Mahler measure of K3 surfaces (Lecture 4)

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Introduced by Mahler in 1962, the logarithmic Mahler measure of a polynomial $P$ is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \cdots, x_n)| \left| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \right|$$

and its Mahler measure

$$M(P) = \exp(m(P))$$

where

$$\mathbb{T}^n = \{(x_1, \cdots, x_n) \in \mathbb{C}^n/\{|x_1| = \cdots = |x_n| = 1\}\}.$$
Remarks

- $n = 1$
  
  By Jensen’s formula, if $P \in \mathbb{Z}[X]$ is monic, then

  $$M(P) = \prod_{P(\alpha) = 0} \max(|\alpha|, 1).$$

  So it is related to Lehmer’s question (1933)
  Does there exist $P \in \mathbb{Z}[X]$, monic, non cyclotomic, satisfying

  $$1 < M(P) < M(P_0) = 1.1762 \cdots$$

  The polynomial

  $$P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

  is the Lehmer polynomial, in fact a Salem polynomial.
Lehmer’s problem is still open.

A partial answer by Smyth (1971)

\[ M(P) \geq 1.32 \cdots \]

if \( P \) is non reciprocal.
The story can be explained with polynomials

\[ x_0 + x_1 + x_2 + \cdots + x_n. \]

- \( m(x_0 + x_1) = 0 \) (by Jensen’s formula)
- \( m(x_0 + x_1 + x_2) = \frac{3 \sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \) Smyth (1980)
- \( m(x_0 + x_1 + x_2 + x_3) = \frac{7}{2\pi^2} \zeta(3) \) Smyth (1980)

These are the first explicit Mahler measures.
$m(x_0 + x_1 + x_2 + x_3 + x_4) = \frac{675 \sqrt{15}}{16 \pi^3} L(f, 4)$ conjectured by Villegas (2004)

$f$ cusp form of weight 3 and conductor 15
$L(f, s)$ is also the L-series of the K3 surface defined by

\[
x_0 + x_1 + x_2 + x_3 + x_4 = 0
\]

\[
\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = 0
\]
How such a conjecture possible?
Because of deep insights of two people.

- **Deninger (1996)** who conjectured

\[
m(x + \frac{1}{x} + y + \frac{1}{y} + 1) \equiv \frac{15}{4\pi^2} L(E, 2) = L'(E, 0)
\]

*E* elliptic curve of conductor 15 defined by the polynomial

This conjecture was proved recently (May 2011) by Rogers and Zudilin thanks to a previous result due to Lalin. Here the polynomial is reciprocal.

A new proof is just posted on the arXiv (April 2013) by Zudilin.
Maillot (2003) using a result of Darboux (1875): the Mahler measure of $P$ which is the integration of a differential form on a variety, when $P$ is non reciprocal, is in fact an integration on a smaller variety and the expression of the Mahler measure is encoded in the cohomology of the smaller variety.
Examples

- $n = 2$ The smaller variety is defined by

  \[ x_0 + x_1 + x_2 = 0 \]
  \[ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} = 0 \iff x_1^2 + x_2^2 + x_1x_2 = 0 \]

  It is a curve of genus 0. So $m(x_0 + x_1 + x_2)$ is expressed as a Dirichlet L-series.

- $n = 3$ The smaller variety is defined by

  \[ x_0 + x_1 + x_2 + x_3 = 0 \]
  \[ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = 0 \iff (x_1 + x_2)(x_1 + x_3)(x_2 + x_3) = 0 \]

  It is the intersection of 3 planes. Thus Smyth’s result.
- **$n = 4$ (Villegas’s Conjecture)** The smaller variety is defined by
  \[ x_0 + x_1 + x_2 + x_3 + x_4 = 0 \]
  \[ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = 0 \]
  It is the modular $K3$-surface studied by Peters, Top, van der Vlugt defined by a reciprocal polynomial. Its $L$-series is related to $f$.

- **$n = 5$ (Villegas’s Conjecture again)**
  \[ m(x_0 + x_1 + x_2 + x_3 + x_4 + x_5) = \ast \ast L(g, 5) \]
  $g$ cusp form of weight 4 and conductor 6 related to $L$-series of the Barth-Nieto quintic.
Barth-Nieto quintic

It the 3-fold compactification of the complete intersection of

\[ x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0 \]
\[ \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5} = 0 \]

It has been studied by Hulek, Spandaw, Van Geemen, Van Straten in 2001. They proved that the $L$-function of the quintic (i.e. of their third etale cohomology group) is modular, a fact predicted by a conjecture of Fontaine and Mazur.

The modular form is the newform of weight 4 for $\Gamma_0(6)$

\[ f = (\eta(q)\eta(q^2)\eta(q^3)\eta(q^6))^2 \]
Motivation

Briefly, to guess the Mahler measure of a non reciprocal polynomial we need results on reciprocal ones.
In particular, it is very important to collect many examples of Mahler measures of $K3$-hypersurfaces.
Notice that Maillot’s insight predicts only the type of formula expected. Also Deninger’s guess comes from Beilinson’s Conjectures.
So replace $E$ by a surface $X$ which is also a **Calabi-Yau variety**, i.e. a $K3$-surface and try to answer the questions:

**What are the analog of Deninger, Boyd, R-Villegas’ results and conjectures?**

**Which type of Eisenstein-Kronecker series corresponds to $L(X, 3)$?**
Our results concern polynomials of the family

\[ P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k \]

defining K3-surfaces \( Y_k \). What’s a K3-surface?

It is a smooth surface \( X \) satisfying

- \( H^1(X, \mathcal{O}_X) = 0 \) i.e. \( X \) simply connected
- \( K_X = 0 \) i.e. the canonical bundle is trivial i.e. there exists a unique, up to scalars, holomorphic 2-form \( \omega \) on \( X \).
A double covering branched along a plane sextic for example defines a K3-surface $X$.

In our case

$$(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4$$

**Main properties**

- $H_2(X, \mathbb{Z})$ is a free group of rank 22.
With the intersection pairing, \( H_2(X, \mathbb{Z}) \) is a lattice and
\[
H_2(X, \mathbb{Z}) \cong U_2^3 \perp (-E_8)^2 := \mathcal{L}
\]
\( \mathcal{L} \) is the K3-lattice, \( U_2 \) the hyperbolic lattice of rank 2, \( E_8 \) the unimodular lattice of rank 8.

\[
\text{Pic}(X) \subset H_2(X, \mathbb{Z}) \cong \text{Hom}(H^2(X, \mathbb{Z}), \mathbb{Z})
\]
where \( \text{Pic}(X) \) is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles (since for K3 surfaces linear and algebraic equivalence are the same).

\[
\text{Pic}(X) \cong \mathbb{Z}^{\rho(X)}
\]
\( \rho(X) := \text{Picard number of } X \)
\[
1 \leq \rho(X) \leq 20
\]
\[ T(X) := (\text{Pic}(X))^\perp \]

is the transcendental lattice of dimension \(22 - \rho(X)\)

If \(\{\gamma_1, \cdots, \gamma_{22}\}\) is a \(\mathbb{Z}\)-basis of \(H_2(X, \mathbb{Z})\) and \(\omega\) the holomorphic 2-form,

\[ \int_{\gamma_i} \omega \]

is called a period of \(X\) and

\[ \int_{\gamma} \omega = 0 \text{ for } \gamma \in \text{Pic}(X). \]

If \(\{X_z\}\) is a family of K3 surfaces, \(z \in \mathbb{P}^1\) with generic Picard number \(\rho\) and \(\omega_z\) the corresponding holomorphic 2-form, then the periods of \(X_z\) satisfy a Picard-Fuchs differential equation of order \(k = 22 - \rho\). For our family \(k = 3\).
In fact, by Morrison, a $\mathcal{M}$-polarized $K3$-surface, with Picard number 19 has a Shioda-Inose structure, that means

$$X \quad \quad \quad \quad A = E \times E / C_N$$

$$Y = \text{Kum}(A/ \pm 1)$$

If the Picard number $\rho = 20$, then the elliptic curve is CM.
Mahler measure of $P_k$

**Theorem**

*(B. 2005) Let $k = t + \frac{1}{t}$ and

$$t = \left( \frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} \right)^6, \quad \eta(\tau) = e^{\frac{\pi i t}{12}} \prod_{n \geq 1} (1 - e^{2\pi i n\tau}), \quad q = \exp 2\pi i \tau$$

$$m(P_k) = \frac{\zeta(\tau)}{8\pi^3} \left\{ \sum_{m, \kappa} \left\{ \begin{array}{c} -4(2\Re) \frac{1}{(m\tau + \kappa)^3(m\bar{\tau} + \kappa)} + \frac{1}{(m\tau + \kappa)^2(m\bar{\tau} + \kappa)^2} \\ + 16(2\Re) \frac{1}{(2m\tau + \kappa)^3(2m\bar{\tau} + \kappa)} + \frac{1}{(2m\tau + \kappa)^2(2m\bar{\tau} + \kappa)^2} \\ - 36(2\Re) \frac{1}{(3m\tau + \kappa)^3(3m\bar{\tau} + \kappa)} + \frac{1}{(3m\tau + \kappa)^2(3m\bar{\tau} + \kappa)^2} \\ + 144(2\Re) \frac{1}{(6m\tau + \kappa)^3(6m\bar{\tau} + \kappa)} + \frac{1}{(6m\tau + \kappa)^2(6m\bar{\tau} + \kappa)^2} \end{array} \right\} \right\}
Sketch of proof

Let

\[ P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k \]

defining the family \((X_k)\) of \(K3\)-surfaces.

- For \(k \in \mathbb{P}^1\), generically \(\rho = 19\).
- The family is \(\mathcal{M}_k\)-polarized with

\[ \mathcal{M}_k \cong U_2 \perp (-E_8)^2 \perp \langle -12 \rangle \]

- Its transcendental lattice satisfies

\[ T_k \cong U_2 \perp \langle 12 \rangle \]

- The Picard-Fuchs differential equation is

\[ (k^2 - 4)(k^2 - 36)y''' + 6k(k^2 - 20)y'' + (7k^2 - 48)y' + ky = 0 \]
The family is modular in the following sense if \( k = t + \frac{1}{t} \), \( \tau \in \mathcal{H} \) and \( \tau \) as in the theorem

\[
t \left( \frac{a\tau + b}{c\tau + d} \right) = t(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6, 2)^* \subset \Gamma_0(12)^* + 12
\]

where

\[
\Gamma_1(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \ (6) \ c \equiv 0 \ (6) \right\}
\]

\[
\Gamma_1(6, 2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6) \mid c \equiv 6b \ (12) \right\}
\]

and

\[
\Gamma_1(6, 2)^* = \langle \Gamma_1(6, 2), w_6 \rangle
\]
The P-F equation has a basis of solutions $G(\tau), \tau G(\tau), \tau^2 G(\tau)$ with

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

satisfying

$$G(\tau) = F(t(\tau)), \quad F(t) = \sum_{n \geq 0} v_n t^{2n+1}, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$$\frac{dm(P_k)}{dk}$$ is a period, hence satisfies the P-F equation

$$\frac{dm(P_k)}{dk} = G(\tau)$$

$$dm(P_k) = -G(\tau) \frac{dt}{t} \frac{1-t^2}{1}$$

is a weight 4 modular form for $\Gamma_1(6,2)^*$

so can be expressed as a combination of $E_4(n\tau)$ for $n = 1, 2, 3, 6$
By integration you get

\[ m(P_k) = \Re\left(-\pi i \tau + \sum_{n \geq 1} \left( \sum_{d | n} d^3 \right) \left( 4 \frac{q^n}{n} - 8 \frac{q^{2n}}{2n} + 12 \frac{q^{3n}}{3n} - 24 \frac{q^{6n}}{6n} \right) \right) \]

Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series.
For some values of $k$, the corresponding $\tau$ is imaginary quadratic. For example

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>10</th>
<th>18</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>$\frac{-3+\sqrt{-3}}{6}$</td>
<td>$\frac{-2+\sqrt{-2}}{6}$</td>
<td>$\frac{-3+\sqrt{-15}}{12}$</td>
<td>$\frac{\sqrt{-6}}{6}$</td>
<td>$\frac{\sqrt{-2}}{2}$</td>
<td>$\sqrt{\frac{-5}{6}}$</td>
</tr>
</tbody>
</table>

For these quadratic $\tau$ called “singular moduli”, the corresponding K3-surface is singular, that means its Picard number is $\rho = 20$ and the elliptic curve $E$ of the Shioda-Inose is CM.

So, an expression of the Mahler measure in terms of Hecke L-series (arithmetic aspect) and perhaps in terms of the L-series of the hypersurface K3 (geometric aspect).
Theorem

Let $Y_k$ the K3 hypersurface associated to the polynomial $P_k$, $L(Y_k, s)$ its L-series, $T_Y$ its transcendental lattice and $f_N$ the unique, up to twist, CM-newform, CM by $\mathbb{Q}(\sqrt{-N})$, of weight 3 and level $N$ with rational coefficients. Then
Theorem

\[ m(P_0) = d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) \quad (B.2005) \]

\[ m(P_2) = \frac{|\det T(Y_2)|^{3/2}}{\pi^3} L(Y_2, 3) = \frac{8\sqrt{8}}{\pi^3} L(f_8, 3) \quad (B.2005) \]

\[ m(P_{10}) = \frac{|\det T(Y_{10})|^{3/2}}{9\pi^3} L(Y_{10}, 3) + 2d_3 = \frac{72\sqrt{72}}{9\pi^3} L(f_8, 3) + 2d_3 \quad (B.2009) \]

\[ m(P_3) = 2\frac{|\det T(Y_3)|^{3/2}}{4\pi^3} L(T(Y_3), 3) = \frac{15\sqrt{15}}{2\pi^3} L(f_{15}, 3) \quad (BFFLM 2013) \]

\[ m(P_6) = \frac{|\det T(Y_6)|^{3/2}}{2\pi^3} L(Y_6, 3) = \frac{24\sqrt{24}}{2\pi^3} L(f_{24}, 3) \quad (BFFLM 2013) \]

\[ m(P_{18}) = \frac{1}{5} \frac{|\det T(Y_{18})|^{3/2}}{4\pi^3} L(Y_{18}, 3) + \frac{14}{5} d_3 = \frac{120\sqrt{120}}{20\pi^3} L(f_{120}, 3) + \frac{14}{5} d_3 \quad (BFFLM 2013) \]
Let $Y$ be a surface. The zeta function is defined by

$$Z(Y, u) = \exp\left( \sum_{n=1}^{\infty} N_n(Y) \frac{u^n}{n} \right), \quad |u| < \frac{1}{p},$$

where $N_n(Y)$ denotes the number of points on $Y$ in $F_{p^n}$. If $Y$ is a $K3$-surface defined over $Q$, then $Y$ gives a $K3$-surface over $F_p$ for almost all $p$ and

$$Z(Y, u) = \frac{1}{(1 - u)(1 - p^2u)P_2(u)},$$

where $\deg P_2(u) = 22$. In fact,

$$P_2(u) = Q_p(u)R_p(u),$$

where the polynomial $R_p(u)$ comes from the algebraic cycles and $Q_p(u)$ comes from the transcendental cycles. Hence, for a singular $K3$-surface, $\deg Q_p = 2$ and $\deg R_p = 20$. 
Finally, we will work with the part of the $L$-function of $Y$ coming from the transcendental lattice, which is given by

$$L(T(Y), s) = (∗) \prod_{p \text{ good}} \frac{1}{Q_p(p^{-s})} = \sum_{n=1}^{\infty} \frac{A_n}{n^s},$$

where (∗) represents finite factors coming from the primes of bad reduction.
Strategy of the proof

- Understand the transcendental lattice and the group of sections.
- Relate the Mahler measure $m(P_k)$ to the $L$-function of a modular form.
- Relate the $L$-function of the surface $Y_k$ to the $L$-function of that same modular form.
Let $S$ be a K3-surface defined over $\mathbb{Q}$, with Picard number 20 and discriminant $N$. Its transcendental lattice $T(S)$ is a dimension 2 $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module thus defines a L series, $L(T(S), s)$.

There exists a weight 3 modular form, $f$, CM over $\mathbb{Q}(\sqrt{-N})$ satisfying

$$L(T(S), s) \doteq L(f, s).$$

Moreover, if $\text{NS}(S)$ is generated by divisors defined over $\mathbb{Q}$,

$$L(S, s) \doteq \zeta(s - 1)^{20} L(f, s).$$
The last ingredient: Schütt’s classification of CM-newforms of weight 3

**Theorem**

Consider the following classifications of singular K3 surfaces over $\mathbb{Q}$:

- by the discriminant $d$ of the transcendental lattice of the surface up to squares,
- by the associated newform up to twisting,
- by the level of the associated newform up to squares,
- by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.

Then, all these classifications are equivalent. In particular, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.
References


