Mahler measure from Number Theory To Algebraic Geometry (ICPAMS2022)

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Introduction

Introduced by Mahler in 1962,
the logarithmic Mahler measure of a polynomial P is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log \mid P(x_1, \cdots, x_n) \mid \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and its Mahler measure

$$M(P) = \exp(m(P))$$

where

$$\mathbb{T}^n = \{(x_1, \cdots, x_n) \in \mathbb{C}^n / |x_1| = \cdots = |x_n| = 1\}.$$

Remarks

- $n = 1$
  By Jensen's formula, if $P \in \mathbb{Z}[X]$ is monic, then

$$M(P) = \prod_{P(\alpha)=0} \max(|\alpha|, 1).$$

So it is related to Lehmer's question (1933)
Does there exist $P \in \mathbb{Z}[X]$, monic, non cyclotomic, satisfying

$$1 < M(P) < M(P_0) = 1.1762\cdots ?$$

The polynomial

$$P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

is the Lehmer polynomial, in fact a Salem polynomial.

Lehmer's problem is still open.
A partial answer by Smyth (1971)

$$M(P) \geq 1.32\cdots$$

if P is non reciprocal.

First explicit Mahler measures

- $m(1+x) = 0$ (by Jensen's formula)
- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) =: L'(\chi_{-3}, -1) \ \text{ Smyth (1980)}$$
- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) \ \text{ Smyth (1980)}$$

$$L(\chi_{-3}, s) = \sum_{n\geq 1} \frac{\chi_{-3}(n)}{n^s}$$

$$\chi_{-3}(3k) = 0, \quad \chi_{-3}(3k+1) = 1, \quad \chi_{-3}(3k+2) = -1$$

Deninger (1996) conjectured

$$m(x + \frac{1}{x} + y + \frac{1}{y} + 1) \stackrel{?}{=} \frac{15}{4\pi^2} L(E, 2) =: L'(E, 0)$$

E elliptic curve of conductor 15 defined by the polynomial
This conjecture was proved (May 2011) by Rogers and Zudilin thanks to a previous result due to Lalin.
Deninger's guess comes from Beilinson's Conjectures.

Villegas's results (1998)

$$m(x+1/x+y+1/y-k) = \frac{1}{2}\Re[-2\pi i\tau + 4\sum_{n=1}^{\infty}\sum_{d|n}\chi(d)d^2\frac{q^n}{n}]$$

or in terms of Eisenstein's series

$$\Re[\frac{16\Im(\tau)}{\pi^2}\sum_{m,n\in\mathbb{Z}}\chi(n)\frac{1}{(m4\tau+n)^2(m4\bar{\tau}+n)}]$$

where $q = \exp 2\pi i\tau$ and $\chi(n) = \left(\frac{n}{4}\right)$

$$k^2 = 1/\mu(\tau) \qquad \mu = q - 8q^2 + 44q^3 - 192q^4 + \ldots$$

When k defines a CM elliptic curve, namely $k = 4\sqrt{2}$ defining

$$A : y^2 = x^3 - 44x + 112 \qquad \text{with conductor} \qquad 64$$

it follows

$$m(x + 1/x + y + 1/y - 4\sqrt{2}) = \frac{64}{4\pi^2} L(A, 2)$$

Also, if $k = 4/\sqrt{2}$ defining

$$B : y^2 = x^3 + 4x \qquad \text{with conductor} \qquad 32$$

it follows

$$m(x + 1/x + y + 1/y - 4/\sqrt{2}) = \frac{32}{4\pi^2} L(B, 2)$$

Finally for $k = 3\sqrt{2}$ we get the modular elliptic curve $X_0(24)$ and using Beilinson's theorem it is possible to get a formula of the same type for the Mahler measure.

A similar result was proved by Benferhat (2009) (one of my former students) concerning the family

$$x + 1/x + y + 1/y + x/y + y/x - k = 0$$

written as

$$1/xy[(x + y + 1)(xy + y + x) - (k + 3)xy] = 0$$

Hints of proof

From Verrill we know that putting $k + 3 = 1/t$, it defines an elliptic modular surface for the congruence group $\Gamma_1(6)$ with Picard-Fuchs equation near 0 (satisfied by the periods)

$$t(t-1)(9t-1)f'' + (27t^2 - 20t + 1)f' + 3(3t-1)f = 0$$

with two properties

- For the Hauptmodul
  $t = \frac{\eta(6\tau)^8\eta(\tau)^4}{\eta(3\tau)^4\eta(2\tau)^8} = q - 4q^2 + 10q^3 - 20q^4 + 39q^5 + \ldots$
- the solution near 0 is expressed as

$$f = \frac{\eta(2\tau)^6\eta(3\tau)}{\eta(\tau)^3\eta(6\tau)^2}$$

- With $k + 3 = 1/t$ it follows that

$$\tilde{m}'(k) = \frac{1}{2i(\pi)^2}\int_{(\mathbb{T})^2} \frac{t}{-1 + \frac{(x+y+1)(xy+y+x)}{xy}} \frac{dx}{x}\frac{dy}{y}$$

is a period of the elliptic curve. Hence it satisfies the Picard-Fuchs equation; moreover it can be identified with the solution near 0. Thus

$$\tilde{m}'(k) = -tf \qquad d\tilde{m} = -f\frac{dt}{t} = -f\frac{t'(q)dq}{t}$$

$$-f(t)\frac{q\frac{dt}{dq}}{t} = 1 + L(q) + 8L(q^2) \qquad L(q) = \sum_{n\geq 1}(\sum_{d|n}\chi(d)d^2)q^n$$

Finally by integration we get

$$
\begin{aligned}
m(k) = \; & \Re\left(-2i\pi\tau + \sum_{n\geq 1}\left(\sum_{d|n}\chi(d)d^2\right)\frac{\exp 2i\pi n\tau}{n}\right)\\
& +8\left(\Re\sum_{n\geq 1}\left(\sum_{d|n}\chi(d)d^2\right)\frac{\exp 4i\pi n\tau}{2n}\right)
\end{aligned}
$$

and in terms of Eisenstein-Kronecker series

$$
\begin{aligned}
m(k) = \; & \Re\left(\frac{9\sqrt{3}\Im\tau}{4\pi^2}\sum_{(m,n)\neq(0,0)}\frac{\chi(n)}{(3m\tau+n)^2(3m\bar{\tau}+n)}\right)\\
& +8\Re\left(\frac{9\sqrt{3}\Im\tau}{4\pi^2}\sum_{(m,n)\neq(0,0)}\frac{\chi(n)}{(6m\tau+n)^2(6m\bar{\tau}+n)}\right)
\end{aligned}
$$

For $k = 0$ the elliptic curve is CM with conductor 36 more precisely $36a1$ with $j = 0$, τ is imaginary quadratic and we can recover $m(0) = 2L'(E_{36}, 0)$.

CM elliptic curves and elliptic modular curves are rare in these families. Other people Mellit, Zudilin, Brunault used other techniques.
For example Mellit obtained results on the same modular surface, that is

$$m(1) = b_{14} \qquad m(-5) = 6b_{14} \qquad m(10) = 10b_{14}$$

(all these conjectured by Boyd.) We recall Boyd's notation
$b_N = \frac{N}{4\pi^2} L(E_N, 2)$
A new technique was elaborated by Zudilin and Brunault parametrizing the elliptic curves with modular units. Based on regulators and modular units I obtained (August 2015, unpublished)

$$m(4) = 3b_{20} \qquad m(-2) = 2b_{20}$$

thus solving Touafek's conjectures on regulators.

Finally with similar techniques Brunault considered the family

$$y^2 + kx + y - x^3$$

and proved (arXiv 2015)

$$m(-1) = 2b_{14} \qquad m(-2) = b_{35} \qquad m(-3) = b_{54}$$

Remark that this defines an elliptic surface with 4 singular fibers $[9, 1, 1, 1]$ which is precisely one of Beauville modular elliptic surface for the congruence group $\Gamma_0(9) \cap \Gamma_1(3)$.
While preparing this talk I noticed that one of my former students Rémi Trannoy studied experimentally this family and conjectured $m(-5) \stackrel{?}{=} 7b_{20}$, $m(6) \stackrel{?}{=} 3b_{27}$. Combining all the previous methods can we prove these conjectures?

From elliptic curves to $K3$ surfaces

So replace E by a surface X which is also a Calabi-Yau variety, i.e. a $K3$-surface and try to answer the questions:

What are the analog of Deninger, Boyd, R-Villegas 's results and conjectures?

Which type of Eisenstein-Kronecker series corresponds to $L(X, 3)$?

Basic facts on K3-surfaces

Our results concern polynomials of the family

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining K3-surfaces Y_k. What's a K3-surface?
It is a smooth surface X satisfying

- $H^1(X, \mathcal{O}_X) = 0$ i.e. X simply connected
- $K_X = 0$ i.e. the canonical bundle is trivial i.e. there exists a unique, up to scalars, holomorphic 2-form ω on X.

- A double covering branched along a plane sextic for example defines a K3-surface X.
  In our case

$$(2z+x+\frac{1}{x}+y+\frac{1}{y}-k)^2=(x+\frac{1}{x}+y+\frac{1}{y}-k)^2-4$$

Main properties

- $H_2(X,\mathbb{Z})$ is a free group of rank 22.

Main properties (continued)

- With the intersection pairing, $H_2(X,\mathbb{Z})$ is a lattice and

$$H_2(X,\mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

$\mathcal{L}$ is the $K3$-lattice, U_2 the hyperbolic lattice of rank 2, E_8 the unimodular lattice of rank 8.

- $$Pic(X) \subset H_2(X,\mathbb{Z}) \simeq Hom(H^2(X,\mathbb{Z}),\mathbb{Z})$$

where $Pic(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles (since for $K3$ surfaces linear and algebraic equivalence are the same).

- $$Pic(X) \simeq \mathbb{Z}^{\rho(X)}$$

$$\rho(X) := \text{Picard number of } X$$

$$1 \leq \rho(X) \leq 20$$

- $$T(X) := (Pic(X))^{\perp}$$

  is the transcendental lattice of dimension $22 - \rho(X)$
- If $\{\gamma_1, \cdots, \gamma_{22}\}$ is a $\mathbb{Z}$-basis of $H_2(X, \mathbb{Z})$ and ω the holomorphic 2-form,

  $$\int_{\gamma_i} \omega$$

  is called a period of X and

  $$\int_{\gamma} \omega = 0 \text{ for } \gamma \in Pic(X).$$
- If $\{X_z\}$ is a family of $K3$ surfaces, $z \in \mathbb{P}^1$ with generic Picard number ρ and ω_z the corresponding holomorphic 2-form, then the periods of X_z satisfy a Picard-Fuchs differential equation of order $k = 22 - \rho$. For our family $k = 3$.

- In fact, by Morrison, a $\mathcal{M}$-polarized $K3$-surface, with Picard number ≥ 19 has a Shioda-Inose structure, that means

$$\begin{array}{ccc} X & & A = E \times E/C_N \\ & \searrow \quad \swarrow & \\ & Y = Kum(A/\pm 1) & \end{array}$$

- If the Picard number $\rho = 20$, then the elliptic curve is CM.

Mahler measure of P_k

Theorem

(B. 2005) Let $k = t + \frac{1}{t}$ *and*

$$t = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^6, \quad \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n\geq 1}(1 - e^{2\pi i n\tau}), \quad q = \exp 2\pi i\tau$$

$$\begin{aligned} m(P_k) =& \frac{\Im\tau}{8\pi^3}\Big\{\sum_{m,\kappa}{}' \Big(-4\Big(2\Re\frac{1}{(m\tau+\kappa)^3(m\bar\tau+\kappa)} + \frac{1}{(m\tau+\kappa)^2(m\bar\tau+\kappa)^2}\Big) \\ &+ 16\Big(2\Re\frac{1}{(2m\tau+\kappa)^3(2m\bar\tau+\kappa)} + \frac{1}{(2m\tau+\kappa)^2(2m\bar\tau+\kappa)^2}\Big) \\ &- 36\Big(2\Re\frac{1}{(3m\tau+\kappa)^3(3m\bar\tau+\kappa)} + \frac{1}{(3m\tau+\kappa)^2(3m\bar\tau+\kappa)^2}\Big) \\ &+ 144\Big(2\Re\frac{1}{(6m\tau+\kappa)^3(6m\bar\tau+\kappa)} + \frac{1}{(6m\tau+\kappa)^2(6m\bar\tau+\kappa)^2}\Big)\Big)\Big\} \end{aligned}$$

Sketch of proof

Let

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining the family (X_k) of $K3$-surfaces.

- For $k \in \mathbb{P}^1$, generically $\rho = 19$.
- The family is $\mathcal{M}_k$-polarized with

$$\mathcal{M}_k \simeq U_2 \perp (-E_8)^2 \perp \langle -12 \rangle$$

- Its transcendental lattice satisfies

$$T_k \simeq U_2 \perp \langle 12 \rangle$$

- The Picard-Fuchs differential equation is

$$(k^2-4)(k^2-36)y''' + 6k(k^2-20)y'' + (7k^2-48)y' + ky = 0$$

(Peters and Stienstra's results)

- The family is modular in the following sense
if $k = t + \frac{1}{t}$, $\tau \in \mathcal{H}$ and τ as in the theorem

$$t(\frac{a\tau + b}{c\tau + d}) = t(\tau) \ \forall \ (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_1(6,2)^* \subset \Gamma_0(12)^* + 12$$

where

$$\Gamma_1(6) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in Sl_2(\mathbb{Z}) \ / \ a \equiv d \equiv 1 \ (6) \ \ c \equiv 0 \ (6)\}$$

$$\Gamma_1(6,2) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_1(6) \ \ c \equiv 6b \ \ (12)\}$$

and

$$\Gamma_1(6,2)^* = \langle \Gamma_1(6,2), w_6 \rangle$$

.

- The P-F equation has a basis of solutions $G(\tau)$, $\tau G(\tau)$, $\tau^2 G(\tau)$ with

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

satisfying

$$G(\tau) = F(t(\tau)), \quad F(t) = \sum_{n\geq 0} v_n t^{2n+1}, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

- $\frac{dm(P_k)}{dk}$ is a period, hence satisfies the P-F equation
-
$$\frac{dm(P_k)}{dk} = G(\tau)$$
-
$$dm(P_k) = -G(\tau)\frac{dt}{t}\frac{1-t^2}{t}$$

is a weight 4 modular form for $\Gamma_1(6,2)^*$
- so can be expressed as a combination of $E_4(n\tau)$ for $n = 1, 2, 3, 6$

- By integration you get

$$m(P_k) = \Re(-\pi i \tau + \sum_{n \geq 1} (\sum_{d|n} d^3)(4\frac{q^n}{n} - 8\frac{q^{2n}}{2n} + 12\frac{q^{3n}}{3n} - 24\frac{q^{6n}}{6n}))$$

- Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series

The singular $K3$ surfaces of the Apéry-Fermi's family (Y_k) correspond to imaginary quadratic τ such that

$$t = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau\eta(3\tau)} \right)^6, \qquad k = t + \frac{1}{t}.$$

They have been computed by Boyd.

k	τ	Equation of τ
0	$\frac{-3+\sqrt{-3}}{6}$	$3\tau^2+3\tau+1=0$
2	$\frac{-2+\sqrt{-2}}{6}$	$6\tau^2+4\tau+1=0$
3	$\frac{-3+\sqrt{-15}}{12}$	$6\tau^2+3\tau+1=0$
6	$\frac{\sqrt{-6}}{6}$	$6\tau^2+1=0$
10	$\frac{\sqrt{-2}}{2}$	$2\tau^2+1=0$
18	$\frac{\sqrt{-30}}{6}$	$6\tau^2+5=0$
102	$\frac{\sqrt{-6\times 13}}{6}$	$6\tau^2+13=0$
198	$\frac{\sqrt{-17\times 6}}{6}$	$6\tau^2+17=0$
$2\sqrt{5}$	$\frac{-1+\sqrt{-5}}{6}$	$6\tau^2+2\tau+1=0$
$3\sqrt{6}$	$\frac{\sqrt{-3}}{3}$	$3\tau^2+1=0$
$2\sqrt{-3}$	$\frac{-1+\sqrt{-1}}{2}$	$2\tau^2+2\tau+1=0$
$3\sqrt{-5}$	$\frac{-3+\sqrt{-15}}{6}$	$3\tau^2+3\tau+2=0$

Livné's modulary theorem

Theorem

Let S be a $K3$-surface defined over $\mathbb{Q}$, with Picard number 20 and discriminant N. Its transcendental lattice $T(S)$ is a dimension 2 $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$-module thus defines a L series, $L(T(S), s)$.
There exists a weight 3 modular form , f, CM over $\mathbb{Q}(\sqrt{-N})$ satisfying

$$L(T(S), s) \doteq L(f, s) = \sum_{n \geq 1} \frac{A_n}{n^s}.$$

Lemma

(B. 2010) Let Y an elliptic $K3$-surface defined over $\mathbb{Q}$ by a Weierstrass equation $Y(t)$. If rank $(Y(t)) = r$ and the r infinite sections generating the Mordell-Weil lattice are defined respectively over $\mathbb{Q}(\sqrt{d_i})$,$i = 1, ..., r$, then

$$A_p = - \sum_{t \in \mathbb{P}^1(\mathbb{F}_p),\quad Y(t) \quad smooth} a_p(t) - \sum_{t \in \mathbb{P}^1(\mathbb{F}_p),\quad Y(t) \quad singular} \epsilon_p(t) - \sum_{i=1}^{r} \left(\frac{d_i}{p}\right)$$

where

$$a_p(t) = p + 1 - \# Y(t)(\mathbb{F}_p)$$

and $\epsilon_p(x)$ defined by

$$\epsilon_p(t) = \begin{cases} 0, & \text{if the reduction of } Y(t) \text{ is additive} \\ 1, & \text{if the reduction of } Y(t) \text{ is split multiplicative} \\ -1, & \text{if the reduction of } Y(t) \text{ is non split multiplicative} \end{cases} .$$

Theorem

(Schütt's classification) Consider the following classification of singular $K3$-surfaces over $\mathbb{Q}$

1. *by the discriminant d of the transcendental lattice of the surface up to squares,*
2. *by the associated newform up to twisting,*
3. *by the level of the associated newform up to squares,*
4. *by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.*

Then, all these classifications are equivalent. In particuliar, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.

Let

$$P_k = x^2yz + xy^2z + xyz^2 + t^2(xy + xz + yz) - kxyzt.$$

Y_k is the desingularization of the set of zeroes of P_k.

Computation of the discriminant N

Different methods allow us to obtain the transcendental lattice, hence its determinant equal to N, of the singular $K3$ lattice. (see Bertin and Lecacheux,arxiv 2022)

Y_0	$[4 \quad 2 \quad 4]$
Y_2	$[2 \quad 0 \quad 4]$
Y_3	$[2 \quad 1 \quad 8]$
Y_6	$[2 \quad 0 \quad 12]$
Y_{10}	$[6 \quad 0 \quad 12]$
Y_{18}	$[10 \quad 0 \quad 12]$
Y_{102}	$[12 \quad 0 \quad 26]$
Y_{198}	$[12 \quad 0 \quad 34]$
$Y_{k^2=20}$	$[2 \quad 0 \quad 10]$
$Y_{k^2=54}$	$[4 \quad 0 \quad 12]$
$Y_{k^2=-12}$	$[6 \quad 0 \quad 6]$
$Y_{k^2=-45}$	$[8 \quad 2 \quad 8]$

Shimada and Zhang 's notation:

$$[a \quad b \quad c] := \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

- From the expression of the Mahler measure in terms of Eisenstein Kronecker series depending of quadratic imaginary τ it follows the Mahler measure as the L-series of a modular form (modular part) plus eventually a Dirichlet L-series.

$m(P_0) =$	d_3 Boyd, Bertin (2005)
$m(P_2) =$	$\frac{8\sqrt{8}}{\pi^3}L(f_8,3)$ (B-2009)
$m(P_3) =$	$\frac{15\sqrt{15}}{2\pi^3}L(f_{15},3)$ (BFFLM-2013)
$m(P_6) =$	$\frac{24\sqrt{24}}{2\pi^3}L(f_{24},3)$ (BFFLM-2013)
$m(P_{10}) =$	$\frac{72\sqrt{72}}{9\pi^3}L(f_8,3)+2d_3$ (B-2010)
$m(P_{18}) =$	$\frac{120\sqrt{120}}{9\pi^3}L(f_{120},3)+\frac{14}{5}d_3$ (BFFLM-2013)
$m(P_{k^2=20})$	$2\frac{20\sqrt{20}}{4\pi^3}L(f_{20}\otimes\chi_5,3)$ (B-2018)
$m(P_{k^2=-45})$	$\frac{6}{5}\frac{15\sqrt{15}}{2\pi^3}L(f_{15}\otimes\chi_5,3)+\frac{d_{15}}{10}$ (B-2018)
$m(P_{k^2=-12})$	$\frac{36}{2\pi^3}(L(f_{36},3)+L(g_{36},3))$ +Dirichlet L-series(B-2022)

The L-series of $Y_{k^2=-45}$ and $Y_{k^2=-12}$

$$L(Y_{k^2=-45},3) = L(f_{15} \otimes \chi_5, 3) \tag{1}$$

$$L(Y_{k^2=-12},3) = L(f_{36} \otimes \chi_3, 3) \tag{2}$$

To compute these L-series we apply the lemma. Thus we need

- an elliptic fibration with Weierstrass equation defined over $\mathbb{Q}$;
- the r infinite sections generating the Mordell-Weil lattice

For both Weierstrass equations defining $Y_{k^2=-45}$ and $Y_{k^2=-12}$ we get $r=2$. For both, from results of Bertin and Lecacheux, we obtain one infinite section.

In the first case, $Y_{k^2=-45}$ is the Kummer surface of another surface Z_{-3} since $T_{Z_{-3}} = [4 \quad 1 \quad 4]$. Thus there exists a 2-isogeny between the surface and its Kummer.

Since Z_{-3} has an elliptic fibration with $r=0$ its L-series can be easily computed and gives (1).

Proof of (2)

$Y_{k^2=-12}$ has an elliptic fibration with Weierstrass equation

$$y^2 = x^3 - (t^3 + 3t^2 - 6t + 4)x^2 + t^3 x$$

with two infinite sections

$$(1, (t-1)\sqrt{-3}) \text{ from(B-L)},(\left(\frac{t-4}{t+2}\right)^2, \frac{3(t^2-16)t(t-1)}{(t+2)^3})\text{(Sage)}$$

One infinite section defined over $\mathbb{Q}(\sqrt{-3})$ and the other over $\mathbb{Q}$.
The $A(p)$ are computed using the Pari order

$$A(p) = -sum(t = 2, p-1, ellak(e(t), p)) - kronecker(-3, p)p$$

-p-kronecker(-1,p)

Proof of (2)

Now we must compare to the $\alpha(p)$ given by the CM newform of level 36 and weight 3 (36.3.d.a in LMFDB)

$$f_{36}(q) = q - 2q^2 + 4q^4 + 8q^5 - 8q^8 - 16q^{10} - 10q^{13} + 16q^{16} - 16q^{17} + 32q^{20} + 39q^{25}$$

p	5	11	13	17	19	23	29	31	37	41	43	47	53
$\alpha(p)$	-8	0	-10	-16	0	0	-40	0	-70	80	0	0	56
A(p)	-8	0	-10	16	0	0	40	0	-70	-80	0	0	-56

Hence

$$L(Y_{k^2=-12}, 3) = L(f_{36} \oplus \chi_{-3}, 3)$$

where $\chi_{-3}(3n) = 0$, $\chi_{-3}(3n+1) = 1$, $\chi_{-3}(3n+2) = -1$

$m(P_{k^2=-12}) = \frac{36}{2\pi^3}\left(L(f_{36},3) + L(g_{36})\right) +$ Dirichlet L-series

For $k^2 = -12$ we get $\tau = \frac{-1+i}{2}$. We must write $m(P_{k^2=-12})$ as a modular part and a Dirichlet part.

$$m(P_{k^2=-12}) = \frac{36 \times 6}{12\pi^3} \sum_{(k,m)\neq(0,0)} \frac{4m^2 - 2k^2 + 2km}{(5m^2 - 2km + 2k^2)^3} + \frac{k^2 - 9m^2}{(k^2 + 9m^2)^3} + D$$

$$D = \frac{9}{\pi^3} \sum_{(k,m)\neq(0,0)} \frac{-1}{(5m^2 - 2km + 2k^2)^2} + \frac{1}{(k^2 + 9m^2)^2}$$

that is

$$m(P_{k^2=-12}) = \frac{36}{2\pi^3}\left(L(f_{36},3) + L(g_{36},3)\right) + D$$

g_{36} is the weight 3 modular form for $\Gamma_0(4)$

$$g_{36} = \frac{1}{2} \sum_{k,m \in \mathbb{Z}} (k^2 - 9m^2) q^{k^2+9m^2}$$

It is the Rankin-Cohen bracket $[\theta_1, \theta_9] := \frac{1}{2}\theta_1\theta_9' - \frac{1}{2}\theta_1'\theta_9$
where $\theta_a = \sum_{n\in\mathbb{Z}} q^{an^2}$

$$D = \frac{4}{3} d_3 + \frac{7}{27} d_4$$

where the Dirichlet L-series is defined by

$$d_f = \frac{f\sqrt{f}}{4\pi} \sum_{n \geq 1} \frac{\chi_{-f}(n)}{n^2}$$