

Mahler measure, regulators and modular units

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We are interested in Boyd's conjectures relating the logarithmic Mahler measure of **certain** two-variables polynomials defining elliptic curves and the L-series at $s = 2$ of this elliptic curve.

The **logarithmic Mahler measure** m of a non-zero Laurent polynomial $A \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is defined as

$$m(A) := \int_0^1 \dots \int_0^1 \log |A(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n})| d\theta_1 \dots d\theta_n$$

and its Mahler measure is the exponential of the latter.

If $A(x, y)$ is in two variables we can write

$$A(x, y) = a_0(y) \prod_{j=1}^d (x - x_j(y))$$

with $x_j(y)$ algebraic functions in y .

By Jensen's formula

$$m(A) = m(a_0) + \sum_{j=1}^d \frac{1}{2\pi i} \int_{|y|=1} \log^+ |x_j(y)| \frac{dy}{y}$$

where $\log^+ |z| = \log |z|$ if $|z| \geq 1$ and 0 otherwise.

Defining

$$\eta(x, y) := \log |x| d \arg y - \log |y| d \arg x$$

a real differential 1-form on $X \setminus S$ (X the variety defined by the polynomial A , smooth projective completion of Y zero locus of A , S points of X where x or y has a zero or a pole), we get

$$m(A) = m(a_0) + \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

γ oriented path on X projecting to $Y \cap \{|y|=1, |x| \geq 1\}$

Deninger's guess (1996)

To illustrate which kind of polynomials and which kind of conjectures or results, let us give **Deninger's guess (1996)** proved in 2011 by Rogers and Zudilin, then again by Zudilin in 2013

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) = \frac{15}{4\pi^2} L(E, 2) =: L'(E, 0) = b_{15}$$

The elliptic curve E is 15a8 (Cremona's notation) of conductor 15 defined by

$$Y^2 + XY + Y = X^3 + X^2$$

Its L-series is given by the modular form

$$f_{15A}(z) = \eta(z)\eta(3z)\eta(5z)\eta(15z)$$

The polynomial

$$x + \frac{1}{x} + y + \frac{1}{y} + 1$$

is tempered

“Tempered” means the roots of all the face polynomials of the Newton polygon of A are roots of unity.

The polynomial

$$Y^2 + XY + Y - (X^3 + X^2)$$

is also tempered.

Very important to obtain formulas “à la Deninger”.

Just after Deninger's guess, **Boyd obtained a lot of conjectures based on numerical computations.**

What is proved now concerning non CM elliptic curves?

Denote

$$b_{N_k} := \frac{N_k}{4\pi^2} L(E^k, 2) \quad m_k = m(P_k)$$

with N_k conductor of the elliptic curve E^k defined by P_k .

In the family

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + k$$

Zudilin-Rogers (2011) then **Zudilin (2013)** $m(1) = b_{15}$

Zudilin arxiv (2013) $m_{2i} = b_{40}$, $m_2 = b_{24}$, $m_i = 2b_{17}$, $m_{\sqrt{2}} = \frac{1}{4}b_{56}$

Brunault arxiv (april 2015) $m_3 = 2b_{21}$, $m_{12} = 2b_{48}$

Lalin-Samart-Zudilin arxiv (july 2015) $m_3 = 2b_{21}$ (another proof)

In the family

$$P_k = x^3 + y^3 + 1 - kxy$$

Mellit preprint (2009) arxiv (2012) $m_{-1} = 2b_{14}$, $m_5 = 7b_{14}$

In the family

$$P_k = (x + 1)y^2 + (x^2 + kx + 1)y + x(x + 1)$$

Mellit preprint (2009) arxiv (2012) $m_1 = b_{14}$, $m_{-5} = 6b_{14}$, $m_{10} = 10b_{14}$

In the family

$$P_k = y^2 + kxy + y - x^3$$

Brunault arxiv (april 2015) $m_{-1} = 2b_{14}$, $m_{-2} = b_{35}$, $m_{-3} = b_{54}$

Related also to the family

$$P_k(x, y) = (x + 1)y^2 + (x^2 + kx + 1)y + x(x + 1)$$

Boyd conjectured the two formulae

$$m_4 = 3b_{20} \quad \text{and} \quad m_{-2} = 2b_{20}.$$

In fact, E^4 is isomorphic to the curve $20a2$ $[0, 1, 0, -1, 0]$, E^{-2} is isomorphic to the curve $20a1$ $[0, 1, 0, 4, 4]$, 2-isogenous to $20a2$.

The corresponding modular form on $\Gamma_0(20)$ thus giving the L -series is

$$f_{20A} = \eta(2z)^2 \eta(10z)^2 = q - 2q^3 - q^5 + 2q^7 + q^9 + 2q^{13} + 2q^{15} \dots$$

We proved recently these conjectures and the main ingredients are regulators and modular units

“tempered” and the K_2 of the elliptic curve

Let X be a **smooth** projective algebraic curve defined over \mathbb{C} and let $\mathbb{C}(X)$ be its function field. Let $x, y \in \mathbb{C}(X)$ be two non-constant rational functions and let $S \subset X$ be the set of zeros and poles of x or y . The image of the rational map $(x, y) : X \setminus S \rightarrow \mathbb{C}^* \times \mathbb{C}^*$ is of dimension 1; let $A \in \mathbb{C}[x, y]$ be a defining equation.

$$\{x, y\} \in K_2(X) \otimes \mathbb{Q} \Leftrightarrow A \text{ “tempered”}$$

Integral expression of the regulator

The regulator r can be expressed as an integral

$$r : K_2(E) \rightarrow \mathbb{C}$$
$$\{f, g\} \mapsto \frac{1}{2\pi} \int_{\gamma} \eta(f, g)$$

with

$$\eta(f, g) = \log |f| d(\arg g) - \log |g| d(\arg f),$$

f and $g \in \mathbb{Q}(E)$ and γ closed path not going through zeros and poles of f and g and generating the subgroup of cycles $H_1(E, \mathbb{Z})^-$

Regulator and Mahler measure

The Mahler measure can be expressed as a regulator if we can prove that the path of integration in the expression of the Mahler measure belongs to $H_1(E, \mathbb{Z})^-$.

This is precisely the case for the polynomial P_{-2} .

Set $P_{-2}(x_2, y_2)$ the polynomial

$$P_{-2}(x_2, y_2) = (x_2 + 1)y_2^2 + (x_2^2 - 2x_2 + 1)y_2 + x_2(x_2 + 1).$$

Then

$$2m_{-2} = \pm r(\{x_2, y_2\}).$$

The diamond operator

Let $\mathbb{Z}\langle P \rangle$ the subgroup of $\mathbb{Z}[E(\mathbb{Q})]$ generated by $P \in E(\mathbb{Q})$ and $\mathbb{Z}[E(\mathbb{Q})]^-$ its quotient by the relation $cl(-P) = -cl(P)$.

Define

$$\begin{aligned} \diamond : \mathbb{Z}\langle P \rangle \times \mathbb{Z}\langle P \rangle &\rightarrow \mathbb{Z}[E(\mathbb{Q})]^- \\ ((f), (g)) &\mapsto (f) \diamond (g) = \sum_{m,n} a_n b_m cl((n-m)P) \end{aligned}$$

$$(f) = \sum_{n \in \mathbb{Z}} a_n [nP], (g) = \sum_{n \in \mathbb{Z}} b_n [nP]$$

The elliptic dilogarithm (introduced by Bloch)

E elliptic curve on \mathbb{Q}

On $E(\mathbb{C})$, we have the representations

$$\begin{aligned} E(\mathbb{C}) &\xrightarrow{\sim} \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) \xrightarrow{\sim} \mathbb{C}^*/q^{\mathbb{Z}} \\ (\wp(u), \wp'(u)) &\mapsto u(\bmod \Lambda) \mapsto z = \exp 2\pi i u \end{aligned}$$

The elliptic dilogarithm D^E is

$$D^E(P) = \sum_{n=-\infty}^{+\infty} D(q^n z)$$

where D denotes the Bloch-Wigner dilogarithm.

Theorem

Let f and g functions on the elliptic curve E with divisors elements of $\mathbb{Z}\langle P \rangle$ such that $\{f, g\} \in K_2(E)$, then

$$\pi r(\{f, g\}) = D^E((f) \diamond (g))$$

Touafek's results

Some years later (2008), in his thesis (not published in extenso), Touafek considered the elliptic curve E_2 defined by the equation

$$Y_2^2 + 2X_2Y_2 + 2Y_2 = (X_2 - 1)^3$$

exhibited the isomorphisms between E_2 , 20a1 and E^{-2} , remarked that

$$\{X_2, Y_2\} \in K_2(E_2) \otimes \mathbb{Q}$$

$$\{x_2, y_2\} \in K_2(E^2) \otimes \mathbb{Q}$$

and used Bloch's theorem to derive the equality

$$r(\{X_2, Y_2\}) = r(\{x_2, y_2\})$$

and conjectured their common value $4b_{20}$.

Beilinson's result, Zagier's conjecture

For elliptic modular curves E , **Beilinson** proved

$$L(E, 2) = \pi D^E(\xi), \quad \xi \in \mathbb{Z}[E(\mathbb{C})]_{\text{tors}}$$

For a general elliptic curve E , **Zagier** conjectured

$$L(E/\mathbb{Q}, 2) \stackrel{?}{=} \pi D^E(\xi), \quad \xi \in \mathbb{Z}[E(\bar{\mathbb{Q}})]^{\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})}$$

Proof of Touafek's conjecture: modular units

The idea is the parametrization by modular units. (Brunault, Mellit, Zudilin).

Recall that a **modular unit** is a modular function whose all zeros and poles are cusps, for example certain quotient of eta functions for $\Gamma_0(20)$.

We proved the lemma

Lemma

The elliptic curve E_2 defined by

$$Y_2^2 + 2X_2Y_2 + 2Y_2 = (X_2 - 1)^3$$

is isomorphic to the curve '20a1' $[0, 1, 0, 4, 4]$ in Cremona's classification and can be parametrized by eta quotients, modular units on $X_0(20)$. More precisely

$$\begin{aligned} X_2 &= \frac{\eta(4\tau)^4}{\eta(20\tau)^4} \frac{\eta(10\tau)^2}{\eta(2\tau)^2} \\ Y_2 &= -\frac{\eta(4\tau)}{\eta(\tau)} \frac{\eta(5\tau)^5}{\eta(20\tau)^5} \end{aligned}$$

Let us recall first the definition of the modular unit g_a :

$$g_a(\tau) := q^{NB(a/N)/2} \prod_{\substack{n \geq 1 \\ n \equiv a \pmod{N}}} (1 - q^n) \prod_{\substack{n \geq 1 \\ n \equiv -a \pmod{N}}} (1 - q^n)$$

Now it follows from the definition of a modular unit:

$$X_2 = \left(\frac{g_4 g_8}{g_2 g_6} \right)^2$$
$$Y_2 = -\frac{g_5^4 g_{10}^2}{g_1 g_2 g_3 g_6 g_7 g_9}$$

Theorem

For integers a, b, c with ac and bc not divisible by N , we have the formula

$$\int_{c/N}^{i\infty} \eta(g_a, g_b) = \frac{1}{4\pi} L(f(\tau) - f(i\infty), 2)$$

where $f(\tau) = f_{a,b;c}(\tau)$, $f_{a,b;c} := e_{a,bc}e_{b,-ac} - e_{a,-bc}e_{b,ac}$ and

$$e_{a,b}(\tau) = \frac{1}{2} \left(\frac{1 + \zeta_N^a}{1 - \zeta_N^a} + \frac{1 + \zeta_N^b}{1 - \zeta_N^b} \right) + \sum_{m,n \geq 1} (\zeta_N^{am+bn} - \zeta_N^{-(am+bn)}) q^{mn}$$

$\zeta_N := \exp(2\pi i/N)$, $q := \exp(2\pi i\tau)$.

How to choose c : the path of integration

If $\alpha, \beta \in \mathcal{H}^*$ satisfy $\beta = M(\alpha)$, $M \in \Gamma_0(N)$ (α and β are said equivalent under the action of $\Gamma_0(N)$).

Any smooth path (for instance a geodesic path) projects to a closed path in the quotient space $X_0(N)$, hence determines an integral homology class in $H_1(X_0(N), \mathbb{Z})$, which depends only on α and β and not on the path chosen. In fact the class depends only on the matrix M . This homology class is denoted by the modular symbol $\{\alpha, \beta\}_{\Gamma_0(N)}$. Conversely, every homology class $\gamma \in H_1(X_0(N), \mathbb{Z})$ can be represented by such a modular symbol $\{\alpha, \beta\}_{\Gamma_0(N)}$.

For $f \in S_2(\Gamma_0(N))$,

$$\langle \gamma, f \rangle := \int_{\gamma} 2\pi i f(z) dz = 2\pi i \int_{\alpha}^{\beta} f(z) dz$$

is called a period of the cusp form f .

Elements of $H_1^-(X_0(N), \mathbb{R})$ are identified by

$$\langle \gamma, f \rangle \in i\mathbb{R} \iff \gamma \in H_1^-(X_0(N), \mathbb{R}).$$

Recall also that by the Manin-Drinfeld theorem, the rational homology $H_1(X_0(N), \mathbb{Q})$ is generated by paths between cusps.

The closed path of integration γ generating $H_1(E, \mathbb{Z})^-$ in the expression of the regulator becomes under the parametrization a closed path generator of $H_1^-(X_0(20), \mathbb{Z})$, hence an appropriate modular symbol we can compute using Sage. We can take the closed path $\{-3/20, 3/20\}$ and apply theorem B-M-Z. So

$$\begin{aligned}
 & r(\{X_2, Y_2\}) \\
 &= \frac{1}{2\pi} \left(\int_{-3/20}^{i\infty} - \int_{3/20}^{i\infty} \right) \eta \left(\left(\frac{g_4 g_8}{g_2 g_6} \right)^2, \frac{g_5^4 g_{10}^2}{g_1 g_2 g_3 g_6 g_7 g_9} \right) \\
 &= \frac{1}{2\pi} \frac{4}{4\pi} (4L(f_{4,5;-3}) + 2L(f_{4,10;-3}) - L(f_{4,1;-3}) - L(f_{4,2;-3}) \\
 &\quad - L(f_{4,3;-3}) - L(f_{4,6;-3}) - L(f_{4,7;-3}) - L(f_{4,7;-3}) \\
 &\quad + 4L(f_{8,5;-3}) + 2L(f_{8,10;-3}) - L(f_{8,1;-3}) - L(f_{8,2;-3}) - L(f_{8,3;-3}) \\
 &\quad - L(f_{8,6;-3}) - L(f_{8,7;-3}) - L(f_{8,7;-3}) \\
 &\quad \dots) \\
 &= \frac{1}{4\pi^2} 4 \times 20L(f, 2)
 \end{aligned}$$

f is the newform of conductor 20

$$f(q) = q - 2q^3 - q^5 + 2q^7 + q^9 + \dots$$

The end

We have just proved Touafek's conjecture

$$r(\{X_2, Y_2\}) = \frac{1}{2\pi^2} 40L(f, 2) = 4b_{20}$$

and previously it was obtained

$$r(\{X_2, Y_2\}) = r(\{x_2, y_2\})$$

$$2m_{-2} = \pm r(\{x_2, y_2\}).$$

We deduce Boyd's conjecture

$$m_{-2} = m(P_{-2}) = 2b_{20}$$

where $b_{20} = \frac{20}{4\pi^2} L(E^{-2}, 2)$.

Proof of the second conjecture

Similarly, Touafek considered the isomorphic curves E^4 defined by

$$(x_1 + 1)y_1^2 + (x_1^2 + 4x_1 + 1)y_1 + x_1(x_1 + 1) = 0$$

and the elliptic curve E_1 defined by

$$Y_1^2 + 2X_1 Y_1 - X_1^3 + X_1 = 0.$$

Both polynomials are tempered; so the respective regulators $r(\{x_1, y_1\})$ and $r(\{X_1, Y_1\})$ can be defined and from Touafek's computations we can also deduce the equality

$$r(\{x_1, y_1\}) = \frac{3}{2}r(\{X_1, Y_1\}).$$

Touafek proved also the relation

$$r(\{X_2, Y_2\}) = r(\{X_1, Y_1\}).$$

As previously we get

$$2m_4 = r(\{x_1, y_1\}).$$

Finally, it follows

$$\begin{aligned} 2m_4 = r(\{x_1, y_1\}) &= \frac{3}{2} r(\{X_1, Y_1\}) \\ &= \frac{3}{2} r(\{X_2, Y_2\}) \\ &= \frac{3}{2} 4b_{20} \end{aligned}$$