Mahler measure of curves and surfaces

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Introduced by Mahler in 1962, 
the logarithmic Mahler measure of a polynomial $P$ is 

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and its Mahler measure 

$$M(P) = \exp(m(P))$$

where 

$$\mathbb{T}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n/|x_1| = \cdots = |x_n| = 1\}.$$
n = 1

By Jensen’s formula, if \( P \in \mathbb{Z}[X] \) is monic, then

\[
M(P) = \prod_{P(\alpha)=0} \max(|\alpha|, 1).
\]

So it is related to Lehmer’s question (1933)
Does there exist \( P \in \mathbb{Z}[X] \), monic, non cyclotomic, satisfying

\[1 < M(P) < M(P_0) = 1.1762 \cdots?\]

The polynomial

\[P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1\]

is the Lehmer polynomial, in fact a Salem polynomial.
Lehmer’s problem is still open.
A partial answer by Smyth (1971)

\[ M(P) \geq 1.32 \cdots \]

if \( P \) is non reciprocal.
\begin{itemize}
\item $n > 1,$
\item $m(x_0 + x_1) = 0$ (by Jensen’s formula)
\item 
\end{itemize}

$$m(x_0 + x_1 + x_2) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \quad \text{Smyth (1980)}$$

\begin{itemize}
\item 
\end{itemize}

$$m(x_0 + x_1 + x_2 + x_3) = \frac{7}{2\pi^2} \zeta(3) \quad \text{Smyth (1980)}$$

These are the first explicit Mahler measures.

Now, many explicit Mahler measures from Boyd, Smyth, Villegas, Lalin, Rogers, ...
But I want to point out two results.  
For $n = 2$, there is Deninger’s guess (1996)

$$m(X + \frac{1}{X} + Y + \frac{1}{Y} + 1) \approx \frac{15}{4\pi^2} L(E, 2) = L'(E, 0)$$

where $E$ is the elliptic curve of conductor 15 defined by the Laurent polynomial,
and also a pioneer result by Cassaigne-Maillot (1997)

$$m(a + bx + cy) = \begin{cases} \frac{1}{\pi}(D(\frac{|a|}{|b|} e^{i\gamma}) + \alpha \log |a| + \beta \log |b| + \gamma \log |c|) & \text{if } \Delta \\ \max\{\log |a|, \log |b|, \log |c|\} & \text{if not } \Delta \end{cases}$$
\( \Delta \) means that \( |a|, |b|, |c| \) are the sides of a triangle with angles \( \alpha, \beta, \gamma \).

\( D \) is the Bloch-Wigner dilogarithm.
The Bloch-Wigner dilogarithm

In Maillot’s result we notice the Bloch-Wigner dilogarithms, so perhaps a link with the **Bloch group of number fields and zeta functions of number fields**.

The Bloch-Wigner dilogarithm of a complex number $z$ is defined as

$$D(z) := \Im Li_2(z) + \log |z| \arg(1 - z)$$

where $Li_2$ is the ordinary dilogarithm. It is a univalued, real analytic function in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

**Properties**

- $$D\left(\frac{1}{z}\right) = D(1 - z) = D(\bar{z}) = -D(z)$$

- the distribution relation

$$D(z^n) = n \sum_{k=0}^{n-1} D(\zeta^k z)$$

where $\zeta$ denotes a primitive $n$-th root of unity.
Properties of the Bloch-Wigner dilogarithm

- the five-term relation

\[ D(x) + D(y) + D(1 - xy) + D\left(\frac{1 - x}{1 - xy}\right) + D\left(\frac{1 - y}{1 - xy}\right) = 0. \]

Moreover, if the wedge differential \( \eta \) is defined by

\[ \eta(x, y) = \log(|x|)d \arg y - \log(|y|)d \arg x, \]

the derivative of \( D \) verifies

\[ dD(x) = \eta(x, 1 - x). \]
Notice that the differential $\eta$ is

- multiplicative in each variable
- antisymmetric
- and if $\alpha \neq \beta$ satisfies the Tate’s relation

$$\eta(t - \alpha, t - \beta) = \eta\left(\frac{t - \alpha}{t - \beta}, 1 - \frac{t - \alpha}{t - \beta}\right) + \eta(t - \alpha, \alpha - \beta) + \eta(\beta - \alpha, t - \beta).$$
Let $F$ be a field and define the abelian groups

$$C(F) \subset A(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$$

where $A(F) = \ker \beta$ if

$$\beta : \mathbb{Z}[\mathbb{P}^1_F] \to \Lambda^2 F^\times$$

is defined by

$$\beta(0) = \beta(1) = \beta(\infty) = 0$$
$$\beta(x) = (x) \wedge (1 - x)$$

and

$$C(F) := \langle [x] + [y] + [1 - xy] + \left[ \frac{1 - x}{1 - xy} \right] + \left[ \frac{1 - y}{1 - xy} \right] \rangle$$

is generated by the five-term relation.
The Bloch group is now defined by the exact sequence

\[ 0 \to \mathcal{B}_2(F) \to \mathbb{Z}[[\mathbb{P}_F]]/\mathcal{C}(F) \xrightarrow{\beta} \Lambda^2 F^\times \]

The class of \( x \) in \( \mathcal{B}_2(F) \), \([x]\), behaves like a Bloch-Wigner dilogarithm. The complex

\[ \mathcal{B}_F(2) \bigotimes \mathbb{Q} : \mathcal{B}_2(F)_{\mathbb{Q}} \xrightarrow{\delta_1^2} (\Lambda^2 F^\times)_{\mathbb{Q}} \]

has a cohomology related to \( K \)-theory by Matsumoto’s theorem

\[ H^2(\mathcal{B}_F(2)) \cong K_2(F). \]

These things are related to results by Zagier on \( \zeta_F(2) \).
Zagier’s theorem

Theorem

(Zagier) Let $F$ be a number field, $[F : \mathbb{Q}] = n_+ + n_-$. Then the Bloch group modulo torsion satisfies

1. $\mathcal{B}(F)/\text{tors} \cong \mathbb{Z}^{n-}$

2. If $\xi_1, \ldots, \xi_{n-}$ is a basis of $\mathcal{B}(F)/\text{tors}$ and $\sigma_1, \ldots, \sigma_{n-}$ the complex embeddings of $F$ into $\mathbb{C}$, then

$$\det(D(\sigma_i(\xi_j)))_{i,j=1,\ldots,n-} \frac{\pi^{2n_+}}{\sqrt{|D_F|}} = r\zeta_F(2).$$
Maillot’s insight

Maillot (2003) using a result of Darboux (1875): the Mahler measure of $P$ which is the integration of a differential form on a variety, when $P$ is non-reciprocal, is in fact an integration on a smaller variety and the expression of the Mahler measure is encoded in the cohomology of the smaller variety. This insight can be realized for curves of genus 0 through Vandervelde’s theorem.
Theorem

**Vandervelde** Let \( P \in \mathbb{C}[x, y] \) such \( P = 0 \) parametrized by \( x = f(t) = \lambda_1 \prod((t - \alpha_r)^{l_r} \text{ and } y = g(t) = \lambda_2 \prod(t - \beta_s)^{m_s} \). Let \( S = \{(x, y) \in \mathbb{C}^2/ \mid x \mid = 1 \mid y \mid \geq 1\} \) and \( \gamma_1, \ldots, \gamma_n \) be the paths which map to \( S \) under \( t \mapsto (f(t), g(t)) \), oriented via \( f(\gamma_j) \subset \mathbb{T}^1 \). If \( u_j \) and \( v_j \) denote the initial and terminal points of \( \gamma_j \) and \( P(x, y) \) has leading coefficient \( \lambda(x) \) as a polynomial in \( y \), then

\[
2\pi m(P) = 2\pi m(\lambda(x)) + \sum_{j=1}^{n} \left( \sum_{r, s} l_r m_s \left[ D\left( \frac{u_j - \alpha_r}{\beta_s - \alpha_r} \right) - D\left( \frac{v_j - \alpha_r}{\beta_s - \alpha_r} \right) \right] \right)
\]

\[
+ \sum_r l_r \log | \tilde{g}(\alpha_r) | \ wind(\gamma_j, \alpha_r) - \sum_m l_s \log | \tilde{f}(\beta_s) | \ wind(\gamma_j, \beta_s),
\]

where

\[
wind(\gamma, \beta) = \int_\gamma \frac{dz}{z - \beta}.
\]
A new theorem

Theorem

*Bertin (2008)*

\[
m((x + 1)^2 y + x^2 + x + 1) = \frac{1}{3} L'(\chi_{-7}, -1) = \frac{1}{3} \frac{7^{3/2}}{4\pi} L(\chi_{-7}, 2) = \frac{7\sqrt{7}}{2\pi^3} \zeta_{\mathbb{Q}}(\sqrt{-7})(2) \]

\[
m((x^2 + x + 1)y + x^2 + 1) = \frac{1}{12} L'(\chi_{-15}, -1) = \frac{1}{12} \frac{15^{3/2}}{4\pi} L(\chi_{-15}, 2) = \frac{15\sqrt{15}}{8\pi^3} \zeta_{\mathbb{Q}}(\sqrt{-15})(2). \]
Proof.

- By Vandervelde, only one path $\gamma_1$ with initial point $u_1 = \frac{-3 + \sqrt{-7}}{4} = \alpha$ and terminal point $\bar{\alpha}$.

$$2\pi m(P) = 2D\left(\frac{-3 + \sqrt{-7}}{4} j^2\right) + 2D\left(\frac{-3 + \sqrt{-7}}{4} j\right) + 4D\left(\frac{3 + \sqrt{-7}}{4}\right).$$

- By distribution formula for $n = 3$ and $n = 2$,

$$\pi m(P) = D(\alpha) - D(\alpha^2) + \frac{1}{3} D(\alpha^3).$$

- Easy to prove

$$\xi = [\alpha] - [\alpha^2] + \frac{1}{3} [\alpha^3] \in B_2(\mathbb{Q}(\sqrt{-7})).$$
Proof.

- Thus, by Zagier
  \[ \pi^2 D(\xi) = r' \sqrt{7} \zeta_F(2) \]
  for \( r' \in \mathbb{Q}^\times \), that is \( m(P) = rd_7 \). We must prove now that \( r = \frac{1}{3} \).

- From five-term relation, distribution relations and the equation
  \[ 1 + \alpha + \alpha^2 = -\frac{\alpha}{2}, \]
  \[ D(\xi) = -2D(-\alpha) - \frac{1}{3} D(\frac{\alpha}{2}). \]

- Thus
  \[ D(\xi) = \frac{4}{3} D\left(\frac{1 + \sqrt{-7}}{2}\right) + \frac{2}{3} D\left(\frac{-1 + \sqrt{-7}}{4}\right) \]
There is a similar proof for the second relation but we have to use a result by Gangl, namely

$$\pi d_{15} = \frac{15\sqrt{15}}{4} L(\chi_{-15}, 2) = 6D\left(\frac{a + 3}{4}\right) + 8D(a - 1) + 8D\left(\frac{a}{2}\right)$$

where $a = \frac{1+\sqrt{-15}}{2}$. His proof uses the triangulation into ideal tetrahedra of the orbifold quotient $M = \Gamma \backslash \mathbb{H}^3$ with $\Gamma = PSL_2(\mathcal{O}_K)$ and $K = \mathbb{Q}(\sqrt{-15})$.

If there was a similar formula for $\mathbb{Q}(\sqrt{-11})$, we could prove

$$m((x + 1)^2(x^2 + x + 1)y + (x^2 - x + 1)^2) \equiv \frac{2}{3} L'(\chi_{-11}, -1).$$
Elliptic curves

Results by Boyd (mostly experimental), Villegas (Deninger-type when E is CM), Brunault (modular case)

- confirming Beilinson’s conjecture
- expressing the Mahler measure in terms of elliptic dilogarithm $D^E$
- related to elliptic regulators when the polynomial is “tempered”

So perhaps the level 1 link to regulators in Algebraic Geometry (Lewis, Kerr)
Remarks

- For the notion of “tempered” and the link with the $K_2(E)$, CCGLS’s paper is crucial (Cooper, Culler, Gillet, Long, Shalen Plane curves associated to character varieties of 3-manifolds).

- The fact that an elliptic curve is a Calabi-Yau variety is also fundamental.

- Maillot’s insight for $P$ non reciprocal but using Deninger-type results for reciprocal ones is also very important.

These are my motivations for studying $m(P)$ for $P$ reciprocal defining a $K3$-surface.
Basic facts on K3-surfaces

Our results concern polynomials of the family

\[ P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k \]

defining K3-surfaces \( Y_k \). What’s a K3-surface?

- A double covering branched along a plane sextic for example defines a K3-surface \( X \).
- In our case

\[
(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4
\]

- There is a unique holomorphic 2-form \( \omega \) on \( X \) up to a scalar.
- \( H_2(X, \mathbb{Z}) \) is a free group of rank 22.
With the intersection pairing, $H_2(X, \mathbb{Z})$ is a lattice and

$$H_2(X, \mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

$\mathcal{L}$ is the $K3$-lattice, $U_2$ the hyperbolic lattice of rank 2, $E_8$ the unimodular lattice of rank 8.

$$\text{Pic}(X) \subset H^2(X, \mathbb{Z}) \simeq \text{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z})$$

where $\text{Pic}(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles.

$$\text{Pic}(X) \simeq \mathbb{Z}^{\rho(X)}$$

$\rho(X) := $ Picard number of $X$

$$1 \leq \rho(X) \leq 20$$
\[ T(X) := (Pic(X))^\perp \]

is the transcendental lattice of dimension \( 22 - \rho(X) \)

If \( \{ \gamma_1, \cdots, \gamma_{22} \} \) is a \( \mathbb{Z} \)-basis of \( H_2(X, \mathbb{Z}) \) and \( \omega \) the holomorphic 2-form,

\[ \int_{\gamma_i} \omega \]

is called a period of \( X \) and

\[ \int_{\gamma} \omega = 0 \text{ for } \gamma \in Pic(X). \]

If \( \{ X_z \} \) is a family of K3 surfaces, \( z \in \mathbb{P}^1 \) with generic Picard number \( \rho \) and \( \omega_z \) the corresponding holomorphic 2-form, then the periods of \( X_z \) satisfy a Picard-Fuchs differential equation of order \( k = 22 - \rho \). For our family \( k = 3 \).
In fact, by Morrison, a $\mathcal{M}$-polarized $K3$-surface, with Picard number 19 has a Shioda-Inose structure, that means

$$X \quad \quad \quad \quad A = E \times E / C_N$$

$$Y = Kum(A/ \pm 1)$$

If the Picard number $\rho = 20$, then the elliptic curve is CM.
Theorem

(B. 2005) Let $k = t + \frac{1}{t}$ and

$$t = \left( \frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} \right)^6, \quad \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n \geq 1} (1 - e^{2\pi in\tau}), \quad q = \exp 2\pi i \tau$$

$$m(P_k) = \frac{\zeta(3)}{8\pi^3} \left\{ \sum_{m,\kappa} \left\{ -4 \left( 2\Re \frac{1}{(m\tau + \kappa)^3(m\bar{\tau} + \kappa)} + \frac{1}{(m\tau + \kappa)^2(m\bar{\tau} + \kappa)^2} \right) ight. ight.$$

$$+ 16 \left( 2\Re \frac{1}{(2m\tau + \kappa)^3(2m\bar{\tau} + \kappa)} + \frac{1}{(2m\tau + \kappa)^2(2m\bar{\tau} + \kappa)^2} \right)$$

$$- 36 \left( 2\Re \frac{1}{(3m\tau + \kappa)^3(3m\bar{\tau} + \kappa)} + \frac{1}{(3m\tau + \kappa)^2(3m\bar{\tau} + \kappa)^2} \right)$$

$$+ 144 \left( 2\Re \frac{1}{(6m\tau + \kappa)^3(6m\bar{\tau} + \kappa)} + \frac{1}{(6m\tau + \kappa)^2(6m\bar{\tau} + \kappa)^2} \right) \right\}$$
Sketch of proof

Let

\[ P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k \]

defining the family \((X_k)\) of \(K3\)-surfaces.

- For \(k \in \mathbb{P}^1 \setminus \{\infty, \pm2, \pm6\}\), \(\rho \geq 19\).
- The family is \(\mathcal{M}_k\)-polarized with

\[ \mathcal{M}_k \cong U_2 \perp (-E_8)^2 \perp \langle -12 \rangle \]

- Its transcendental lattice satisfies

\[ T_k \cong U_2 \perp \langle 12 \rangle \]

- The Picard-Fuchs differential equation is

\[ (k^2 - 4)(k^2 - 36)y''' + 6k(k^2 - 20)y'' + (7k^2 - 48)y' + ky = 0 \]
The family is modular in the following sense
if \( k = t + \frac{1}{t} \), \( \tau \in \mathcal{H} \) and \( \tau \) as in the theorem

\[
t\left(\frac{a\tau + b}{c\tau + d}\right) = t(\tau) \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6,2)^* \subset \Gamma_0(12)^* + 12
\]

where

\[
\Gamma_1(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) / \ a \equiv d \equiv 1 \ (6) \ c \equiv 0 \ (6) \right\}
\]

\[
\Gamma_1(6,2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6) / c \equiv 6b \ (12) \right\}
\]

and

\[
\Gamma_1(6,2)^* = \langle \Gamma_1(6,2), w_6 \rangle
\]
The P-F equation has a basis of solutions $G(\tau)$, $\tau G(\tau)$, $\tau^2 G(\tau)$ with

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

satisfying

$$G(\tau) = F(t(\tau)), \quad F(t) = \sum_{n \geq 0} v_n t^{2n+1}, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

$\frac{dm(P_k)}{dk}$ is a period, hence satisfies the P-F equation

$$\frac{dm(P_k)}{dk} = G(\tau)$$

$$dm(P_k) = -G(\tau) \frac{dt}{t} \frac{1 - t^2}{t}$$

is a weight 4 modular form for $\Gamma_1(6, 2)^*$

so can be expressed as a combination of $E_4(n\tau)$ for $n = 1, 2, 3, 6$
By integration you get

\[ m(P_k) = \Re \left( -\pi i \tau + \sum_{n \geq 1} \left( \sum_{d \mid n} d^3 \right) \left( 4 \frac{q^n}{n} - 8 \frac{q^{2n}}{n} + 12 \frac{q^{3n}}{n} - 24 \frac{6n}{n} \right) \right) \]

Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series.
For some values of $k$, the corresponding $\tau$ is imaginary quadratic. For example

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>6</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau$</td>
<td>$\frac{-3 + \sqrt{-3}}{6}$</td>
<td>$\frac{-2 + \sqrt{-2}}{6}$</td>
<td>$\frac{-3 + \sqrt{-15}}{12}$</td>
<td>$\frac{\sqrt{-6}}{6}$</td>
<td>$\frac{\sqrt{-2}}{2}$</td>
</tr>
</tbody>
</table>

For these quadratic $\tau$ called “singular moduli”, the corresponding K3-surface is singular, that means its Picard number is $\rho = 20$ and the elliptic curve $E$ of the Shioda-Inose is CM.

So, an expression of the Mahler measure in terms of Hecke L-series (arithmetic aspect) and perhaps in terms of the L-series of the hypersurface K3 (geometric aspect).
Mahler measure and Hecke L-series

$K = \mathbb{Q}(\sqrt{d})$ an imaginary quadratic field, $O_K$ its ring of integers, $D$ its discriminant

**Definition** A Hecke “Grössencharacter” of weight $k$, $k \geq 2$, and conductor $\Lambda$, $\Lambda$ being an ideal of $O_K$, is an homomorphism $\phi$

$$\phi : I(\Lambda) \rightarrow \mathbb{C}^*$$

such that

$$\phi(\alpha O_K) = \alpha^{k-1} \text{ if } \alpha \equiv 1(\Lambda)$$

The corresponding Hecke $L$-series is defined by

$$L(\phi, s) := \sum_P \frac{\phi(P)}{N(P)^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

the summation being on the prime ideals $P \subset O_K$ prime to $\Lambda$.

Replacing $O_K$ by an order $R$, you get a similar definition.
Theorem (B. 2007)

\[ m(P_0) = d_3 \]

\[ m(P_2) = \frac{16\sqrt{2}}{\pi^3} L_{\mathbb{Q}(\sqrt{-2})}(\phi, 3) \]

\[ m(P_3) = \frac{15\sqrt{15}}{2\pi^3} L_{\mathbb{Q}(\sqrt{-15})}(\phi_1, 3) \]

where \( \phi_1(P) = -\omega \) if \( P = (2, \omega) \) and \( \omega = \frac{1+\sqrt{-15}}{2}, P \) being a representative of the second ideal class of the number field \( \mathbb{Q}(\sqrt{-15}) \) of class number 2.

\[ m(P_6) = \frac{24\sqrt{6}}{\pi^3} L_{\mathbb{Q}(\sqrt{-6})}(\phi_2, 3) \]

where \( \phi_2(P) = -2 \) if \( P = (2, \sqrt{-6}) \), \( P \) being a representative of the second ideal class of the number field \( \mathbb{Q}(\sqrt{-6}) \) of class number 2.
where
\[ d_3 := \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = \frac{2\sqrt{3}}{\pi^3} \sum_{m,k} \frac{1}{(m^2 + 3k^2)^2} \]

and under the following relations, proved by Zagier,

\[ \frac{9\sqrt{2}}{\pi^3} \left( \sum' \frac{-1}{(9m^2 + 2k^2)^2} + \frac{1}{(k^2 + 18m^2)^2} \right) = 2d_3, \]

\[ \sum' \frac{k^2 - 18m^2}{(k^2 + 18m^2)^3} + \sum' \frac{9m^2 - 2k^2}{(9m^2 + 2k^2)^3} = \]

\[ \frac{10}{9} \sum' \frac{m^2 - 2k^2}{(m^2 + 2k^2)^3}. \]
Theorem
(B. 2008) Let $Y_k$ the K3 hypersurface associated to the polynomial $P_k$, $L(Y_k, s)$ its $L$-series and $T_Y$ its transcendental lattice. Then,

\[ L(Y_2, s) = L(Y_{10}, s) = L(f, s) \]

where $f(z) = \eta(z)^2 \eta(2z) \eta(4z) \eta(8z)^2 \in S_3(\Gamma_0(8), \epsilon_8)$, $\epsilon_8$ being some Dirichlet character modulo 8,

\[ m(P_2) = 4 \frac{|\det T_{Y_2}|^{3/2}}{4\pi^3} L(Y_2, 3) \]

Under the previous relations

\[ m(P_{10}) = 2d_3 + \frac{1}{9} \frac{|\det T_{Y_{10}}|^{3/2}}{2\pi^3} L(Y_{10}, 3), \]
Some ingredients in the proof

Let

\[ P_k = x^2yz + xy^2z + xyz^2 + t^2(xy + xz + yz) - kxyzt. \]

- \( Y_k \) is the desingularization of the set of zeroes of \( P_k \).
- With some fibration, \( Y_k \) is an elliptic surface with singular fibers of type \( I_n \).
- Use Shioda’s theorems on elliptic surfaces to compute the determinant of \( NS(Y_k) \), in particular, the formula

\[ \rho_k = r_k + 2 + \sum_{\nu} (m_{\nu,k} - 1) \]

where \( r_k \) is the rank of \( MW(Y_k) \).
Some ingredients in the proof (continued)

- If $k = 2$, since $\rho_2 = 20$ and fibers are of type $l_{12}, l_6, l_2, l_1, l_1$, $r_2 = 0$ (easy case).

- So,

$$| \det NS(Y_2) | = \frac{\prod m_{\nu,2}}{\text{Torsion}^2} = 8$$

- If $k = 10$, since $\rho_2 = 20$ and fibers are of type $l_{12}, l_3, l_3, l_2, l_2, l_1, l_1$, $r_{10} = 1$ (difficult case).

  - So, have to guess an infinite section,
  - have to use Néron’s desingularization.
The value of \( \det NS(Y_{10}) \) gives the CM-field of the elliptic curve in the Shioda-Inose structure.

Have to count the number of points of the reduction of \( Y_k \) modulo \( q \) (\( q = p^r \)).

In case \( Y_k \) modular this allows to determine which modular form gives the equality

\[
L(Y_k, s) = L(f, s).
\]

Compare to the expression of the Mahler measure and conclude.
We have

\[ \det NS(Y_2) = -8 \]
\[ \det NS(Y_{10}) = -72 \]

so the underlying elliptic curves \( E_2 \) and \( E_{10} \) are both CM on \( \mathbb{Q}(\sqrt{-2}) \).

Since

\[ L(Y_2, s) = L(Y_{10}, s) = L(f, s), \]

by Tate’s conjecture, \( Y_2 \) and \( Y_{10} \) are related by an algebraic correspondance.