Mahler's measure
from Number Theory to Geometry

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I INTRODUCTION

The first interested in was D.H.Lehmer

" On factorization of certain cyclotomic functions" (1933)

with his famous question (still unsolved): does there exist a monic irreducible polynomial P, non cyclotomic, with integer coefficients such that

$$\Omega(P) := \prod_{P(\alpha)=0} max(|\alpha|, 1) < \Omega(P_0) \simeq 1.1762...$$

where P_0 is the Lehmer's polynomial

$$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1?$$

In fact

$$\Omega(P) = M(P)$$

the Mahler measure of P (introduced by Mahler in 1962).

The logarithmic Mahler's measure of a polynomial P is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} log \mid P(x_1, \cdots, x_n) \mid \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and the Mahler's measure

$$M(P) = exp(m(P)).$$

By Jensen's formula, if $P \in \mathbb{Z}[X]$ is monic, then

$$M(P) = \prod_{P(\alpha)=0} max(\mid \alpha \mid, 1).$$

The first partial answer to Lehmer's question is due to Smyth (1971)

$$M(P) \geq M(X^3 - X - 1) \simeq 1.32..$$

if P is non reciprocal. The obstruction for Lehmer's question is therefore the reciprocal polynomials.

Boyd's limit formula (1981)

$$m(P(x, x^N)) \longrightarrow m(P(x, y))$$

is a hope to get small measures in one variable from small measures in two variables.

$$M((x+1)y^2+(x^2+x+1)y+x(x+1)) = 1.25...$$

$$M(y^2+(x^2+x+1)y+x^2) = 1.28..$$

are the smallest known measures in two variables.

At the same time Smyth obtained the first explicit Mahler measures:

$$m(x+y+1) = L'(\chi_{-3}, -1)$$

$$m(x+y+z+1) = \frac{7}{2\pi^2}\zeta(3)$$

Deninger (1996) guessed

$$m(x+\frac{1}{x}+y+\frac{1}{y}+1) \stackrel{?}{=} \frac{15}{4\pi^2}L(E,2) = L'(E,0)$$

Since then, an abundant literature in this area, three Conferences on the Mahler measure and developments in many mathematics domains.

I want to focus on two questions.

- **There are experimental relations between the Mahler measure of different polynomials. Can we prove these relations? What they encode?**

- **How the geometry of the curve or the surface is involved in the explicit expressions?**

II CURVES OF GENUS 0

Let me take an example.

For

$$P = (y^2(x+1)^2 + 2y(x^2-6x+1) + (x+1)^2$$

Boyd guessed (1998)

$$m(P) \stackrel{?}{=} 4L'(\chi_{-4}, -1) = \frac{8}{\pi} L(\chi_{-4}, 2)$$

$$L(\chi_{-4}, 2) = 1 - \frac{1}{3^2} + \frac{1}{5^2} + \cdots = G$$

G being the Catalan's constant.

P defines a singular elliptic curve with $(1,1)$ as double point.

Put $x = 1 + X$ and $y = 1 + Y$, complete the square,

$$(Y(X+2)^2 + 2X^2)^2 = -16X^2(X+1).$$

Hence the parametrization of the two branches of the curve

$$x_1 = -t^2 \qquad y_1 = -(\frac{1+t}{1-t})^2$$

$$x_2 = -t^2 \qquad y_1 = -(\frac{1-t}{1+t})^2$$

But,

$$\begin{aligned} m(P) &= \frac{1}{(2\pi i)^2}\int_{|x|=1}\int_{|y|=1} \log|P(x,y)|\frac{dx}{x}\frac{dy}{y} \\ &= \frac{1}{2\pi i}\int_{|x|=1} \log(\max(|y_1|,|y_2|))\frac{dx}{x} \\ &= \frac{1}{2\pi i}\int_{\Gamma} \eta_2(2)(x,y) \end{aligned}$$

where

$$\eta_2(2)(x,y) = \log|y| di \arg x - \log|x| di \arg y$$

is a differential form on the variety Γ

$$\Gamma = \{(x,y) \in \mathbb{C}^2 / \ (x,y) \in C \ |x| = 1, \quad |y| = 1\}.$$

By parametrization,

$$m(P) = -\frac{1}{2\pi i}(\int_{\gamma_1} \eta_2(2)(x_1(t), y_1(t)) + \frac{1}{2\pi i}(\int_{\gamma_2} \eta_2(2)(x_2(t), y_2(t)))$$

Now $\eta_2(2)$ has something to do with the Bloch-Wigner dilogarithm D

$$D(x) := \Im Li_2(x) + \log|x| \arg(1-x)$$

univalued, real analytic in $\mathbb{P}^1(\mathbb{C}) \backslash \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$

$$\widehat{D}(x) = iD(x)$$

$$d\widehat{D}(x) = \eta_2(2)(x, 1-x)$$

Because of properties of $\eta_2(2)$

- multiplicative in each variable
- antisymetric
- if $\alpha \neq \beta$

$$\begin{aligned}\eta_2(2)(t-\alpha, t-\beta) &= \eta_2(2)(\frac{t-\alpha}{t-\beta}, 1-\frac{t-\alpha}{t-\beta}) \\ &+ \eta_2(2)(t-\alpha, \alpha-\beta) \\ &+ \eta_2(2)(\beta-\alpha, t-\beta)\end{aligned}$$

so

$$
\begin{aligned}
m(P) &= -\frac{1}{2\pi i}\int_{\gamma_1} 4d\widehat{D}(-t) - 4d\widehat{D}(t) \\
&\quad - \frac{1}{2\pi i}\int_{\gamma_2} -4d\widehat{D}(-t) + 4d\widehat{D}(t) \\
&= \frac{2}{\pi}[D(t) - D(-t)]_{-i}^{i} + \frac{2}{\pi}[D(-t) - D(t)]_{i}^{-i} \\
&= \frac{16}{\pi}D(i) \\
&= 4d_4
\end{aligned}
$$

So for some “good” genus 0 curves, the Mahler measure encodes the Bloch-Wigner dilogarithm hence the Bloch groups.

The Bloch groups

Let F a field and define

$$R_2(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$$

$$R_2(F) := [x] + [y] + [1 - xy] + [\frac{1-x}{1-xy}] + [\frac{1-y}{1-xy}]$$

$$B_2(F) := \mathbb{Z}[\mathbb{P}^1_F]/R_2(F)$$

$$B_2(F) \stackrel{\delta_1^2}{\to} \Lambda^2 F^*$$

$$\delta_1^2([x]_2) = x \wedge (1 - x)$$

The class of x in $B_2(F)$, $[x]_2$, behaves like a Bloch-Wigner dilogarithm.

The complex

$$B_F(2) \bigotimes \mathbb{Q} : B_2(F)_{\mathbb{Q}} \stackrel{\delta_1^2}{\to} (\Lambda^2 F^*)_{\mathbb{Q}}$$

has a cohomology related to K-theory by Matsumoto's theorem

$$H^2(B_F(2)) \simeq K_2(F)$$

The first Smyth's result can be treated in that context (Lalin).

There are "good" curves for this method (Vandervelde).

For them, the Mahler measure are dilogarithms of algebraic numbers plus eventually a term in $\zeta_F(2)$.

One of the "good" condition for a polynomial is to be "tempered".

Definition A polynomial in two variables is tempered if the polynomials of the faces of its Newton polygon has only roots of unity.

$$y^2 + y + x^2 + x + 1$$

is tempered

But

$$P := (x^2+x-1)y^2+(x^2+5x+1)y-x^2+x+1$$

is not tempered.

Boyd guessed for $m(P)$ the formula

$$A := m(P) \stackrel{?}{=\!=} \frac{2}{3}\log\phi + \frac{1}{6}d_{15}$$

$$d_{15} = L'(\chi_{-15}, 1)$$

where χ_{-15} is the odd primitive character of conductor 15.

By Bloch's formula,

$$\frac{1}{6}L'(\chi_{-15}, -1)$$

$$= \frac{5}{4\pi}[D(\zeta_{15} + D(\zeta_{15}^2) + D(\zeta_{15}^4) + D(\zeta_{15}^8)]$$

Using some parametrization, I got

$$A = \frac{2}{3}\log\phi - D(-j^2\frac{1-\sqrt{5}}{2}) - D(j\frac{1-\sqrt{5}}{2}) \\ - D(-j\frac{1+\sqrt{5}}{2}) - D(j^2\frac{1+\sqrt{5}}{2})$$

Numerically

$$C := -D(-j^2\frac{1-\sqrt{5}}{2}) - D(j\frac{1-\sqrt{5}}{2})$$

$$-D(-j\frac{1+\sqrt{5}}{2}) - D(j^2\frac{1+\sqrt{5}}{2})$$

$$\stackrel{?}{=}$$

$$E := \frac{5}{4}[D(\zeta_{15} + D(\zeta_{15}^2) + D(\zeta_{15}^4) + D(\zeta_{15}^8)]$$

So, even if the polynomial is not "tempered", there is underneath a relation between dilogarithms hence the objects are living in a Bloch group.

And by Galois descent in the Bloch group, C and E live in $B_2(\mathbb{Q}(\sqrt{-15})_{\mathbb{Q}}$.

I have not yet guessed to which element in $B_2(\mathbb{Q}(\sqrt{-15})_{\mathbb{Q}}$, C and E are equal.

III CURVES OF GENUS 1

Let

$$P = (x+1)(y+1)(x+y+1) + xy$$

and

$$Q = y'^2 - (x'^2 + 2x' - 1)y' + x'^3$$

Theorem (B. (2004))

$$7m(Q) = 5m(P)$$

Why?

If P defines a "good" curve of genus 1, such an equality encodes the K-theory of an elliptic curve.

The elliptic regulator

Let K be a field. By Matsumoto, $K_2(K)$ can be described in terms of symbols $\{f, g\}$, f and $g \in K^*$ and relations.

The relations are

- $\{f_1 f_2, g\} = \{f_1, g\} + \{f_2, g\}$
- $\{f, g_1 g_2\} = \{f, g_1\} + \{f, g_2\}$
- $\{1 - f, f\} = 0$

For example, if v is a discrete valuation on K with maximal ideal $\mathcal{M}$ and residual field k, Tate's tame symbol

$$(x, y)_v \equiv \ (-1)^{v(x)v(y)} \frac{x^{v(y)}}{y^{v(x)}} \qquad (mod \mathcal{M})$$

defines a homomorphism

$$\lambda_v : K_2(F) \to k^*.$$

Let E be an elliptic curve on $\mathbb{Q}$ and $\mathbb{Q}(E)$ its rational function field. To any $P \in E(\bar{\mathbb{Q}})$ is associated a valuation on $\mathbb{Q}(E)$ that gives the homomorphism

$$\lambda_P : K_2(\mathbb{Q}(E)) \to \mathbb{Q}(P)^*$$

and the exact sequence

$$0 \to K_2(E) \otimes \mathbb{Q} \to K_2(\mathbb{Q}(E)) \otimes \mathbb{Q} \xrightarrow{\lambda}$$

$$\coprod_{P \in E(\bar{\mathbb{Q}})} \mathbb{Q}(P)^* \otimes \mathbb{Q} \to \cdots .$$

By definition $K_2(E)$ is modulo torsion

$$K_2(E) \simeq \ker \lambda = \cap_P \ker \lambda_P \subset K_2(\mathbb{Q}(E)).$$

By a theorem due to Villegas, under some hypothesis, if $P \in \mathbb{Q}[x^{\pm}, y^{\pm}]$ defines a smooth curve C, we get

$$\{x, y\} \in K_2(C).$$

In particuliar, if

$$P(x, y) = (x + y + 1)(x + 1)(y + 1) + xy$$

we get

$$\{x, y\} \in K_2(E).$$

Let f et g dans $\mathbb{Q}(E)^*$ and define

$$\eta(f,g) = \log \mid f \mid d\arg g - \log \mid g \mid d\arg f.$$

Definition The elliptic regulator r of E is given by

$$\begin{array}{rcl} r : K_2(E) & \to & \mathbb{R} \\ \{f, g\} & \mapsto & \frac{1}{2\pi} \int_\gamma \eta(f,g) \end{array}$$

for a suitable loop γ generating $H_1(E,\mathbb{Z})^- \subset H_1(E,\mathbb{Z})$ where the complex conjugation acts by -1.

The elliptic dilogarithm $D^E(P)$

$$\begin{array}{rl} E(\mathbb{C}) \simeq & \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \to \mathbb{C}^*/q^{\mathbb{Z}} \\ (\mathcal{P}(u), \mathcal{P}'(u)) \to & u(mod.\lambda) \mapsto e^{2\pi i u} = z \end{array}$$

Define

$$D^E(P) = \sum_{n=-\infty}^{+\infty} D(q^n z)$$

for $P \in E(\mathbb{C})$ and D the Bloch-Wigner dilogarithm.

D^E can be extended to divisors on $E(\mathbb{C})$ and is also related to the elliptic regulator.

Now

$$\begin{aligned} m(P) &= -\frac{1}{2\pi i}\int_{|x|=1} \log|y_1|\frac{dx}{x} \\ &= \frac{1}{2\pi}\int_{\sigma} \eta(x,y) \end{aligned}$$

But σ generates $H_1(E,\mathbb{Z})^-$, so

$$m(P) = \pm r(\{x,y\})$$

For the same reasons,

$$m(Q) = \pm r(\{x',y'\})$$

Comparing these regulators with the regulator of the isomorphic elliptic curve $X_1(11)$

$$Y^2 + Y - X^3 + X^2 = 0$$

one gets

$$7r(\{X,Y\}) + r(\{x,y\}) = 0$$

$$-5r(\{X,Y\}) + r(\{x',y'\}) = 0$$

that is

$$5m(P) = 7m(Q)$$

Moreover, you get more, the proof of an "exotic" relation suspected by Bloch and Grayson

$$3D^E(P) = 2D^E(2P)$$

if $P = (0,0)$ is a 5-torsion point of $X_1(11)$.

SURFACES

Consider the family of Laurent polynomials

$$Q_k = X + \tfrac{1}{X} + Y + \tfrac{1}{Y} + Z + \tfrac{1}{Z}$$
$$+XY + \frac{1}{XY} + ZY + \frac{1}{ZY} + XYZ + \frac{1}{XYZ} - k.$$

and the relation guessed by Boyd

$$2m(Q_{-36}) \stackrel{?}{=} 4m(Q_{-6}) + m(Q_0)$$

What is under?

Computations possible thanks to the following result

Theorem 1. *(B. 2005) Let* $k = -(t + \frac{1}{t}) - 2$ *and*

$$t = \frac{\eta(3\tau)^4 \eta(12\tau)^8 \eta(2\tau)^{12}}{\eta(\tau)^4 \eta(4\tau)^8 \eta(6\tau)^{12}}.$$

$$
\begin{aligned}
m(Q_k) = \frac{\Im\tau}{8\pi^3}\{\sum_{m,\kappa}{}'(\\
&2(2\Re\frac{1}{(m\tau+\kappa)^3(m\bar{\tau}+\kappa)}\\
&+\frac{1}{(m\tau+\kappa)^2(m\bar{\tau}+\kappa)^2})\\
&-32(2\Re\frac{1}{(2m\tau+\kappa)^3(2m\bar{\tau}+\kappa)}\\
&+\frac{1}{(2m\tau+\kappa)^2(2m\bar{\tau}+\kappa)^2})\\
&-18(2\Re\frac{1}{(3m\tau+\kappa)^3(3m\bar{\tau}+\kappa)}\\
&+\frac{1}{(3m\tau+\kappa)^2(3m\bar{\tau}+\kappa)^2})\\
&+288(2\Re\frac{1}{(6m\tau+\kappa)^3(6m\bar{\tau}+\kappa)}\\
&+\frac{1}{(6m\tau+\kappa)^2(6m\bar{\tau}+\kappa)^2}))\}
\end{aligned}
$$

Brief comments

- More geometry is necessary since Q_k define $K3$-surfaces X_k.

- Since X_k is $K3$, there is on X_k a unique (up to scalars) holomorphic 2-form

- Since X_k is $K3$, one can define periods

- The family of periods satisfy a P.F. differential equation of order 3.

- $\frac{dm(P_k)}{dk}$ is a period of X_k

- The family is modular, so the previous formulae.

- For some X_k, the singular ones, $m(P_k)$ is related to the L-series of the variety.

– One of the most important result on $K3$-surfaces is a theorem of Morrison:

A $K3$-surface, $\mathcal{M}$-polarized, with Picard number 19, has a Shioda-Inose structure, that is

$$\begin{tikzcd} X \arrow[dr, "\iota"'] & & A = E \times E/C_N \arrow[dl] \\ & Y = Kum(A/\pm) & \end{tikzcd}$$

where C_N is a cyclic group of isogeny and $X/\langle\iota\rangle$ is birationally isomorphic to Y.

– If X is singular (Picard number 20), then E has complex multiplication.

Now we get the theorem

Theorem 2. *(B. 2005)*

Let $\mathbb{Q}(\sqrt{-3})$ *and* $R = (1, 2\sqrt{-3}) \subset R' = (1, \sqrt{-3})$ *two orders of discriminants* -48 *(resp.* -12*), with class numbers* 2 *(resp.* 1*).*

Let Φ_R *(resp.* $\Phi_{R'}$*) the Hecke Grössencharacter of weight* 3*:*

$$\Phi_R(\alpha R) = \alpha^2 \quad \Phi_R(P) = -3 \quad P = (3, 2\sqrt{-3})$$

$$\Phi_{R'}(\beta R') = \beta^2$$

Then the relation

$$2m(Q_{-36}) = 4m(Q_{-6}) + m(Q_0)$$

is equivalent to

$$\frac{9}{8}\sum_{m,\kappa}{}' \frac{m^2 - 3\kappa^2}{(m^2 + 3\kappa^2)^3}$$

$$= \sum_{m,\kappa}{}' \left(\frac{4m^2 - 3\kappa^2}{(4m^2 + 3\kappa^2)^3} - \frac{12m^2 - \kappa^2}{(12m^2 + \kappa^2)^3}\right.$$

•Zagier proved that this is in fact a relation between the L-series of weight 3 modular forms for $\Gamma_0(4)$

$$(1+2\times 4^{1-s})L(f,s) = L(f_1,s) + L(f_2,s)$$

where $f = [\theta_1, \theta_3]$, $f_1 = [\theta_1, \theta_{12}]$ and $f_2 = [\theta_4, \theta_3]$ are Rankin-Cohen brackets.

$$\theta_a = \sum_{n\in\mathbb{Z}} q^{an^2}$$

$$RC(g,h) = [g,h] = kgh' - lg'h$$

is a modular form of weight $k+l+2$ if g is of weight k and h of weight l.

Final remarks

Using Zagier-Goncharov trilogarithm, Lalin generalized the wedge product to 3 variables, explaining for instance the second Smyth's relation and also

$$m((1+x+y^{-1})-(1+x+y)z)=\frac{14}{3\pi^2}\zeta(3)$$

(Smyth)

$$m((1+x)(1+y)-(1-x)(1-y)z))=\frac{7}{3\pi^2}\zeta(3)$$

(Lalin)

For all these examples the surfaces are rational of a certain type.

What are the explicit formulae for rational elliptic surfaces such as

$$x(x-1)(y-1)=zy(x-y),$$

the rational elliptic modular surface associated to $\Gamma_0(6)$?