $i=\pm 8$, $i=0$ and ∞.
For π_1 we compute $k\pi_1$ with $2 \leq k \leq 6$. For $i=\pm1$, only $6\pi_1$ intersects $\theta_{i,0}$ so π_1 intersects $\theta_{i,1}$. (this choice 1, not 5, fixes the numbering of components). For $i=\pm8$, only $3\pi_1$ intersects $\theta_{i,0}$, so π_i intersects $\theta_{i,1}$. Modulo t, we get $\pi_1=(-16,64)$, so π_1 intersects $\theta_{0,1}$, and π_1 intersects $\theta_{\infty,0}$.
As for the 3-torsion section $s_3=(0,0)$, s_3 intersects $\theta_{i,2}$ or $\theta_{i,4}$ if $i=\pm1$. Computing $2\pi_1-s_3$, we see that s_3 intersects $\theta_{1,2}$ and $\theta_{-1,4}$.
For $i=\pm8$, we compute π_1-s_3, for $t=8$ and show that s_3 intersects $\theta_{8,2}$ and $\theta_{-8,1}$. For $t=0$ and $t=\infty$ s_3 intersects the 0 component.
So we can compute the relation between the section s_3 and the $\theta_{i,j}$ and find that $3s_3 \approx -2\theta_{1,1} - 4\theta_{1,2} - 3\theta_{1,3} - 2\theta_{1,4} - \theta_{1,5}$. Thus, we can choose the following base of the Néron-Severi lattice ordered as $s_0, F, \theta_{1,j}$, with $1 \leq j \leq 4$, $s_3, \theta_{-1,k}$ with $1 \leq k \leq 5, \theta_{8,k}, k=1,2$, $\theta_{-8,k}, k=1,2$ and $\theta_{0,1}$, $\theta_{\infty,1}, \omega, \pi_1$.

The last remark is that only the two sections ω and π_1 intersect. So we can write the Gram matrix NS of the Néron-Severi lattice,

$$\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 1 & -2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0\\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1\\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & -2
\end{pmatrix}.$$

According to Shimada's lemma 2.1, $G_{NS} \equiv \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/36\mathbb{Z}$ is generated by the vectors L_1 and L_2 satisfying $q_{L_1} = -\frac{1}{2}$, $q_{L_2} = \frac{37}{36}$, and $b(L_1, L_2) = \frac{1}{2}$.
Moreover the following generators of the discriminant group of the lattice with Gram matrix $M_{18} = \left(\begin{smallmatrix} 4 & 0 \\ 0 & 18 \end{smallmatrix}\right)$ namely $f_1 = (0, \frac{1}{2})$, $f_2 = (\frac{1}{4}, \frac{7}{18})$ verify $q_{f_1} = \frac{1}{2}$, $q_{f_2} = -\frac{37}{36}$ and $b(f_1, f_2) = -\frac{1}{2}$.
So the Gram matrix of the transcendental lattice of the surface is M_{18}.
A 3-isogeny between two elliptic fibrations of Y_{10} and the $K3$ surface with transcendental lattice $[2 \quad 0 \quad 36]$ can also be obtained. We have shown in remark 4.2 that H_c has a $(Z/3Z)^2$- torsion group and exhibited a 3-isogeny between some elliptic fibrations of Y_{10} and Y_2. Notice that with the Weierstrass equation H_c the point σ_3 of X coordinate $-\left(c^2+c+7\right)\left(c^2-c+7\right)$ defines a 3-torsion section. After a translation to put this point in $(0,0)$ and scaling, we obtain a Weierstrass equation

$$Y'^2 - (t^2+11)Y'X' - (t^2+t+7)(t^2-t+7)Y' = X'^3.$$

The 3-isogenous curve of kernel $<\sigma_3>$ has a Weierstrass equation

$$y^2 + 3\left(t^2+11\right)xy + \left(t^2+2\right)^3 y = x^3,$$

with singular fibers $I_2(\infty)$, $2I_9\left(t^2+2\right)$, $4I_1\left(t^4+13t^2+49\right)$.
The section $P_c = \left(-\frac{1}{4}\left(t^4+t^2+1\right), -\frac{1}{8}\left(t-j\right)^3\left(t+j^2\right)^3\right)$ where $j = \frac{-1+i\sqrt{3}}{2}$, of infinite order, generates the Mordell-Weil lattice.
We consider the components of the reducible fibers in the following order $\theta_{i\sqrt{2},j}$, $j \leq 1 \leq 8$, $\theta_{-i\sqrt{2},k}$, $1 \leq k \leq 8$, $\theta_{\infty,1}$.
The 3-torsion section $s_3 = (0,0)$ and the previous components are linked by the relation

$$3s_3 \approx -\theta_{i\sqrt{2},8} + \sum a_{i,j}\theta_{i,j}.$$

So we can replace, in the previous ordered sequence of components, the element $\theta_{i\sqrt{2},8}$ by s_3. We notice that $(s_3.P_c) = 2$, $\left(P_c.\theta_{\pm i\sqrt{2},0}\right) = 1$ and $(P_c.\theta_{\infty,0}) = 1$, so the Gram matrix of the Néron-Severi lattice NS is

$$\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2
\end{pmatrix}.$$

Its determinant is -72. According to Shimada's lemma 2.1, $G_{NS} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/36$ is generated by the vectors L_1 and L_2 satisfying $q_{L_1} = \frac{-1}{2}, q_{L_2} = \frac{5}{36}$ and $b(L_1, L_2) = \frac{1}{2}$.
Moreover the following generators of the discriminant group of the lattice with Gram matrix $M_{36} = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 36 \end{smallmatrix}\right)$, namely $f_1 = (\frac{1}{2}, 0)$, $f_2 = (\frac{-1}{2}, \frac{-7}{36})$ verify $q_{f_1} = \frac{1}{2}$, $q_{f_2} = -\frac{5}{36}$ and $b(f_1, f_2) = \frac{-1}{2}$.
So the transcendental lattice is $M_{36} = \begin{pmatrix} 2 & 0 \\ 0 & 36 \end{pmatrix}$.

□

5. Isogenies and L-series of singular $K3$ surfaces

We notice that, along all the previous computations, the discriminants of the $K3$ surface Y_2 (resp. Y_{10}) and their 2 or 3 -isogenous $K3$ surfaces are the same up to square. It is indeed a corollary of the following theorem about the L-series of a singular $K3$ surface and some of its 2-or 3-isogenous. Let us recall first the following results.

Theorem 5.1 (Tate's isogeny theorem)**.** [23] *The fact that two elliptic curves E_1 and E_2 defined over $\mathbb{F}_q$ are isogenous is equivalent to the fact they have the same number of $\mathbb{F}_q$ points.*

Lemma 5.1. [1] *Let Y an elliptic $K3$-surface defined over $\mathbb{Q}$ by a Weierstrass equation $Y(t)$.
If rank $(Y(t)) = r$ and the r infinite sections generating the Mordell-Weil lattice are defined respectively over $\mathbb{Q}(\sqrt{d_i})$, $i = 1, ..., r$, then*

$$A_p = -\sum_{t \in \mathbb{P}^1(\mathbb{F}_p), \quad Y(t) \quad smooth} a_p(t) - \sum_{t \in \mathbb{P}^1(\mathbb{F}_p), \quad Y(t) \quad singular} \epsilon_p(t) - \sum_{i=1}^{r} \left(\frac{d_i}{p}\right) p$$

where

$$a_p(t) = p + 1 - \#Y(t)(\mathbb{F}_p)$$

and $\epsilon_p(x)$ defined by

$$\epsilon_p(t) = \begin{cases} 0, & \text{if the reduction of } Y(t) \text{ is additive} \\ 1, & \text{if the reduction of } Y(t) \text{ is split multiplicative} \\ -1, & \text{if the reduction of } Y(t) \text{ is non split multiplicative} \end{cases}.$$

Theorem 5.2. *The L-series of the transcendental lattice of a singular $K3$ surface Y defined over $\mathbb{Q}$ is inchanged by a 2 or a 3-isogeny whose kernel is defined over $\mathbb{Q}(t)$ and obtained from an elliptic fibration whose infinite sections (if any) are defined on $\mathbb{Q}$ or on a quadratic number field.*

Proof. Denote $Y(t)$ (resp. $\widetilde{Y(t)}$) a Weierstrass equation of a singular $K3$ surface (resp. of its 2 or 3 isogenous curve).
The coefficients of the newform associated to the L-series of the K3 surface are given in the previous lemma.
1) Suppose $\widetilde{Y(t)}$ is the Weierstrass equation of its 2-isogenous.
We get

$$(Y(t)) \qquad y^2 = x^3 + a(t)x^2 + b(t)x \qquad\qquad (\widetilde{Y(t)}) \qquad Y^2 = X^3 - 2a(t)X^2 + (a(t)^2 - 4b(t))X$$

and

$$(X, Y) = (\frac{x^2 + a(t)x + b(t)}{x}, y\frac{b(t) - x^2}{x^2}).$$

Hence, since the ranks of $Y(t)$ and its 2-isogenous are the same, if $Y(t)$ has r infinite sections defined on $\mathbb{Q}(\sqrt{d_i})$ it is similar for $\widetilde{Y(t)}$.
If $t \in \mathbb{P}^1(\mathbb{F}_p)$ satisfies $Y(t)$ smooth, t is not a root of $\Delta = 16b^2(a^2 - 4b)$ and also not a root of $\widetilde{\Delta} = 256b(a^2 - 4b)^2$; hence $\widetilde{Y(t)}$ is also smooth. For these t, using Tate's isogeny theorem, we find $a_p(t) = \widetilde{a_p(t)}$.
Suppose $Y(t_0)$ singular i.e. either $b(t_0) = 0$ or $a(t_0)^2 = 4b(t_0)$ (in these cases the reduction of $Y(t_0)$ is multiplicative).
Suppose $b(t_0) = 0$, we get

$$(Y(t_0)) \qquad y^2 = x^2(x + a(t_0)) \qquad\qquad (\widetilde{Y(t_0)}) \qquad Y^2 = (X - a(t_0))^2 X = U^2(U + a(t_0)).$$

Hence $Y(t_0)$ and $\widetilde{Y(t_0)}$ have the same multiplicatice reduction, either split if $a(t_0)$ is a square modulo p or non split if $a(t_0)$ is not a square modulo p.
Suppose now $a(t_0)^2 = 4b(t_0)$, we get

$$(Y(t_0)) \qquad y^2 = x(x + a(t_0)/2)^2 = U^2(U - a(t_0)/2) \qquad\qquad (\widetilde{Y(t_0)}) \qquad Y^2 = X^2(X - 2a(t_0)).$$

Similarly, if $-a(t_0)/2$ or equivalently $-2a(t_0)$ is a square (resp. not a square) modulo p, the reduction is split (resp. non split) multiplicative. Thus $Y(t_0)$ and $\widetilde{Y(t_0)}$ have the same type of multiplicative reduction.
Finally when both $a(t_0) = 0$ and $b(t_0) = 0$, the reduction of $Y(t_0)$ and $\widetilde{Y(t_0)}$ is additive.
Thus we have proved that $A_p = \widetilde{A_p}$, that is the 2-isogenous $K3$ surface has the same transcendental L-series as Y.
2) Suppose $\widetilde{Y(t)}$ is the Weierstrass equation of its 3-isogenous.
Since we want to apply Tate's isogeny theorem we need a 3-isogeny defined over $\mathbb{Q}$ whose kernel is defined over $\mathbb{Q}(t)$. Using the formulae of 2.5.1, we get

$$(Y(t)) \qquad y^2 + a(t)xy + b(t)y = x^3$$

$$(\widetilde{Y(t)}) \qquad Y^2 + a(t)XY + 3b(t)Y = X^3 - 6a(t)b(t)X - b(t)(a(t)^3 + 9b(t))$$

and

$$X = \frac{x^3 + a(t)b(t)x + b(t)^2}{x^2}$$

$$Y = \frac{y(x^3 - a(t)b(t)x - 2b(t)^2) - b(t)(x^3 + a(t)^2x^2 + 2a(t)b(t)x + b(t)^2)}{x^3}.$$

Hence, since the ranks of $Y(t)$ and its 3-isogenous curve are the same, if $Y(t)$ has r infinite sections defined on $\mathbb{Q}(\sqrt{d_i})$ it is similar for $\widetilde{Y(t)}$.

If $t \in \mathbb{P}^1(\mathbb{F}_p)$ satisfies $Y(t)$ smooth, t is not a root of $\Delta = b^3(a^3 - 27b)$ and also not a root of $\widetilde{\Delta} = b(a^3 - 27b)^3/16$; hence $\widetilde{Y(t)}$ is also smooth. For these t, using Tate's isogeny theorem, we find $a_p(t) = \widetilde{a_p(t)}$.
Suppose $Y(t_0)$ singular i.e. either $b(t_0) = 0$ or $a(t_0)^3 = 27b(t_0)$ (in these cases the reduction of $Y(t_0)$ is multiplicative).
Suppose $b(t_0) = 0$. We get

$$(Y(t_0)) \qquad y^2 + a(t_0)xy = x^3 \qquad\qquad (\widetilde{Y(t_0)}) \qquad Y^2 + a(t_0)XY = X^3.$$

Hence the two curves have the same multiplicative reduction.
Suppose $b(t_0) = a(t_0)^3/27$. Putting at the origin the singular point $(-\frac{a(t_0)^2}{9}, \frac{a(t_0)^3}{27})$ of $Y(t_0)$ (resp. $(-\frac{a(t_0)^2}{3}, \frac{a(t_0)^3}{9})$ of $\widetilde{Y(t_0)}$, it follows

$$(Y(t_0)) \qquad y_1^2 + a(t_0)x_1y_1 = x_1^3 - a(t_0)^2x_1^2/3 \qquad ((\widetilde{Y(t_0)}) \qquad y_2^2 + a(t_0)x_2y_2 = x_2^3 - a(t_0)^2x_2^2.$$

Since their respective discriminants are $x_1^2(4x_1 - a(t_0)^2/3)$ and $x_2^2(4x_2 - 3a(t_0)^2)$, the two curves have the same multiplicative reduction.
Thus we have proved that $A_p = \widetilde{A_p}$, that is the 3-isogenous $K3$ surface has the same transcendental L-series as Y.

□

Corollary 5.1. *A singular $K3$ surface Y as in Theorem 5.2 and its 2 or 3-isogenous surface have their discriminants equal up to square.*

Proof. This is a consequence of a Schütt's theorem.

Theorem 5.3. *(Schütt's classification)* [15] *Consider the following classification of singular $K3$-surfaces over $\mathbb{Q}$*

(1) by the discriminant d of the transcendental lattice of the surface up to squares,
(2) by the associated newform up to twisting,
(3) by the level of the associated newform up to squares,
(4) by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.

Then, all these classifications are equivalent. In particuliar, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.

□

5.1. **Isogenies as isometries of the rational transcendental lattice.** Denoting the rational transcendental lattice $T(X)_{\mathbb{Q}} := T(X) \otimes \mathbb{Q}$, we recall that $T(X)_{\mathbb{Q}}$ and $T(Y)_{\mathbb{Q}}$ are isometric if they define congruent lattices, that is if there exists $M \in Gl_n(\mathbb{Q})$ satisfying $T(X)_{\mathbb{Q}} =^t MT(Y)_{\mathbb{Q}}M$.
Bessière, Sarti and Veniani proved the following theorem [5].

Theorem 5.4. [5] *Let $\gamma : X \to Y$ be a p-isogeny between complex projective $K3$ surfaces X and Y. Then $rk(T(Y)_{\mathbb{Q}}) = rk(T(X)_{\mathbb{Q}}) =: r$ and*

(1) If r is odd, there is no isometry between $T(Y)_{\mathbb{Q}}$ and $T(X)_{\mathbb{Q}}$.
(2) If r is even, there exists an isometry between $T(Y)_{\mathbb{Q}}$ and $T(X)_{\mathbb{Q}}$ if and only if $T(Y)_{\mathbb{Q}}$ is isometric to $T(Y)_{\mathbb{Q}}(p)$. This property is equivalent to the following:
   a) If $p = 2$, for every prime number q congruent to 3 or 5 modulo 8, the q-adic valuation $\nu_q(\det T(Y))$ is even.
   b) If $p > 2$, for every prime number $q > 2$, $q \neq p$, such that p is not a square in $\mathbb{F}_q$, the number $\nu_q(det(T_Y)$ is even and the following equation holds in $\mathbb{F}_p^*/(\mathbb{F}_p^*)^2$

$$res_p(det(T_y) = (-1)^{\frac{p(p-1)}{2} + \nu_p(det(T_Y))}$$

where $res_p(det(T_Y)) = \frac{det(T_Y)}{p^{\nu_p(det(T_Y))}}$.

This theorem allows us to find 2-isogenies as self isogenies on Y_2 and Y_{10}. In a previous paper we gave all the 2-isogenies of Y_2 and exhibited self isogenies on Y_2. In section 3 we also exhibited 2-isogenies as self isogenies on Y_{10}.
In section 4 we proved that all the 3- isogenies on Y_2 are between Y_2 and Y_{10} and some 3-isogenies on Y_{10} are between Y_{10} and other $K3$ surfaces with dicriminant 72, namely $[4 \quad 0 \quad 18]$ or $[2 \quad 0 \quad 36]$. These results illustrate Bessière, Sarti, Veniani's theorem. Indeed $det(T(Y_2)) = 8$, hence $res_3(8) = 8$ which is congruent modulo 3 to $(-1)^3$ and $det(T(Y_{10})) = 8 \times 9$, hence $res_3(8 \times 9) = 8$ which is congruent modulo 3 to $(-1)^{3+2}$. And, since

$$T_{\mathbb{Q}}(Y_2) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \qquad T_{\mathbb{Q}}(Y_{10}) = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix} \qquad T_{\mathbb{Q}}([4 \quad 0 \quad 18]) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \qquad T_{\mathbb{Q}}([2 \quad 0 \quad 36]) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

we find, as expected, these matrices are isometric since

$$\begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Some remarks
As a consequence of Bessière, Sarti and Veniani's theorem, there could be 2 or 3-self (either PF or EF) elliptic fibrations on Y_2 and on Y_{10}. Indeed we found 2-self isogenies on both Y_2 and Y_{10}. As for 3-isogenies, there is no self-isogeny on Y_2 and also probably none on Y_{10}. Concerning rank 0 elliptic fibrations, using Shimada and Zhang's table [17], we recover easily all our results without using Weierstrass equations. We have only to know the transform by a 2- or a 3-isogeny of a type of singular fiber. This can be obtained using Tate's algorithm [24] and an analog of Dockchitzer's remark [7].

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