Mahler measure and L-series of K3-hypersurfaces

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Historic background and motivation

The first reference is Lehmer's question (1933) (still unsolved):
Let $P \in \mathbb{Z}[X]$, monic, non cyclotomic and define:

$$\Omega(P) = \prod_{P(\alpha)=0} \max(|\alpha|, 1),$$

does there exist such a P satisfying

$$1 < \Omega(P) < \Omega(P_0) = 1.1762\cdots ?$$

where

$$P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

is the Lehmer polynomial, in fact a Salem polynomial.

Introduced by Mahler in 1962,
the logarithmic Mahler measure of a polynomial P is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log \mid P(x_1, \cdots, x_n) \mid \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and its Mahler measure

$$M(P) = \exp(m(P))$$

where

$$\mathbb{T}^n = \{(x_1, \cdots, x_n) \in \mathbb{C}^n / |x_1| = \cdots = |x_n| = 1\}.$$

By Jensen's formula , $\Omega(P) = M(P)$ if P a one variable polynomial.

Smyth's result

Lehmer's problem is still open.
A partial answer by Smyth (1971)

$$M(P) \geq 1.32\cdots$$

if P is non reciprocal.
Thus the focus on reciprocal polynomials.

Boyd's limit formula

In 1981, Boyd's limit formula was a great hope:

$$m(P(x, x^n)) \to m(P(x, y))$$

since small measures in one variable could be obtained from small measures in two variables.

Boyd computed:

$$M(x + \frac{1}{x} + y + \frac{1}{y} + \frac{x}{y} + \frac{y}{x} + 1) = 1.25...$$

$$M(x + \frac{1}{x} + y + \frac{1}{y} + 1) = 1.28...$$

$$M(x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + 1) = 1.4483... \text{Boyd or Mossinghoff (2006)}$$

I obtained

$$M(x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + xy + \frac{1}{xy} + zy + \frac{1}{zy} + xyz + \frac{1}{xyz} + 1) = 1.4351...$$

These are the smallest known measures in 2 or 3 variables.

Smyth explicit measures (1981)

At the same time (1981) Smyth was visiting Boyd and found his first explicit Mahler measures

$$m(x+y+1) = \frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) = L'(\chi_{-3},-1) =: d_3$$

$$d_3 := \frac{3\sqrt{3}}{4\pi}\sum_{n\geq 1}\frac{\chi_{-3}(n)}{n^2}.$$

$$m(x+y+z+1) = \frac{7}{2\pi^2}\zeta(3)$$

It would be tempting to know $m(x+y+z+t+1)$ and $m(x+y+z+t+w+1)$ but they are only conjectures by Rodriguez-Villegas.

The Calgary CMS Summer meeting (1996)

There, Boyd met Deninger and Deninger guessed the famous explicit Mahler measure

$$m(x + \frac{1}{x} + y + \frac{1}{y} + 1) = \frac{15}{4\pi^2} L(E, 2) =: L'(E, 0),$$

E elliptic curve,algebraic closure of the zero set of the polynomial, denoted $15a8$ (Cremona's notation), of conductor 15, defined by

$$Y^2 + XY + Y = X^3 + X^2$$

with L-series given by the modular form

$$f_{15A}(z) = \eta(z)\eta(3z)\eta(5z)\eta(15z)$$

(Deninger's guess was proved in 2011 by Rogers and Zudilin and again in 2013 by Zudilin.)

It was the starting point of intensive research, first by Boyd, then by Rodriguez-Villegas and others.
Boyd studied many pencils of elliptic curves and curves of genus 2. He conjectured lots of explicit measures and found a necessary condition: the polynomial P must be tempered

Basic facts on elliptic K3-surfaces

It was very tempting to generalise the above results or conjectures to the family

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + k$$

My own reference was, at that time "A pencil of $K3$-surfaces related to Apéry's recurrence for $\zeta(3)$ and Fermi surfaces for potential zero" by Peters and Stienstra (1988)

What's a K3-surface?

- A double covering branched along a plane sextic for example defines a K3-surface X.

  In case of the Apéry-Fermi pencil

$$(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4$$

- An elliptic $K3$ surface X admits a fibration $\pi : X \to \mathbb{P}^1$ such that the fiber $\pi^{-1}(t)$ is an elliptic curve for all but a finite number of t giving the singular fibers classified by Kodaira.
- Given an elliptic surface as

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

  you recognise an elliptic fibration on a $K3$ surface if the degree of polynomials a_i is $\leq 2i$ and is exactly $2i$ for one i.
- How to get an elliptic fibration on the Apéry-Fermi pencil?
  First, as did Peters and van Vglut, in cutting by a pencil of planes $X + Y + Z = t$. This gives the elliptic fibration:

$$y^2 - xy(t^2 - kt + 1) = x(x-1)(x + t^2 - tk)$$

  t is called an elliptic parameter
  But, if we want to obtain all the elliptic fibrations of the pencil, we must use the technique of Elkies's neighbors. (see our recent preprint B. and Lecacheux (2018)).

- There is a unique holomorphic 2-form ω on X up to a scalar.
- $H_2(X, \mathbb{Z})$ is a free group of rank 22.
- With the intersection pairing, $H_2(X, \mathbb{Z})$ is a lattice and

$$H_2(X, \mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

  $\mathcal{L}$ is the $K3$-lattice, U_2 the hyperbolic lattice of rank 2, E_8 the unimodular lattice of rank 8.
- $$Pic(X) \subset H_2(X, \mathbb{Z}) \simeq Hom(H^2(X, \mathbb{Z}), \mathbb{Z})$$

  where $Pic(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles.
- $$Pic(X) \simeq \mathbb{Z}^{\rho(X)}$$

$$\rho(X) := \text{Picard number of } X$$

$$1 \leq \rho(X) \leq 20$$

- $$T(X) := (Pic(X))^{\perp}$$
  is the transcendental lattice of dimension $22 - \rho(X)$
- If $\{\gamma_1, \cdots, \gamma_{22}\}$ is a $\mathbb{Z}$-basis of $H_2(X, \mathbb{Z})$ and ω the holomorphic 2-form,
  $$\int_{\gamma_i} \omega$$
  is called a period of X and
  $$\int_{\gamma} \omega = 0 \text{ for } \gamma \in Pic(X).$$
- If $\{X_z\}$ is a family of $K3$ surfaces, $z \in \mathbb{P}^1$ with generic Picard number ρ and ω_z the corresponding holomorphic 2-form, then the periods of X_z satisfy a Picard-Fuchs differential equation of order $k = 22 - \rho$. For our family $k = 3$.

- In fact, by Morrison, a $\mathcal{M}$-polarized $K3$-surface, with Picard number 19 or 20 has a Shioda-Inose structure, that means

$$\begin{array}{ccc} X & & A = E \times E/C_N \\ & \searrow \quad \swarrow & \\ & Y = Kum(A/\pm 1) & \end{array}$$

- If the Picard number $\rho = 20$, then the elliptic curve is CM.

Mahler measure of P_k

Theorem

(B. 2005) Let $k = t + \frac{1}{t}$ *and*

$$t = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^6, \;\; \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n\geq 1}(1 - e^{2\pi i n\tau}), \;\; q = \exp 2\pi i\tau$$

$$\begin{aligned} m(P_k) = & \frac{\Im\tau}{8\pi^3}\Big\{\sum_{m,\kappa}{}' \Big(-4\Big(2\Re\frac{1}{(m\tau+\kappa)^3(m\bar{\tau}+\kappa)} + \frac{1}{(m\tau+\kappa)^2(m\bar{\tau}+\kappa)^2}\Big) \\ & + 16\Big(2\Re\frac{1}{(2m\tau+\kappa)^3(2m\bar{\tau}+\kappa)} + \frac{1}{(2m\tau+\kappa)^2(2m\bar{\tau}+\kappa)^2}\Big) \\ & - 36\Big(2\Re\frac{1}{(3m\tau+\kappa)^3(3m\bar{\tau}+\kappa)} + \frac{1}{(3m\tau+\kappa)^2(3m\bar{\tau}+\kappa)^2}\Big) \\ & + 144\Big(2\Re\frac{1}{(6m\tau+\kappa)^3(6m\bar{\tau}+\kappa)} + \frac{1}{(6m\tau+\kappa)^2(6m\bar{\tau}+\kappa)^2}\Big)\Big)\Big\} \end{aligned}$$

Sketch of proof

Let

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining the family (Y_k) of $K3$-surfaces.

- For $k \in \mathbb{P}^1$, generically $\rho = 19$.
- The family is $\mathcal{M}_k$-polarized with

$$\mathcal{M}_k \simeq U_2 \perp (-E_8)^2 \perp \langle -12 \rangle$$

- Its transcendental lattice satisfies

$$T_k \simeq U_2 \perp \langle 12 \rangle$$

- The Picard-Fuchs differential equation is

$$(k^2-4)(k^2-36)y''' + 6k(k^2-20)y'' + (7k^2-48)y' + ky = 0$$

- The family is modular in the following sense
if $k = t + \frac{1}{t}$, $\tau \in \mathcal{H}$ and τ as in the theorem

$$t(\frac{a\tau + b}{c\tau + d}) = t(\tau) \; \forall \; (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_1(6,2)^* \subset \Gamma_0(12)^* + 12$$

where

$$\Gamma_1(6) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in Sl_2(\mathbb{Z}) \; / \; a \equiv d \equiv 1 \; (6) \;\; c \equiv 0 \; (6)\}$$

$$\Gamma_1(6,2) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_1(6) \;\; c \equiv 6b \;\; (12)\}$$

and

$$\Gamma_1(6,2)^* = \langle \Gamma_1(6,2), w_6 \rangle$$

.

- The P-F equation has a basis of solutions $G(\tau)$, $\tau G(\tau)$, $\tau^2 G(\tau)$ with

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

  satisfying

$$G(\tau) = F(t(\tau)), \quad F(t) = \sum_{n\geq 0} v_n t^{2n+1}, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

- $\frac{dm(P_k)}{dk}$ is a period, hence satisfies the P-F equation
- $$\frac{dm(P_k)}{dk} = G(\tau)$$
- $$dm(P_k) = -G(\tau)\frac{dt}{t}\frac{1-t^2}{t}$$

  is a weight 4 modular form for $\Gamma_1(6,2)^*$
- so can be expressed as a combination of $E_4(n\tau)$ for $n = 1, 2, 3, 6$

- By integration you get

$$m(P_k) = \Re(-\pi i \tau + \sum_{n\geq 1}(\sum_{d|n} d^3)(4\frac{q^n}{n} - 8\frac{q^{2n}}{n} + 12\frac{q^{3n}}{n} - 24\frac{6n}{n}))$$

- Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series

For some values of k, the corresponding τ is imaginary quadratic.
For example

k	0	2	3	6	10	18
τ	$\frac{-3+\sqrt{-3}}{6}$	$\frac{-2+\sqrt{-2}}{6}$	$\frac{-3+\sqrt{-15}}{12}$	$\frac{\sqrt{-6}}{6}$	$\frac{\sqrt{-2}}{2}$	$\sqrt{\frac{-5}{6}}$

For these quadratic τ called "singular moduli", the corresponding K3-surface is singular, that means its Picard number is $\rho = 20$ and the elliptic curve E of the Shioda-Inose is CM

Mahler measure and L-series of the K3-hypersurfaces

Theorem

Let Y_k the $K3$ hypersurface associated to the polynomial P_k, and T_{Y_k} its transcendental lattice. Then,

$$m(P_0) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) \quad m(P_2) = \frac{|\det T(Y_2)|^{3/2}}{\pi^3} L(Y_2, 3) = \frac{8\sqrt{8}}{\pi^3} L(f_8, 3)$$

$$m(P_3) = \frac{15\sqrt{15}}{2\pi^3} L(f_{15}, 3) = \frac{15\sqrt{15}}{2\pi^3} L(Y_3, 3)$$

$$m(P_6) = \frac{|\det T(Y_6)|^{3/2}}{2\pi^3} L(Y_6, 3) = \frac{24\sqrt{24}}{2\pi^3} L(f_{24}, 3)$$

$$m(P_{10}) = \frac{|\det T(Y_{10})|^{3/2}}{9\pi^3} L(Y_{10}, 3) + 2d_3 = \frac{72\sqrt{72}}{9\pi^3} L(f_8, 3) + 2d_3$$

$$m(P_{18}) = \frac{|\det T(Y_{18})|^{3/2}}{9\pi^3} L(Y_{18}, 3) + \frac{14}{5} d_3 = \frac{120\sqrt{120}}{9\pi^3} L(f_{120}, 3) + \frac{14}{5} d_3$$

L-series of a singular $K3$-hypersurface

$$L^*(X,s) := \prod_{p \nmid N}^{*} Z(X|\mathbb{F}_p, p^{-s}) = \sum_{n \geq 1} \frac{A(n)}{n^s}$$

N is the determinant of the transcendental lattice. Giving a suitable value to the local factors, the L-series of the surface X can be expressed in terms of the Mellin transform of a modular form.

Main ingredient: Livné's modularity theorem

Theorem

Let S be a $K3$-surface defined over $\mathbb{Q}$, with Picard number 20 and discriminant N. Its transcendental lattice $T(S)$ is a dimension 2 Gal$(\bar{\mathbb{Q}}/\mathbb{Q})$-module thus defines a L series, $L(T(S),s)$.
There exists a weight 3 modular form , f, CM over $\mathbb{Q}(\sqrt{-N})$ satisfying

$$L(T(S),s) \doteq L(f,s).$$

Moreover, if $NS(S)$ is generated by divisors defined over $\mathbb{Q}$,

$$L(S,s) \doteq \zeta(s-1)^{20} L(f,s).$$

Schütt's classification of CM-newforms of weight 3

Theorem

Consider the following classifications of singular $K3$ surfaces over $\mathbb{Q}$:

1. *by the discriminant d of the transcendental lattice of the surface up to squares,*
2. *by the associated newform up to twisting,*
3. *by the level of the associated newform up to squares,*
4. *by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.*

Then, all these classifications are equivalent. In particuliar, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.

Computing $L^*(X,s)$

Lemma

Let X be a singular elliptic $K3$ surface defined over $\mathbb{Q}$. If a Weierstrass model E of an elliptic fibration of X has rank 1 and possess an infinite section defined over $\mathbb{Q}(\sqrt{d})$, then

$$A_p = -\sum_{x\in\mathbb{P}^1(\mathbb{F}_p),\quad E_x \quad \textit{smooth}} a_p(x) - \sum_{x\in\mathbb{P}^1(\mathbb{F}_p),\quad E_x \quad \textit{singular}} \epsilon_p(x) - \left(\frac{d}{p}\right) p$$

with $a_p(x) = p - 1 - \#E_x(\mathbb{F}_p)$ and $\epsilon_p(x)$ such that

$$\epsilon_p(x) = \begin{cases} 0, & \textit{if } E_x \textit{ has additive reduction} \\ 1, & \textit{si } E_x \textit{ has split multiplicative reduction} \\ -1, & \textit{si } E_x \textit{ has non split multiplicative reduction} \end{cases}.$$

New results

Theorem

$$m(P_{2\sqrt{5}}) = 2.\frac{20\sqrt{20}}{4\pi^3} L(f_{20} \otimes \chi_5, 3)$$

$$L(Y_{2\sqrt{5}}, 3) = L(f_{20}, 3)$$

$$m(P_{3\sqrt{-5}}) = \frac{6}{5} \frac{15\sqrt{15}}{2\pi^3} L(f_{15}, 3) + \frac{d_{15}}{10}$$

If $E_{3\sqrt{-5}}$ defined by

$$y^2 = x^3 + (270t + 2025t^4 + 1755t^2 - 3 - 4050t^3)x^2 + 720xt(t-1)$$

has an infinite section defined over $\mathbb{Q}(\sqrt{-15})$

$$L(Y_{3\sqrt{-5}}, 3) = L(f_{15}, 3)$$

Remarks

The $K3$-surfaces Y_2 and Y_{10} have the same L-series $L(f_8, s)$ and transcendental lattices:

$$T_{Y_2} = [2\ 0\ 4] \qquad T_{Y_{10}} = [6\ 0\ 12]$$

in Shimada-Zhang notation. So, by Tate's conjecture an algebraic relation is suspected between the two $K3$-surfaces.
Indeed, there is an 3-isogeny between Y_2 and Y_{10} (B. and Lecacheux).
The $K3$-surfaces Y_3 and $Y_{3\sqrt{-5}}$ have the same L-series $L(f_{15}, 3)$ but transcendental lattices:

$$T_{Y_3} = [2\ 1\ 8] \qquad T_{Y_{3\sqrt{-5}}} = [8\ 2\ 8].$$

(! a wrong computation in BFFLM where is written $T_{Y_3} = [2\ 3\ 12]$) There is no algebraic relation between them but $Y_{3\sqrt{-5}}$ is the Kummer surface of Z_{-3} which is a $K3$-surface of the family Q_k

$$Q_k = (x + y + z + 1)(xy + xz + yz + xyz) - (k + 4)xyz$$

since $T_{Z_{-3}} = [4\ 1\ 4]$. However, $m(Q_{-3}) = \frac{8}{5} d_3$ (B.) but $L(Z_{-3}, 3) = L(f_{15} \otimes \chi, 3)$ (Peters and van Vlugt).

Results concerning the family Q_k

$m(Z_0)$ and $m(Z_{12})$ (B. 2006), $m(Z_{-3})$ (B. 2012)
Based on a Rogers's result, Samart deduced in his paper (arXiv 2013) the following:

$$m(Q_{-36}) = 2(4L'(g,0) + L'(\chi_{-4},-1))$$

$$m(Q_{-6}) = \frac{1}{2}(7L'(g,0) + 2L'(\chi_{-4},-1))$$

$$g = \eta(2\tau)^3\eta(6\tau)^3$$

If interested in the link between the Mahler measure and the L-series of the $K3$-surface, we need a good Weierstrass model for the family.
By chance, quite recently, O. Lecacheux found such a model for the family Q_k, namely

$$(F_k) \quad y^2 = x^3 - ((-k^2+24)t^2 - 2(k-2)(k+4)^2 t - k(k+4)^3)x^2 - 16t^4(t+k+3)x$$

I explain

k	-36	-12	-4	-6	-3	0	4	12	60
rk(MW_k)	1	1	-1	1	0	0	1	0	1

We recover easily my previous results with Shioda's formula.
F_0 has singular fibers III^*, I_4^*, I_3, I_2, 2-torsion hence
$|det T_0| = \frac{2\times4\times3\times2}{2^2} = 12$
F_{12} has singular fibers III^*, I_8, I_3, I_2, I_2, 2-torsion hence
$|det T_{12}| = \frac{2\times8\times3\times4}{2^2} = 12 \times 4$
No algebraic relation between Z_0 and Z_{12} since $T_0 = [2\ 0\ 6]$ and $T_{12} = [2\ 0\ 24]$.

With similar arguments, but not so simple, I have just proved

$$m(Q_4) = \frac{20\sqrt{2}}{\pi^3} \sum{}' \frac{k^2 - 2m^2}{(k^2 + 2m^2)^3} = \frac{20\sqrt{2}}{\pi^3} \times 2 \times L(f_8, 3)$$

$$L(Z_4, 3) = L(f_8, 3)$$

Moreover $\det(T_{Z_4}) = 8 \times 4$.
Z_4 may be the Kummer of Y_2?