Mahler measure of algebraic varieties

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Introduction

Introduced by Mahler in 1962,
the logarithmic Mahler measure of a polynomial P is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log \mid P(x_1, \cdots, x_n) \mid \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and its Mahler measure

$$M(P) = \exp(m(P))$$

where

$$\mathbb{T}^n = \{(x_1, \cdots, x_n) \in \mathbb{C}^n / |x_1| = \cdots = |x_n| = 1\}.$$

Remarks

- $n = 1$
  By Jensen's formula, if $P \in \mathbb{Z}[X]$ is monic, then

$$M(P) = \prod_{P(\alpha)=0} \max(|\,\alpha\,|, 1).$$

So it is related to Lehmer's question (1933)
Does there exist $P \in \mathbb{Z}[X]$, monic, non cyclotomic, satisfying

$$1 < M(P) < M(P_0) = 1.1762\cdots ?$$

The polynomial

$$P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

is the Lehmer polynomial, in fact a Salem polynomial.

Lehmer's problem is still open.
A partial answer by Smyth (1971)

$$M(P) \geq 1.32\cdots$$

if P is non reciprocal.

The story can be explained with polynomials

$$x_0 + x_1 + x_2 + \cdots + x_n.$$

- $m(x_0 + x_1) = 0$ (by Jensen's formula)
- $$m(x_0 + x_1 + x_2) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \quad \text{Smyth (1980)}$$
- $$m(x_0 + x_1 + x_2 + x_3) = \frac{7}{2\pi^2}\zeta(3) \quad \text{Smyth (1980)}$$

  These are the first explicit Mahler measures.

- $$m(x_0+x_1+x_2+x_3+x_4) \stackrel{?}{=} **L(f,3)$$ conjectured by Villegas (2004)

  f cusp form of weight 3 and conductor 15
  $L(f,3)$ is also the L-series of the K3 surface defined by

$$x_0+x_1+x_2+x_3+x_4=0$$
$$\frac{1}{x_0}+\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=0$$

How such a conjecture possible?

Because of deep insights of two people.

- Deninger (1996) who conjectured

$$m(x + \frac{1}{x} + y + \frac{1}{y} + 1) \stackrel{?}{=} \frac{15}{4\pi^2} L(E, 2) = L'(E, 0)$$

  E elliptic curve of conductor 15 defined by the polynomial
  Here the polynomial is reciprocal.

- Maillot (2003) using a result of Darboux (1875): the Mahler measure of P which is the integration of a differential form on a variety, when P is non reciprocal, is in fact an integration on a smaller variety and the expression of the Mahler measure is encoded in the cohomology of the smaller variety.

- $n = 2$ The smaller variety is defined by

$$\begin{aligned} x_0 + x_1 + x_2 &= 0 \\ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} &= 0 \end{aligned} \Leftrightarrow x_1^2 + x_2^2 + x_1 x_2 = 0$$

It is a curve of genus 0. So $m(x_0 + x_1 + x_2)$ is expressed as a Dirichlet L-series.

- $n = 3$ The smaller variety is defined by

$$\begin{aligned} x_0 + x_1 + x_2 + x_3 &= 0 \\ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} &= 0 \end{aligned} \Leftrightarrow (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) = 0$$

It is the intersection of 3 planes. Thus Smyth's result.

- $n = 4$ (Villegas's Conjecture) The smaller variety is defined by

$$x_0 + x_1 + x_2 + x_3 + x_4 = 0$$
$$\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = 0$$

It is the modular $K3$-surface studied by Peters, Top, van der Vlugt defined by a reciprocal polynomial. Its L-series is related to f.

- $n = 5$ (Villegas's Conjecture again)

$$m(x_0 + x_1 + x_2 + x_3 + x_4 + x_5) = **L(g, 4)$$

g cusp form of weight 4 and conductor 6 related to L-series of the Barth-Nieto quintic.

Motivation

Briefly, to guess the Mahler measure of a non reciprocal polynomial we need results on reciprocal ones.
In particuliar, it is very important to collect many examples of Mahler measures of $K3$-hypersurfaces.
Notice that Maillot's insight predicts only the type of formula expected.
Also Deninger's guess comes from Beilinson's Conjectures.

Many Deninger like or Smyth like formulae:

- experimental (Boyd, R-Villegas)
- rare proofs (Smyth, R-Villegas, Lalin, Rogers, Brunault)

Attempt to answer the questions:

Which type of polynomial defining an elliptic curve give a Mahler measure expressed in terms of the L-series of the elliptic curve

Why two types of results?

Let

$$P(x,y) = a_d(x)y^d + \dots + a_0(x) = a_d(x)\prod_{n=1}^{d}(y-\alpha_n(x)).$$

By Jensen's formula

$$\begin{aligned} m(P) &= m(a_d) + \frac{1}{2\pi i}\sum_{n=1}^{d}\int_{\mathbb{T}_1}\log^+ \mid \alpha_n(x) \mid \frac{dx}{x} \\ &= m(P^*) - \frac{1}{2\pi}\int_{\gamma}\eta(x,y) \end{aligned}$$

$$\eta(x,y) := \log \mid x \mid d\arg y - \log \mid y \mid d\arg x$$

is the regulator defined on
$C = \{P(x,y)=0\} \setminus$ ensemble des zéros et pôles de x et y
γ is the union of paths on C where $\mid x \mid = 1$ and $\mid y \mid \geq 1$

$$\partial\gamma = \{P(x,y)=0\} \cap \{\mid x \mid = \mid y \mid = 1\}$$

Now, if $n=2$, two cases

- η is exact and $\partial\gamma \neq 0$ (the integration is made using Stokes) case of Smyth's formula.
- η is not exact and $\partial\gamma = 0$ (the integral is computed with residues's formula) Deninger's case

Bloch (1981) did computations to compare

$$K_2(E) \leftrightsquigarrow L(E,2)$$

and

$$K_2(E) \leftrightsquigarrow K_1(0,\alpha,2) = \sum{}' \frac{\chi(w)}{w^2\bar{w}}$$

W is the lattice corresponding to E
$\chi(w) = \exp(A^{-1}(\bar{\alpha}w - \alpha\bar{w}))$, additive character depending on the image of α in $\mathbb{C}/W$
Precisely, Bloch was looking for relations

$$c.L(E,2) + \sum_{\alpha \in E(\mathbb{Q})_{\text{tors}}} c_\alpha \sum{}' (\Im\tau)^2 \frac{\chi(w)}{w^2\bar{w}} = 0$$

An example (Boyd (1996))

$$P_k = (x^2+x+1)y^2 + kx(x+1)y + x(x^2+x+1)$$

defines generically an hyperelliptic curve Z_k of genus 2. Its Jacobian $J(Z_k)$ is isogenous to the product of elliptic curves $E_k \times F_k$.
For $1 \leq k \leq 33$, $k \neq 3$,

$$m(P_k) \stackrel{?}{=} r_k L'(E_k, 0)$$

$$E_k : \qquad y^2 = x^3 + (k^2-24)x^2 - 16(k^2-9)x$$

But, in fact, we can prove
The derivative of the Mahler measure with respect to k is a period of the elliptic curve E_k (main ingredient for families of elliptic curves)
No example when the Jacobian is not split.
So, probably, the most interesting results for n=2 concern elliptic curves.
And elliptic curves are Calabi-Yau varieties.

So replace E by a surface X which is also a Calabi-Yau variety, i.e. a $K3$-surface and try to answer the questions:
What are the analog of Deninger, Boyd, R-Villegas 's results and conjectures?
Which type of Eisenstein-Kronecker series correspond to $L(X, 3)$?

Our results concern polynomials of the family

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining K3-surfaces Y_k. What's a K3-surface?

- A double covering branched along a plane sextic for example defines a K3-surface X.
  In our case

$$(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4$$

- There is a unique holomorphic 2-form ω on X up to a scalar.
- $H_2(X, \mathbb{Z})$ is a free group of rank 22.

- With the intersection pairing, $H_2(X, \mathbb{Z})$ is a lattice and

$$H_2(X, \mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

$\mathcal{L}$ is the $K3$-lattice, U_2 the hyperbolic lattice of rank 2, E_8 the unimodular lattice of rank 8.

- $$Pic(X) \subset H_2(X, \mathbb{Z}) \simeq Hom(H^2(X, \mathbb{Z}), \mathbb{Z})$$

where $Pic(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles.

- $$Pic(X) \simeq \mathbb{Z}^{\rho(X)}$$

$$\rho(X) := \text{Picard number of } X$$

$$1 \leq \rho(X) \leq 20$$

- $$T(X) := (Pic(X))^{\perp}$$

  is the transcendental lattice of dimension $22 - \rho(X)$

- If $\{\gamma_1, \cdots, \gamma_{22}\}$ is a $\mathbb{Z}$-basis of $H_2(X, \mathbb{Z})$ and ω the holomorphic 2-form,

  $$\int_{\gamma_i} \omega$$

  is called a period of X and

  $$\int_{\gamma} \omega = 0 \text{ for } \gamma \in Pic(X).$$

- If $\{X_z\}$ is a family of $K3$ surfaces, $z \in \mathbb{P}^1$ with generic Picard number ρ and ω_z the corresponding holomorphic 2-form, then the periods of X_z satisfy a Picard-Fuchs differential equation of order $k = 22 - \rho$. For our family $k = 3$.

- In fact, by Morrison, a $\mathcal{M}$-polarized $K3$-surface, with Picard number 19 has a Shioda-Inose structure, that means

$$\begin{array}{ccc} X & & A = E \times E/C_N \\ & \searrow \quad \swarrow & \\ & Y = Kum(A/\pm 1) & \end{array}$$

- If the Picard number $\rho = 20$, then the elliptic curve is CM.

Mahler measure of P_k

Theorem

(B. 2005) Let $k = t + \frac{1}{t}$ *and*

$$t = (\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)})^6, \ \ \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n\geq 1}(1 - e^{2\pi i n\tau}), \ \ q = \exp 2\pi i\tau$$

$$\begin{aligned} m(P_k) =& \frac{\Im\tau}{8\pi^3}\{\sum_{m,\kappa}{}' (-4(2\Re\frac{1}{(m\tau+\kappa)^3(m\bar{\tau}+\kappa)} + \frac{1}{(m\tau+\kappa)^2(m\bar{\tau}+\kappa)^2}) \\ &+ 16(2\Re\frac{1}{(2m\tau+\kappa)^3(2m\bar{\tau}+\kappa)} + \frac{1}{(2m\tau+\kappa)^2(2m\bar{\tau}+\kappa)^2}) \\ &- 36(2\Re\frac{1}{(3m\tau+\kappa)^3(3m\bar{\tau}+\kappa)} + \frac{1}{(3m\tau+\kappa)^2(3m\bar{\tau}+\kappa)^2}) \\ &+ 144(2\Re\frac{1}{(6m\tau+\kappa)^3(6m\bar{\tau}+\kappa)} + \frac{1}{(6m\tau+\kappa)^2(6m\bar{\tau}+\kappa)^2}))\} \end{aligned}$$

Sketch of proof

Let

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining the family (X_k) of $K3$-surfaces.

- For $k \in \mathbb{P}^1$, generically $\rho = 19$.
- The family is $\mathcal{M}_k$-polarized with

$$\mathcal{M}_k \simeq U_2 \perp (-E_8)^2 \perp \langle -12 \rangle$$

- Its transcendental lattice satisfies

$$T_k \simeq U_2 \perp \langle 12 \rangle$$

- The Picard-Fuchs differential equation is

$$(k^2-4)(k^2-36)y''' + 6k(k^2-20)y'' + (7k^2-48)y' + ky = 0$$

- The family is modular in the following sense if $k = t + \frac{1}{t}$, $\tau \in \mathcal{H}$ and τ as in the theorem

$$t(\frac{a\tau + b}{c\tau + d}) = t(\tau) \ \forall \ (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_1(6,2)^* \subset \Gamma_0(12)^* + 12$$

where

$$\Gamma_1(6) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in Sl_2(\mathbb{Z}) \ / \ a \equiv d \equiv 1 \ (6) \ \ c \equiv 0 \ (6)\}$$

$$\Gamma_1(6,2) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_1(6) \ \ c \equiv 6b \ \ (12)\}$$

and

$$\Gamma_1(6,2)^* = \langle \Gamma_1(6,2), w_6 \rangle$$

.

- The P-F equation has a basis of solutions $G(\tau)$, $\tau G(\tau)$, $\tau^2 G(\tau)$ with

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

satisfying

$$G(\tau) = F(t(\tau)), \quad F(t) = \sum_{n\geq 0} v_n t^{2n+1}, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

- $\frac{dm(P_k)}{dk}$ is a period, hence satisfies the P-F equation
- $$\frac{dm(P_k)}{dk} = G(\tau)$$
- $$dm(P_k) = -G(\tau)\frac{dt}{t}\frac{1-t^2}{t}$$

  is a weight 4 modular form for $\Gamma_1(6,2)^*$
- so can be expressed as a combination of $E_4(n\tau)$ for $n = 1, 2, 3, 6$

- By integration you get

$$m(P_k) = \Re(-\pi i\tau + \sum_{n\geq 1}(\sum_{d|n} d^3)(4\frac{q^n}{n} - 8\frac{q^{2n}}{n} + 12\frac{q^{3n}}{n} - 24\frac{6n}{n}))$$

- Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series

For some values of k, the corresponding τ is imaginary quadratic.
For example

k	0	2	3	6	10	18
τ	$\frac{-3+\sqrt{-3}}{6}$	$\frac{-2+\sqrt{-2}}{6}$	$\frac{-3+\sqrt{-15}}{12}$	$\frac{\sqrt{-6}}{6}$	$\frac{\sqrt{-2}}{2}$	$\sqrt{\frac{-5}{6}}$

For these quadratic τ called "singular moduli", the corresponding K3-surface is singular, that means its Picard number is $\rho = 20$ and the elliptic curve E of the Shioda-Inose is CM
So, an expression of the Mahler measure in terms of Hecke L-series (arithmetic aspect) and perhaps in terms of the L-series of the hypersurface K3 (geometric aspect).

Mahler measure and Hecke L-series

$K = \mathbb{Q}(\sqrt{d})$ an imaginary quadratic field, O_K its ring of integers, D its discriminant

Definition A Hecke "Grössencharacter" of weight k, $k \geq 2$, and conductor Λ, Λ being an ideal of O_K, is an homomorphism ϕ

$$\phi : I(\Lambda) \to \mathbb{C}^*$$

such that

$$\phi(\alpha O_K) = \alpha^{k-1} \quad \text{if} \quad \alpha \equiv 1(\Lambda)$$

The corresponding Hecke L-series is defined by

$$L(\phi, s) := \sum_P \frac{\phi(P)}{N(P)^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

the summation being on the prime ideals $P \subset O_K$ prime to Λ.
Replacing O_K by an order R, you get a similar definition.

Theorem

$$m(P_0) = d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = \frac{2\sqrt{3}}{\pi^3} \sum_{m,k}{}' \frac{1}{(m^2+3k^2)^2}$$

$m(P_2) = \frac{16\sqrt{2}}{\pi^3} L_{\mathbb{Q}(\sqrt{-2})}(\phi, 3) m(P_3) = \frac{15\sqrt{15}}{2\pi^3} L_{\mathbb{Q}(\sqrt{-15})}(\phi_1, 3)$ *where* $\phi_1(P) = -\omega$ *if* $P = (2, \omega)$ *and* $\omega = \frac{1+\sqrt{-15}}{2}$, *$P$ being a representative of the second ideal class of the number field* $\mathbb{Q}(\sqrt{-15})$ *of class number* 2.

$$m(P_6) = \frac{24\sqrt{6}}{\pi^3} L_{\mathbb{Q}(\sqrt{-6})}(\phi_2, 3)$$

where $\phi_2(P) = -2$ *if* $P = (2, \sqrt{-6})$, *P being a representative of the second ideal class of the number field* $\mathbb{Q}(\sqrt{-6})$ *of class number* 2.

$$m(P_{10}) = 2d_3 + 3\frac{16\sqrt{2}}{\pi^3} L_{\mathbb{Q}(\sqrt{-2})}(\phi, 3)$$

Mahler measure and L-series of the K3-hypersurfaces

Theorem

Let Y_k the $K3$ hypersurface associated to the polynomial P_k, $L(Y_k, s)$ its L-series and T_Y its transcendental lattice. Then,

$$m(P_0) = d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

$$m(P_2) = \frac{|\det T(Y_2)|^{3/2}}{\pi^3} L(Y_2, 3) = \frac{8\sqrt{8}}{\pi^3} L(f_8, 3)$$

$$m(P_6) = \frac{|\det T(Y_6)|^{3/2}}{2\pi^3} L(Y_6, 3) = \frac{24\sqrt{24}}{2\pi^3} L(f_{24}, 3)$$

$$m(P_{10}) = \frac{|\det T(Y_{10})|^{3/2}}{9\pi^3} L(Y_{10}, 3) + 2d_3 = \frac{72\sqrt{72}}{9\pi^3} L(f_8, 3) + 2d_3$$

$$m(P_{18}) = \frac{|\det T(Y_{18})|^{3/2}}{9\pi^3} L(Y_{18}, 3) + \frac{14}{5} d_3 = \frac{120\sqrt{120}}{9\pi^3} L(f_{120}, 3) + \frac{14}{5} d_3$$

Some ingredients in the proof

Here f_N denotes the unique, up to twist, CM-newform, CM by $\mathbb{Q}(\sqrt{-N})$, of weight 3 and level N with rational coefficients.

$$L^*(X,s) := \prod_{p\nmid N}^{*} Z(X|\mathbb{F}_p, p^{-s}) = \sum_{n\geq 1} \frac{a(n)}{n^s}$$

$$Z(X|\mathbb{F}_p, t) := \exp(\sum_{s=1}^{\infty} N_{p^n} \frac{t^s}{s}) = \frac{1}{(1-t)(1-p^2t)P_2(t)}$$

N is the determinant of the transcendental lattice

$$P_2(t) = \det(1 - tF_p | H^2_{\text{et}}(X, \mathbb{Q}_l))$$

is a degree 22 polynomial

$$H^2_{\text{et}}(X,\mathbb{Q}_l)) = H^2_{\text{alg}}(X,\mathbb{Q}_l)) + H^2_{\text{tr}}(X,\mathbb{Q}_l))$$

$$N_p = \#X(\mathbb{F}_p)$$

$$N_p = 1 + p^2 + \overbrace{\mathrm{Tr} H^2_{\mathrm{alg}}(X, \mathbb{Q}_l))}^{(1)} + \overbrace{\mathrm{Tr} H^2_{\mathrm{tr}}(X, \mathbb{Q}_l))}^{(2)}$$

(1) corresponds to algebraic cycles and depends on whether they are defined over $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$

(2) corresponds to transcendental cycles

For example, suppose X singular and the 20 generators of the Néron-Severi defined over $\mathbb{F}_p$ (case of Y_2 and Y_6)

$$P_2(t) = (1 - pt)^{20}(1 - \beta t)(1 - \beta' t)$$

$$N_p = 1 + p^2 + 20p + \beta + \beta'$$

Serre-Livné 's criterion

Lemma

Let $\rho_l, \rho'_l : G_{\mathbb{Q}} \to Aut\, V_l$ two rational l-adic representations with $TrF_{p,\rho_l} = TrF_{p,\rho'_l}$ for a set of primes p of density one (i.e. for all but finitely many primes). If ρ_l and ρ'_l fit into two strictly compatible systems, the L-functions associated to these systems are the same.

Then the great idea is to replace this set of primes of density one by a finite set.

Definition

A finite set T of primes is said to be an effective test set for a rational Galois representation $\rho_l : G_{\mathbb{Q}} \to \mathrm{Aut}\, V_l$ if the previous lemma holds with the set of density one replaced by T.

Definition

Let $\mathcal{P}$ denote the set of primes, S a finite subset of $\mathcal{P}$ with r elements, $S' = S \cup \{-1\}$. Define for each $t \in \mathcal{P}$, $t \neq 2$ and each $s \in S'$ the function

$$f_s(t) := \frac{1}{2}(1 + \left(\frac{s}{t}\right))$$

and if $T \subset \mathcal{P}$, $T \cap S = \emptyset$,

$$f : T \to (\mathbb{Z}/2\mathbb{Z})^{r+1}$$

such that

$$f(t) = (f_s(t))_{s \in S'} \, .$$

Theorem

(Serre-Livné's criterion) Let ρ *and* ρ' *be two 2-adic* $G_{\mathbb{Q}}$*-representations which are unramified outside a finite set* S *of primes, satisfying*

$$TrF_{p,\rho} \equiv TrF_{p,\rho'} \equiv 0 \ (mod\, 2)$$

and

$$detF_{p,\rho} \equiv detF_{p,\rho'} \ (mod\, 2)$$

for all $p \notin S \cup \{2\}$.
Any finite set T *of rational primes disjoint from* S *with* $f(T) = (\mathbb{Z}/2\mathbb{Z})^{r+1} \setminus \{0\}$ *is an effective test set for* ρ *with respect to* ρ'.

Another ingredient: Livné's modularity theorem

Theorem

Let S be a $K3$-surface defined over $\mathbb{Q}$, with Picard number 20 and discriminant N. Its transcendental lattice $T(S)$ is a dimension 2 Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$-module thus defines a L series, $L(T(S), s)$.
There exists a weight 3 modular form , f, CM over $\mathbb{Q}(\sqrt{-N})$ satisfying

$$L(T(S), s) \doteq L(f, s).$$

Moreover, if $NS(S)$ is generated by divisors defined over $\mathbb{Q}$,

$$L(S, s) \doteq \zeta(s-1)^{20} L(f, s).$$

The last ingredient:Schütt's classification of CM-newforms of weight 3

Theorem

Consider the following classifications of singular $K3$ surfaces over $\mathbb{Q}$:

1. *by the discriminant d of the transcendental lattice of the surface up to squares,*
2. *by the associated newform up to twisting,*
3. *by the level of the associated newform up to squares,*
4. *by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.*

Then, all these classifications are equivalent. In particuliar, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.

Let

$$P_k = x^2yz + xy^2z + xyz^2 + t^2(xy + xz + yz) - kxyzt.$$

- Y_k is the desingularization of the set of zeroes of P_k.
- With some fibration, Y_k is an elliptic surface with singular fibers of type I_n.
- Use Shioda's theorems on elliptic surfaces to compute the determinant of $NS(Y_k)$, in particular, the formula

$$\rho_k = r_k + 2 + \sum_{\nu}(m_{\nu,k} - 1)$$

where r_k is the rank of $MW(Y_k)$.

Some ingredients in the proof (continued)

- If $k = 2$, since $\rho_2 = 20$ and fibers are of type I_{12}, I_6, I_2, I_2, I_1, I_1, $r_2 = 0$ (easy case).
- So,
$$| \det NS(Y_2) | = \frac{\prod m_{\nu,2}}{\text{Torsion}^2} = 8$$
- If $k = 10$, since $\rho_2 = 20$ and fibers are of type I_{12}, I_3, I_3, I_2, I_2, I_1, I_1, $r_{10} = 1$ (difficult case).
  - So, have to guess an infinite section,
  - have to use Néron's desingularization.

Some ingredients in the proof (continued)

- The value of $\det NS(Y_{10})$ gives the CM-field of the elliptic curve in the Shioda-Inose structure.
- Have to count the number of points of the reduction of Y_k modulo q ($q = p^r$).
- In case Y_k modular this allows to determine which modular form gives the equality
$$L(Y_k, s) = L(f, s).$$
- Compare to the expression of the Mahler measure and conclude.

Remarks

- We have

$$\det NS(Y_2) = -8$$

$$\det NS(Y_{10}) = -72$$

so the underlying elliptic curves E_2 and E_{10} are both CM on $\mathbb{Q}(\sqrt{-2})$.

- Since

$$L(Y_2, s) = L(Y_{10}, s) = L(f, s),$$

by Tate's conjecture, Y_2 and Y_{10} are related by an algebraic correspondance.