Mahler measure of K3 surfaces

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Introduced by Mahler in 1962, the logarithmic Mahler measure of a polynomial $P$ is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{T^n} \log |P(x_1, \cdots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and its Mahler measure

$$M(P) = \exp(m(P))$$

where

$$T^n = \{ (x_1, \cdots, x_n) \in \mathbb{C}^n / |x_1| = \cdots = |x_n| = 1 \}.$$
By Jensen’s formula, if $P \in \mathbb{Z}[X]$ is monic, then

$$M(P) = \prod_{P(\alpha)=0} \max(|\alpha|, 1).$$

So it is related to Lehmer’s question (1933)
Does there exist $P \in \mathbb{Z}[X]$, monic, non cyclotomic, satisfying

$$1 < M(P) < M(P_0) = 1.1762 \cdots?$$

The polynomial

$$P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

is the Lehmer polynomial, in fact a Salem polynomial.
Lehmer’s problem is still open.
A partial answer by Smyth (1971)

\[ M(P) \geq 1.32 \cdots \]

if \( P \) is non reciprocal.
The story can be explained with polynomials

\[ x_0 + x_1 + x_2 + \cdots + x_n. \]

- \( m(x_0 + x_1) = 0 \) (by Jensen’s formula)
- \( m(x_0 + x_1 + x_2) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \) Smyth (1980)
- \( m(x_0 + x_1 + x_2 + x_3) = \frac{7}{2\pi^2} \zeta(3) \) Smyth (1980)

These are the first explicit Mahler measures.
\[ m(x_0 + x_1 + x_2 + x_3 + x_4) \geq \frac{675\sqrt{15}}{16\pi^3} L(f, 4) \] conjectured by Villegas (2004)

\( f \) cusp form of weight 3 and conductor 15

\( L(f, s) \) is also the L-series of the K3 surface defined by

\[ x_0 + x_1 + x_2 + x_3 + x_4 = 0 \]

\[ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = 0 \]
How such a conjecture possible?
Because of deep insights of two people.

- **Deninger (1996)** who conjectured

\[
m(x + \frac{1}{x} + y + \frac{1}{y} + 1) = \frac{15}{4\pi^2} L(E, 2) = L'(E, 0)
\]

*E* elliptic curve of conductor 15 defined by the polynomial

This conjecture was proved recently (May 2011) by Rogers and Zudilin thanks to a previous result due to Lalin. Here the polynomial is reciprocal.
Maillot (2003) using a result of Darboux (1875): the Mahler measure of $P$ which is the integration of a differential form on a variety, when $P$ is non reciprocal, is in fact an integration on a smaller variety and the expression of the Mahler measure is encoded in the cohomology of the smaller variety.
Examples

- $n = 2$ The smaller variety is defined by

\[
x_0 + x_1 + x_2 = 0
\]

\[
\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} = 0 \iff x_1^2 + x_2^2 + x_1x_2 = 0
\]

It is a curve of genus 0. So $m(x_0 + x_1 + x_2)$ is expressed as a Dirichlet L-series.

- $n = 3$ The smaller variety is defined by

\[
x_0 + x_1 + x_2 + x_3 = 0
\]

\[
\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0 \iff (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) = 0
\]

It is the intersection of 3 planes. Thus Smyth’s result.
n = 4 (Villegas’s Conjecture) The smaller variety is defined by

\[ x_0 + x_1 + x_2 + x_3 + x_4 = 0 \]
\[ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = 0 \]

It is the modular K3-surface studied by Peters, Top, van der Vlugt defined by a reciprocal polynomial. Its L-series is related to \( f \).

n = 5 (Villegas’s Conjecture again)

\[ m(x_0 + x_1 + x_2 + x_3 + x_4 + x_5) = \ast \ast L(g, 5) \]

\( g \) cusp form of weight 4 and conductor 6 related to L-series of the Barth-Nieto quintic.
Barth-Nieto quintic

It the 3-fold compactification of the complete intersection of

\[ x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0 \]
\[
\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} = 0
\]

It has been studied by Hulek, Spandaw, Van Geemen, Van Straten in 2001. They proved that the $L$-function of the quintic (i.e. of their third etale cohomology group) is modular, a fact predicted by a conjecture of Fontaine and Mazur.

The modular form is the newform of weight 4 for $\Gamma_0(6)$

\[ f = (\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2 \]
Briefly, to guess the Mahler measure of a non reciprocal polynomial we need results on reciprocal ones. In particular, it is very important to collect many examples of Mahler measures of $K3$-hypersurfaces. Notice that Maillot’s insight predicts only the type of formula expected. Also Deninger’s guess comes from Beilinson’s Conjectures.
So replace $E$ by a surface $X$ which is also a Calabi-Yau variety, i.e. a $K3$-surface and try to answer the questions:

What are the analog of Deninger, Boyd, R-Villegas’s results and conjectures?
Which type of Eisenstein-Kronecker series corresponds to $L(X, 3)$?
Basic facts on K3-surfaces

Our results concern polynomials of the family

\[ P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k \]

defining K3-surfaces \( Y_k \). What’s a K3-surface?
It is a smooth surface \( X \) satisfying

- \( H^1(X, \mathcal{O}_X) = 0 \) i.e. \( X \) simply connected
- \( K_X = 0 \) i.e. the canonical bundle is trivial i.e. there exists a unique, up to scalars, holomorphic 2-form \( \omega \) on \( X \).
A double covering branched along a plane sextic for example defines a K3-surface $X$.

In our case

$$(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4$$

Main properties

- $H_2(X, \mathbb{Z})$ is a free group of rank 22.
Main properties (continued)

- With the intersection pairing, $H_2(X, \mathbb{Z})$ is a lattice and

$$H_2(X, \mathbb{Z}) \cong U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

$\mathcal{L}$ is the $K3$-lattice, $U_2$ the hyperbolic lattice of rank 2, $E_8$ the unimodular lattice of rank 8.

- $Pic(X) \subset H_2(X, \mathbb{Z}) \cong Hom(H^2(X, \mathbb{Z}), \mathbb{Z})$

where $Pic(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles (since for $K3$ surfaces linear and algebraic equivalence are the same).

- $Pic(X) \cong \mathbb{Z}^\rho(X)$

$\rho(X) :=$ Picard number of $X$

$1 \leq \rho(X) \leq 20$
\[
T(X) := (Pic(X))^\perp
\]

is the transcendental lattice of dimension \(22 - \rho(X)\).

If \(\{\gamma_1, \cdots, \gamma_{22}\}\) is a \(\mathbb{Z}\)-basis of \(H_2(X, \mathbb{Z})\) and \(\omega\) the holomorphic 2-form,

\[
\int_{\gamma_i} \omega
\]

is called a period of \(X\) and

\[
\int_{\gamma} \omega = 0 \text{ for } \gamma \in Pic(X).
\]

If \(\{X_z\}\) is a family of K3 surfaces, \(z \in \mathbb{P}^1\) with generic Picard number \(\rho\) and \(\omega_z\) the corresponding holomorphic 2-form, then the periods of \(X_z\) satisfy a Picard-Fuchs differential equation of order \(k = 22 - \rho\). For our family \(k = 3\).
In fact, by Morrison, a $\mathcal{M}$-polarized $K3$-surface, with Picard number 19 has a Shioda-Inose structure, that means

\[ X \rightarrow Y = \text{Kum}(A/\pm 1) \]

\[ A = E \times E / C_N \]

If the Picard number $\rho = 20$, then the elliptic curve is CM.
Mahler measure of $P_k$

**Theorem**

(B. 2005) Let $k = t + \frac{1}{t}$ and

$$t = \left( \frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} \right)^6, \quad \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n \geq 1} (1 - e^{2\pi i n \tau}), \quad q = \exp 2\pi i \tau$$

$$m(P_k) = \frac{\mathbb{Q}_\tau}{8\pi^3} \left\{ \sum_{m, \kappa}^\prime \left( -4(2\Re) \frac{1}{(m\tau + \kappa)^3(m\bar{\tau} + \kappa)} + \frac{1}{(m\tau + \kappa)^2(m\bar{\tau} + \kappa)^2} \right) + 16(2\Re) \frac{1}{(2m\tau + \kappa)^3(2m\bar{\tau} + \kappa)} + \frac{1}{(2m\tau + \kappa)^2(2m\bar{\tau} + \kappa)^2} \right)$$

$$- 36(2\Re) \frac{1}{(3m\tau + \kappa)^3(3m\bar{\tau} + \kappa)} + \frac{1}{(3m\tau + \kappa)^2(3m\bar{\tau} + \kappa)^2}$$

$$+ 144(2\Re) \frac{1}{(6m\tau + \kappa)^3(6m\bar{\tau} + \kappa)} + \frac{1}{(6m\tau + \kappa)^2(6m\bar{\tau} + \kappa)^2} \right\}$$
Sketch of proof

Let
\[ P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k \]
defining the family \((X_k)\) of K3-surfaces.

- For \(k \in \mathbb{P}^1\), generically \(\rho = 19\).
- The family is \(\mathcal{M}_k\)-polarized with
  \[ \mathcal{M}_k \cong \mathbb{U}_2 \perp (-E_8)^2 \perp \langle -12 \rangle \]
- Its transcendental lattice satisfies
  \[ T_k \cong \mathbb{U}_2 \perp \langle 12 \rangle \]
- The Picard-Fuchs differential equation is
  \[ (k^2 - 4)(k^2 - 36)y''' + 6k(k^2 - 20)y'' + (7k^2 - 48)y' + ky = 0 \]
The family is modular in the following sense if $k = t + \frac{1}{t}$, $\tau \in \mathcal{H}$ and $\tau$ as in the theorem

$$t\left(\frac{a\tau + b}{c\tau + d}\right) = t(\tau) \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6, 2)^* \subset \Gamma_0(12)^* + 12$$

where

$$\Gamma_1(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \ (6) \ c \equiv 0 \ (6) \right\}$$

$$\Gamma_1(6, 2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6) \ c \equiv 6b \ (12) \right\}$$

and

$$\Gamma_1(6, 2)^* = \langle \Gamma_1(6, 2), w_6 \rangle$$
The P-F equation has a basis of solutions $G(\tau), \tau G(\tau), \tau^2 G(\tau)$ with

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

satisfying

$$G(\tau) = F(t(\tau)), \quad F(t) = \sum_{n \geq 0} v_n t^{2n+1}, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$\frac{dm(P_k)}{dk}$ is a period, hence satisfies the P-F equation

$$\frac{dm(P_k)}{dk} = G(\tau)$$

$dm(P_k) = -G(\tau) \frac{dt}{t} \frac{1 - t^2}{t}$

is a weight 4 modular form for $\Gamma_1(6, 2)^*$

so can be expressed as a combination of $E_4(n\tau)$ for $n = 1, 2, 3, 6$
By integration you get

$$m(P_k) = \Re(-\pi i\tau + \sum_{n \geq 1} \left( \sum_{d|n} d^3 \right) \left( 4\frac{q^n}{n} - 8\frac{q^{2n}}{2n} + 12\frac{q^{3n}}{3n} - 24\frac{q^{6n}}{6n} \right))$$

Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series.
Remark

Such a formula may be quite interesting.

Example

For the family $Q'_k$

$X + 1/X + Y + 1/Y + Z + 1/Z + XY + 1/XY + ZY + 1/ZY + XYZ + 1/XYZ - k$

we get

$$m(Q'_k) = \Re(-\pi i \tau + \sum_{n \geq 1} (\sum_{d|n} d^3)(-2\frac{q^n}{n} + 32\frac{q^{2n}}{2n} + 18\frac{q^{3n}}{3n} - 288\frac{q^{6n}}{6n}))$$

By $X = x$, $Y = y/x$, $Z = z/y$, $Q'_k$ is transformed in $Q_k$

$$(x + y + z + 1)(xy + xz + yz + xyz) - (k + 4)xyz$$

and

$$m(Q_k) = m(Q'_k)$$
Remark (continued)

\[ m(Q_{-4}) = 2m(x + y + z + 1) = \frac{7}{\pi^2} \zeta(3) \ (Smyth) \]

can be recovered from the expression of \( m(Q'_{-4}) \)

- \( k = -4 \) corresponds to \( \tau = 0 \) thus \( q = 1 \) (Verrill)

and

\[ m(Q'_{-4}) = \sum_{n \geq 1} \left( \sum_{d \mid n} d^3 \right) \left( -\frac{2}{n} + \frac{32}{2n} + \frac{18}{3n} - \frac{288}{6n} \right) \]

Lemma

\[ \sum_{n \geq 1} \left( \sum_{d \mid n} \chi(d)d^3 \right) \frac{1}{n^s} = \zeta(s)L(\chi, s - 3) \]
Lemma

\[
\lim_{s \to 1} \zeta(s)L(\chi, s - 3) = -\frac{1}{4\pi^2} L(\chi, 3)
\]

if \( \chi(-1) = 1 \), in particular if \( \chi \) is the trivial character.

Thus

\[
m(Q'_{-4}) = -\frac{\zeta(3)}{4\pi^2} (-2 + 16 + 6 - 48) = \frac{7}{\pi^2}\zeta(3)
\]

Another proof of Smyth’s formula!
For some values of $k$, the corresponding $\tau$ is imaginary quadratic. For example

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>10</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>$-\frac{3+\sqrt{-3}}{6}$</td>
<td>$-\frac{2+\sqrt{-2}}{6}$</td>
<td>$\frac{-3+\sqrt{-15}}{12}$</td>
<td>$\frac{\sqrt{-6}}{6}$</td>
<td>$\frac{\sqrt{-2}}{2}$</td>
<td>$\frac{\sqrt{-5}}{6}$</td>
</tr>
</tbody>
</table>

For these quadratic $\tau$ called “singular moduli”, the corresponding K3-surface is singular, that means its Picard number is $\rho = 20$ and the elliptic curve $E$ of the Shioda-Inose is CM.

So, an expression of the Mahler measure in terms of Hecke L-series (arithmetic aspect) and perhaps in terms of the L-series of the hypersurface K3 (geometric aspect).
Theorem

Let $Y_k$ the K3 hypersurface associated to the polynomial $P_k$, $L(Y_k, s)$ its L-series and $T_Y$ its transcendental lattice. Then,

\[
m(P_0) = d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)
\]

\[
m(P_2) = \frac{|\det T(Y_2)|^{3/2}}{\pi^3} L(Y_2, 3) = \frac{8\sqrt{8}}{\pi^3} L(f_8, 3)
\]

\[
m(P_6) = \frac{|\det T(Y_6)|^{3/2}}{2\pi^3} L(Y_6, 3) = \frac{24\sqrt{24}}{2\pi^3} L(f_{24}, 3)
\]

\[
m(P_{10}) = \frac{|\det T(Y_{10})|^{3/2}}{9\pi^3} L(Y_{10}, 3) + 2d_3 = \frac{72\sqrt{72}}{9\pi^3} L(f_8, 3) + 2d_3
\]

\[
m(P_{18}) = \frac{|\det T(Y_{18})|^{3/2}}{9\pi^3} L(Y_{18}, 3) + \frac{14}{5} d_3 = \frac{120\sqrt{120}}{9\pi^3} L(f_{120}, 3) + \frac{14}{5} d_3
\]
1. Prove the conjectured expressions of $m(P_6), m(P_{18})$
2. Find an analog for $m(P_3)$
3. Give the corresponding expressions in terms of Hecke L-series.
Some ingredients in the proof

Here $f_N$ denotes the unique, up to twist, CM-newform, CM by $\mathbb{Q}(\sqrt{-N})$, of weight 3 and level $N$ with rational coefficients.

$$L^*(X, s) := \prod_{p \nmid N}^* Z(X|\mathbb{F}_p, p^{-s}) = \sum_{n \geq 1} \frac{a(n)}{n^s}$$

$$Z(X|\mathbb{F}_p, t) := \exp\left(\sum_{s=1}^{\infty} Np^n \frac{t^s}{s}\right) = \frac{1}{(1-t)(1-p^2 t)P_2(t)}$$

$N$ is the determinant of the transcendental lattice

$$P_2(t) = \det(1 - tF_p|H^2_{\text{et}}(X, \mathbb{Q}_l))$$

is a degree 22 polynomial

$$H^2_{\text{et}}(X, \mathbb{Q}_l)) = H^2_{\text{alg}}(X, \mathbb{Q}_l)) + H^2_{\text{tr}}(X, \mathbb{Q}_l))$$
\[ N_p = \#X(\mathbb{F}_p) \]

\[ N_p = 1 + p^2 + \underbrace{\text{Tr}H^2_{\text{alg}}(X, \mathbb{Q}_l)}_{(1)} + \underbrace{\text{Tr}H^2_{\text{tr}}(X, \mathbb{Q}_l)}_{(2)} \]

(1) corresponds to algebraic cycles and depends on whether they are defined over \( \mathbb{F}_p \) or \( \mathbb{F}_{p^2} \)

(2) corresponds to transcendental cycles

For example, suppose \( X \) singular and the 20 generators of the Néron-Severi defined over \( \mathbb{F}_p \) (case of \( Y_2 \) and \( Y_6 \))

\[ P_2(t) = (1 - pt)^{20}(1 - \beta t)(1 - \beta' t) \]

\[ N_p = 1 + p^2 + 20p + \beta + \beta' \]
Lemma

Let $\rho_l, \rho'_l : G_Q \to \text{Aut } V_l$ two rational $l$-adic representations with $\text{Tr} F_p, \rho_l = \text{Tr} F_p, \rho'_l$ for a set of primes $p$ of density one (i.e. for all but finitely many primes). If $\rho_l$ and $\rho'_l$ fit into two strictly compatible systems, the $L$-functions associated to these systems are the same.

Then the great idea is to replace this set of primes of density one by a finite set.

Definition

A finite set $T$ of primes is said to be an effective test set for a rational Galois representation $\rho_l : G_Q \to \text{Aut } V_l$ if the previous lemma holds with the set of density one replaced by $T$.
Definition

Let $\mathcal{P}$ denote the set of primes, $S$ a finite subset of $\mathcal{P}$ with $r$ elements, $S' = S \cup \{-1\}$. Define for each $t \in \mathcal{P}$, $t \neq 2$ and each $s \in S'$ the function

$$f_s(t) := \frac{1}{2} \left(1 + \left(\frac{s}{t}\right)\right)$$

and if $T \subset \mathcal{P}$, $T \cap S = \emptyset$,

$$f : T \to (\mathbb{Z}/2\mathbb{Z})^{r+1}$$

such that

$$f(t) = (f_s(t))_{s \in S'}.$$
Theorem

(Serre-Livné’s criterion) Let \( \rho \) and \( \rho' \) be two 2-adic \( G_\mathbb{Q} \)-representations which are unramified outside a finite set \( S \) of primes, satisfying

\[
\text{Tr} F_{p, \rho} \equiv \text{Tr} F_{p, \rho'} \equiv 0 \pmod{2} \\
\text{det} F_{p, \rho} \equiv \text{det} F_{p, \rho'} \pmod{2}
\]

for all \( p \notin S \cup \{2\} \).

Any finite set \( T \) of rational primes disjoint from \( S \) with

\[
f(T) = (\mathbb{Z}/2\mathbb{Z})^{r+1} \setminus \{0\}
\]

is an effective test set for \( \rho \) with respect to \( \rho' \).
Another ingredient: Livné’s modularity theorem

Theorem

Let $S$ be a $K3$-surface defined over $\mathbb{Q}$, with Picard number 20 and discriminant $N$. Its transcendental lattice $T(S)$ is a dimension 2 $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module thus defines a $L$ series, $L(T(S), s)$.

There exists a weight 3 modular form $f$, CM over $\mathbb{Q}(\sqrt{-N})$ satisfying

$$L(T(S), s) \doteq L(f, s).$$

Moreover, if $\text{NS}(S)$ is generated by divisors defined over $\mathbb{Q}$,

$$L(S, s) \doteq \zeta(s - 1)^{20}L(f, s).$$
The last ingredient: Schütt’s classification of CM-newforms of weight 3

Theorem

Consider the following classifications of singular K3 surfaces over $\mathbb{Q}$:

1. by the discriminant $d$ of the transcendental lattice of the surface up to squares,
2. by the associated newform up to twisting,
3. by the level of the associated newform up to squares,
4. by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.

Then, all these classifications are equivalent. In particular, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.
Let

\[ P_k = x^2yz + xy^2z + xyz^2 + t^2(xy + xz + yz) - kxyzt. \]

- \( Y_k \) is the desingularization of the set of zeroes of \( P_k \).
- With some fibration, \( Y_k \) is an elliptic surface with singular fibers of type \( I_n \).
- Use Shioda’s theorems on elliptic surfaces to compute the determinant of \( \text{NS}(Y_k) \), in particular, the formula

\[ \rho_k = r_k + 2 + \sum_{\nu} (m_{\nu,k} - 1) \]

where \( r_k \) is the rank of \( \text{MW}(Y_k) \).
Some ingredients in the proof (continued)

- If \( k = 2 \), since \( \rho_2 = 20 \) and fibers are of type \( l_{12}, l_6, l_2, l_1, l_1 \), \( r_2 = 0 \) (easy case).

- So,

\[
| \det NS(Y_2) | = \frac{\prod m_{\nu,2}}{\text{Torsion}^2} = 8
\]

- If \( k = 10 \), since \( \rho_2 = 20 \) and fibers are of type \( l_{12}, l_3, l_3, l_2, l_2, l_1, l_1 \), \( r_{10} = 1 \) (difficult case).
  
  - So, have to guess an infinite section,
  
  - have to use Néron’s desingularization.
The value of $\det NS(Y_{10})$ gives the CM-field of the elliptic curve in the Shioda-Inose structure.

Have to count the number of points of the reduction of $Y_k$ modulo $q$ ($q = p^r$).

In case $Y_k$ modular this allows to determine which modular form gives the equality

$$L(Y_k, s) = L(f, s).$$

Compare to the expression of the Mahler measure and conclude.
We have

\[
\det NS(Y_2) = -8 \\
\det NS(Y_{10}) = -72
\]

so the underlying elliptic curves \( E_2 \) and \( E_{10} \) are both CM on \( \mathbb{Q}(\sqrt{-2}) \).

Since

\[
L(Y_2, s) = L(Y_{10}, s) = L(f, s),
\]

by Tate’s conjecture, \( Y_2 \) and \( Y_{10} \) are related by an algebraic correspondence.