Mahler measure of curves and surfaces

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Introduced by Mahler in 1962,
the logarithmic Mahler measure of a polynomial P is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log \mid P(x_1, \cdots, x_n) \mid \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and its Mahler measure

$$M(P) = \exp(m(P))$$

where

$$\mathbb{T}^n = \{(x_1, \cdots, x_n) \in \mathbb{C}^n / |x_1| = \cdots = |x_n| = 1\}.$$

Remarks

- $n = 1$

  By Jensen's formula, if $P \in \mathbb{Z}[X]$ is monic, then

$$M(P) = \prod_{P(\alpha)=0} \max(\mid \alpha \mid, 1).$$

So it is related to Lehmer's question (1933)
Does there exist $P \in \mathbb{Z}[X]$, monic, non cyclotomic, satisfying

$$1 < M(P) < M(P_0) = 1.1762 \cdots ?$$

The polynomial

$$P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

is the Lehmer polynomial, in fact a Salem polynomial.

Lehmer's problem is still open.

A partial answer by Smyth (1971)

$$M(P) \geq 1.32\cdots$$

if P is non reciprocal.

- $n > 1$,
- $m(x_0 + x_1) = 0$ (by Jensen's formula)
- $$m(x_0 + x_1 + x_2) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1) \quad \text{Smyth (1980)}$$
- $$m(x_0 + x_1 + x_2 + x_3) = \frac{7}{2\pi^2}\zeta(3) \quad \text{Smyth (1980)}$$

  These are the first explicit Mahler measures.

Now, many explicit Mahler measures from Boyd, Smyth, Villegas, Lalin, Rogers, ...

But I want to point out two results.
For $n = 2$, there is Deninger's guess (1996)

$$m(X + \frac{1}{X} + Y + \frac{1}{Y} + 1) \stackrel{?}{=} \frac{15}{4\pi^2} L(E, 2) = L'(E, 0)$$

where E is the elliptic curve of conductor 15 defined by the Laurent polynomial,
and also a pioneer result by Cassaigne-Maillot (1997)

$$m(a+bx+cy) = \begin{cases} \frac{1}{\pi}(D(\frac{|a|}{|b|}e^{i\gamma}) + \alpha \log \mid a \mid + \beta \log \mid b \mid + \gamma \log \mid c \mid) \text{ if } \Delta \\ \max\{\log \mid a \mid, \log \mid b \mid, \log \mid c \mid\} \text{ if not } \Delta \end{cases}$$

Δ means that $| a |$, $| b |$, $| c |$ are the sides of a triangle with angles α, β, γ
D is the Bloch-Wigner dilogarithm

The Bloch-Wigner dilogarithm

In Maillot's result we notice the Bloch-Wigner dilogarithms, so perhaps a link with the Bloch group of number fields and zeta functions of number fields.

The Bloch-Wigner dilogarithm of a complex number z is defined as

$$D(z) := \Im Li_2(z) + \log|z| \arg(1-z)$$

where Li_2 is the ordinary dilogarithm. It is a univalued, real analytic function in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

Properties

- $$D(\frac{1}{z}) = D(1-z) = D(\bar{z}) = -D(z)$$
- the distribution relation
  $$D(z^n) = n \sum_{k=0}^{n-1} D(\zeta^k z)$$
  where ζ denotes a primitive n-th root of unity

Properties of the Bloch-Wigner dilogarithm

- the five-term relation

$$D(x) + D(y) + D(1 - xy) + D\left(\frac{1-x}{1-xy}\right) + D\left(\frac{1-y}{1-xy}\right) = 0.$$

Moreover, if the wedge differential η is defined by

$$\eta(x, y) = \log(\mid x \mid) d \arg y - \log(\mid y \mid) d \arg x,$$

the derivative of D verifies

$$dD(x) = \eta(x, 1 - x).$$

Notice that the differential η is

- multiplicative in each variable
- antisymetric
- and if $\alpha \neq \beta$ satisfies the Tate's relation

$$\eta(t-\alpha, t-\beta) = \eta(\frac{t-\alpha}{t-\beta}, 1-\frac{t-\alpha}{t-\beta}) + \eta(t-\alpha, \alpha-\beta) + \eta(\beta-\alpha, t-\beta).$$

The Bloch group

Let F be a field and define the abelian groups

$$\mathcal{C}(F) \subset \mathcal{A}(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$$

where $\mathcal{A}(F) = \ker \beta$ if

$$\beta : \mathbb{Z}[\mathbb{P}^1_F] \to \Lambda^2 F^\times$$

is defined by

$$\beta(0) = \beta(1) = \beta(\infty) = 0$$

$$\beta(x) = (x) \wedge (1 - x)$$

and

$$\mathcal{C}(F) := \langle [x] + [y] + [1 - xy] + [\frac{1 - x}{1 - xy}] + [\frac{1 - y}{1 - xy}] \rangle$$

is generated by the five-term relation.

The Bloch group (continued)

The Bloch group is now defined by the exact sequence

$$0 \to \mathcal{B}_2(F) \to \mathbb{Z}[\mathbb{P}^1_F]/\mathcal{C}(F) \xrightarrow{\beta} \Lambda^2 F^\times$$

The class of x in $\mathcal{B}_2(F)$, $[x]$, behaves like a Bloch-Wigner dilogarithm.
The complex

$$\mathcal{B}_F(2) \bigotimes \mathbb{Q} : \mathcal{B}_2(F)_\mathbb{Q} \xrightarrow{\delta_1^2} (\Lambda^2 F^\times)_\mathbb{Q}$$

has a cohomology related to K-theory by Matsumoto's theorem

$$H^2(\mathcal{B}_F(2)) \simeq K_2(F).$$

These things are related to results by Zagier on $\zeta_F(2)$.

Zagier's theorem

Theorem

(Zagier) Let F be a number field, $[F:\mathbb{Q}] = n_+ + n_-$. Then the Bloch group modulo torsion satisfies

1. $$\mathcal{B}(F)/tors \simeq \mathbb{Z}^{n_-}$$

2. *If $\xi_1, \dots, \xi_{n_-}$ is a basis of $\mathcal{B}(F)/tors$ and $\sigma_1, \dots, \sigma_{n_-}$ the complex embeddings of F into $\mathbb{C}$, then*

$$\det(D(\sigma_i(\xi_j)))_{i,j=1,\dots,n_-} \frac{\pi^{2n_+}}{\sqrt{\mid D_F \mid}} = r\zeta_F(2).$$

Maillot's insight

Maillot (2003) using a result of Darboux (1875): the Mahler measure of P which is the integration of a differential form on a variety, when P is non reciprocal, is in fact an integration on a smaller variety and the expression of the Mahler measure is encoded in the cohomology of the smaller variety. This insight can be realized for curves of genus 0 through Vandervelde's theorem

Elementary case of Maillot's insight

Theorem

Vandervelde *Let* $P \in \mathbb{C}[x, y]$ *such* $P = 0$ *parametrized by* $x = f(t) = \lambda_1 \prod((t - \alpha_r)^{l_r}$ *and* $y = g(t) = \lambda_2 \prod(t - \beta_s)^{m_s}$. *Let* $S = \{(x, y) \in \mathbb{C}^2 / \mid x \mid = 1 \quad \mid y \mid \geq 1\}$ *and* $\gamma_1, \ldots, \gamma_n$ *be the paths which map to* S *under* $t \mapsto (f(t), g(t))$, *oriented via* $f(\gamma_j) \subset \mathbb{T}^1$. *If* u_j *and* v_j *denote the initial and terminal points of* γ_j *and* $P(x, y)$ *has leading coefficient* $\lambda(x)$ *as a polynomial in* y, *then*

$$2\pi m(P) = 2\pi m(\lambda(x)) + \sum_{j=1}^{n} (\sum_{r,s}^{l} l_r m_s [D(\frac{u_j - \alpha_r}{\beta_s - \alpha_r}) - D(\frac{v_j - \alpha_r}{\beta_s - \alpha_r})]$$

$$+ \sum_r l_r \log \mid \tilde{g}(\alpha_r) \mid wind(\gamma_j, \alpha_r) - \sum_m l_s \log \mid \tilde{f}(\beta_s) \mid wind(\gamma_j, \beta_s),$$

where

$$wind(\gamma, \beta) = \int_\gamma \frac{dz}{z - \beta}.$$

A new theorem

Theorem

Bertin (2008)

$$m((x+1)^2y + x^2 + x + 1) = \frac{1}{3}L'(\chi_{-7}, -1) = \frac{1}{3}\frac{7^{3/2}}{4\pi}L(\chi_{-7}, 2)$$
$$= \frac{7\sqrt{7}}{2\pi^3}\zeta_{\mathbb{Q}(\sqrt{-7})}(2)$$

$$m((x^2+x+1)y + x^2 + 1) = \frac{1}{12}L'(\chi_{-15}, -1) = \frac{1}{12}\frac{15^{3/2}}{4\pi}L(\chi_{-15}, 2)$$
$$= \frac{15\sqrt{15}}{8\pi^3}\zeta_{\mathbb{Q}(\sqrt{-15})}(2).$$

Proof.

- By Vandervelde, only one path γ_1 with initial point $u_1 = \frac{-3+\sqrt{-7}}{4} = \alpha$ and terminal point $\bar{\alpha}$.
- $$2\pi m(P) = 2D(\frac{-3+\sqrt{-7}}{4}j^2) + 2D(\frac{-3+\sqrt{-7}}{4}j) + 4D(\frac{3+\sqrt{-7}}{4}).$$
- By distribution formula for $n=3$ and $n=2$,
$$\pi m(P) = D(\alpha) - D(\alpha^2) + \frac{1}{3}D(\alpha^3).$$
- Easy to prove
$$\xi = [\alpha] - [\alpha^2] + \frac{1}{3}[\alpha^3] \in \mathcal{B}_2(\mathbb{Q}(\sqrt{-7})).$$

□

Proof.

- Thus, by Zagier

$$\pi^2 D(\xi) = r'\sqrt{7}\zeta_F(2)$$

for $r' \in \mathbb{Q}^\times$, that is $m(P) = rd_7$. We must prove now that $r = \frac{1}{3}$.

- From five-term relation, distribution relations and the equation $1 + \alpha + \alpha^2 = -\frac{\alpha}{2}$,

$$D(\xi) = -2D(-\alpha) - \frac{1}{3}D(\frac{\alpha}{2}).$$

- Thus

$$D(\xi) = \frac{4}{3}D(\frac{1+\sqrt{-7}}{2}) + \frac{2}{3}D(\frac{-1+\sqrt{-7}}{4})$$

□

There is a similar proof for the second relation but we have to use a result by Gangl, namely

$$\pi d_{15} = \frac{15\sqrt{15}}{4} L(\chi_{-15}, 2) = 6D(\frac{a+3}{4}) + 8D(a-1) + 8D(\frac{a}{2})$$

where $a = \frac{1+\sqrt{-15}}{2}$. His proof uses the triangulation into ideal tetrahedra of the orbifold quotient $M = \Gamma \backslash \mathbb{H}^3$ with $\Gamma = PSl_2(\mathcal{O}_K)$ and $K = \mathbb{Q}(\sqrt{-15})$.

If there was a similar formula for $\mathbb{Q}(\sqrt{-11})$, we could prove

$$m((x+1)^2(x^2+x+1)y + (x^2-x+1)^2) \stackrel{?}{=} \frac{2}{3} L'(\chi_{-11}, -1).$$

Elliptic curves

Results by Boyd (mostly experimental), Villegas (Deninger-type when E is CM), Brunault (modular case)

- confirming Beilinson's conjecture
- expressing the Mahler measure in terms of elliptic dilogarithm D^E
- related to elliptic regulators when the polynomial is "tempered"

So perhaps the level 1 link to regulators in Algebraic Geometry (Lewis, Kerr)

Remarks

- For the notion of "tempered" and the link with the $K_2(E)$, CCGLS's paper is crucial (Cooper, Culler, Gillet, Long, Shalen Plane curves associated to character varieties of 3-manifolds).
- The fact that an elliptic curve is a Calabi-Yau variety is also fundamental.
- Maillot's insight for P non reciprocal but using Deninger-type results for reciprocal ones is also very important.

These are my motivations for studying $m(P)$ for P reciprocal defining a $K3$-surface.

Basic facts on K3-surfaces

Our results concern polynomials of the family

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining K3-surfaces Y_k. What's a K3-surface?

- A double covering branched along a plane sextic for example defines a K3-surface X.
  In our case

$$(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4$$

- There is a unique holomorphic 2-form ω on X up to a scalar.
- $H_2(X, \mathbb{Z})$ is a free group of rank 22.

- With the intersection pairing, $H_2(X,\mathbb{Z})$ is a lattice and

$$H_2(X,\mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

$\mathcal{L}$ is the $K3$-lattice, U_2 the hyperbolic lattice of rank 2, E_8 the unimodular lattice of rank 8.

- $$Pic(X) \subset H^2(X,\mathbb{Z}) \simeq Hom(H_2(X,\mathbb{Z}),\mathbb{Z})$$

where $Pic(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles.

- $$Pic(X) \simeq \mathbb{Z}^{\rho(X)}$$

$$\rho(X) := \text{Picard number of } X$$

$$1 \leq \rho(X) \leq 20$$

- $$T(X) := (Pic(X))^{\perp}$$
  is the transcendental lattice of dimension $22 - \rho(X)$
- If $\{\gamma_1, \cdots, \gamma_{22}\}$ is a $\mathbb{Z}$-basis of $H_2(X, \mathbb{Z})$ and ω the holomorphic 2-form,
  $$\int_{\gamma_i} \omega$$
  is called a period of X and
  $$\int_{\gamma} \omega = 0 \text{ for } \gamma \in Pic(X).$$
- If $\{X_z\}$ is a family of $K3$ surfaces, $z \in \mathbb{P}^1$ with generic Picard number ρ and ω_z the corresponding holomorphic 2-form, then the periods of X_z satisfy a Picard-Fuchs differential equation of order $k = 22 - \rho$. For our family $k = 3$.

- In fact, by Morrison, a $\mathcal{M}$-polarized $K3$-surface, with Picard number 19 has a Shioda-Inose structure, that means

$$\begin{array}{ccc} X & & A = E \times E/C_N \\ & \searrow \quad \swarrow & \\ & Y = Kum(A/\pm 1) & \end{array}$$

- If the Picard number $\rho = 20$, then the elliptic curve is CM.

Mahler measure of P_k

Theorem

(B. 2005) Let $k = t + \frac{1}{t}$ *and*

$$t = (\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)})^6, \;\; \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n \geq 1}(1 - e^{2\pi i n\tau}), \;\; q = \exp 2\pi i \tau$$

$$\begin{aligned} m(P_k) = & \frac{\Im\tau}{8\pi^3}\{{\sum_{m,\kappa}}'(-4(2\Re\frac{1}{(m\tau+\kappa)^3(m\bar{\tau}+\kappa)} + \frac{1}{(m\tau+\kappa)^2(m\bar{\tau}+\kappa)^2}) \\ & + 16(2\Re\frac{1}{(2m\tau+\kappa)^3(2m\bar{\tau}+\kappa)} + \frac{1}{(2m\tau+\kappa)^2(2m\bar{\tau}+\kappa)^2}) \\ & - 36(2\Re\frac{1}{(3m\tau+\kappa)^3(3m\bar{\tau}+\kappa)} + \frac{1}{(3m\tau+\kappa)^2(3m\bar{\tau}+\kappa)^2}) \\ & + 144(2\Re\frac{1}{(6m\tau+\kappa)^3(6m\bar{\tau}+\kappa)} + \frac{1}{(6m\tau+\kappa)^2(6m\bar{\tau}+\kappa)^2}))\} \end{aligned}$$

Sketch of proof

Let

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining the family (X_k) of $K3$-surfaces.

- For $k \in \mathbb{P}^1 \setminus \{\infty, \pm 2, \pm 6\}$, $\rho \geq 19$.
- The family is $\mathcal{M}_k$-polarized with

$$\mathcal{M}_k \simeq U_2 \perp (-E_8)^2 \perp \langle -12 \rangle$$

- Its transcendental lattice satisfies

$$T_k \simeq U_2 \perp \langle 12 \rangle$$

- The Picard-Fuchs differential equation is

$$(k^2-4)(k^2-36)y''' + 6k(k^2-20)y'' + (7k^2-48)y' + ky = 0$$

- The family is modular in the following sense
  if $k = t + \frac{1}{t}$, $\tau \in \mathcal{H}$ and τ as in the theorem

$$t(\frac{a\tau + b}{c\tau + d}) = t(\tau) \; \forall \; (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_1(6,2)^* \subset \Gamma_0(12)^* + 12$$

where

$$\Gamma_1(6) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in Sl_2(\mathbb{Z}) \;/\; a \equiv d \equiv 1 \; (6) \;\; c \equiv 0 \; (6)\}$$

$$\Gamma_1(6,2) = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_1(6) \;\; c \equiv 6b \;\; (12)\}$$

and

$$\Gamma_1(6,2)^* = \langle \Gamma_1(6,2), w_6 \rangle$$

.

- The P-F equation has a basis of solutions $G(\tau)$, $\tau G(\tau)$, $\tau^2 G(\tau)$ with

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

satisfying

$$G(\tau) = F(t(\tau)), \quad F(t) = \sum_{n\geq 0} v_n t^{2n+1}, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

- $\frac{dm(P_k)}{dk}$ is a period, hence satisfies the P-F equation
- $$\frac{dm(P_k)}{dk} = G(\tau)$$
- $$dm(P_k) = -G(\tau)\frac{dt}{t}\frac{1-t^2}{t}$$

  is a weight 4 modular form for $\Gamma_1(6,2)^*$
- so can be expressed as a combination of $E_4(n\tau)$ for $n = 1, 2, 3, 6$

- By integration you get

$$m(P_k) = \Re(-\pi i\tau + \sum_{n\geq 1}(\sum_{d|n} d^3)(4\frac{q^n}{n} - 8\frac{q^{2n}}{n} + 12\frac{q^{3n}}{n} - 24\frac{6n}{n}))$$

- Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series

For some values of k, the corresponding τ is imaginary quadratic.
For example

k	0	2	3	6	10
τ	$\frac{-3+\sqrt{-3}}{6}$	$\frac{-2+\sqrt{-2}}{6}$	$\frac{-3+\sqrt{-15}}{12}$	$\frac{\sqrt{-6}}{6}$	$\frac{\sqrt{-2}}{2}$

For these quadratic τ called "singular moduli", the corresponding K3-surface is singular, that means its Picard number is $\rho = 20$ and the elliptic curve E of the Shioda-Inose is CM

So, an expression of the Mahler measure in terms of Hecke L-series (arithmetic aspect) and perhaps in terms of the L-series of the hypersurface K3 (geometric aspect).

Mahler measure and Hecke L-series

$K = \mathbb{Q}(\sqrt{d})$ an imaginary quadratic field, O_K its ring of integers, D its discriminant

Definition A Hecke "Grössencharacter" of weight k, $k \geq 2$, and conductor Λ, Λ being an ideal of O_K, is an homomorphism ϕ

$$\phi : I(\Lambda) \to \mathbb{C}^*$$

such that

$$\phi(\alpha O_K) = \alpha^{k-1} \quad \text{if} \quad \alpha \equiv 1(\Lambda)$$

The corresponding Hecke L-series is defined by

$$L(\phi, s) := \sum_P \frac{\phi(P)}{N(P)^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

the summation being on the prime ideals $P \subset O_K$ prime to Λ.
Replacing O_K by an order R, you get a similar definition.

Theorem

(B. 2007)

$$m(P_0) = d_3$$

$$m(P_2) = \frac{16\sqrt{2}}{\pi^3} L_{\mathbb{Q}(\sqrt{-2})}(\phi, 3)$$

$$m(P_3) = \frac{15\sqrt{15}}{2\pi^3} L_{\mathbb{Q}(\sqrt{-15})}(\phi_1, 3)$$

where $\phi_1(P) = -\omega$ *if* $P = (2, \omega)$ *and* $\omega = \frac{1+\sqrt{-15}}{2}$*, P being a representative of the second ideal class of the number field* $\mathbb{Q}(\sqrt{-15})$ *of class number 2.*

$$m(P_6) = \frac{24\sqrt{6}}{\pi^3} L_{\mathbb{Q}(\sqrt{-6})}(\phi_2, 3)$$

where $\phi_2(P) = -2$ *if* $P = (2, \sqrt{-6})$*, P being a representative of the second ideal class of the number field* $\mathbb{Q}(\sqrt{-6})$ *of class number 2.*

$$m(P_{10}) = 2d_3 + 3\frac{16\sqrt{2}}{\pi^3} L_{\mathbb{Q}(\sqrt{-2})}(\phi, 3)$$

where

$$d_3 := \frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) = \frac{2\sqrt{3}}{\pi^3}\sum_{m,k}{}' \frac{1}{(m^2+3k^2)^2}$$

and under the following relations, proved by Zagier,

$$\frac{9\sqrt{2}}{\pi^3}\left(\sum{}' \frac{-1}{(9m^2+2k^2)^2} + \frac{1}{(k^2+18m^2)^2}\right) = 2d_3,$$

$$\sum{}' \frac{k^2-18m^2}{(k^2+18m^2)^3} + \sum{}' \frac{9m^2-2k^2}{(9m^2+2k^2)^3}$$

$$=$$

$$\frac{10}{9}\sum{}' \frac{m^2-2k^2}{(m^2+2k^2)^3}.$$

Theorem

(B. 2008) Let Y_k the K3 hypersurface associated to the polynomial P_k, $L(Y_k, s)$ its L-series and T_Y its transcendental lattice. Then,

- $$L(Y_2, s) = L(Y_{10}, s) = L(f, s)$$
  where $f(z) = \eta(z)^2\eta(2z)\eta(4z)\eta(8z)^2 \in S_3(\Gamma_0(8), \epsilon_8)$, ϵ_8 *being some Dirichlet character modulo* 8,
- $$m(P_2) = 4\frac{|\det T_{Y_2}|^{3/2}}{4\pi^3}L(Y_2, 3)$$
- *Under the previous relations*
  $$m(P_{10}) = 2d_3 + \frac{1}{9}\frac{|\det T_{Y_{10}}|^{3/2}}{2\pi^3}L(Y_{10}, 3),$$

Some ingredients in the proof

Let

$$P_k = x^2yz + xy^2z + xyz^2 + t^2(xy + xz + yz) - kxyzt.$$

- Y_k is the desingularization of the set of zeroes of P_k.
- With some fibration, Y_k is an elliptic surface with singular fibers of type I_n.
- Use Shioda's theorems on elliptic surfaces to compute the determinant of $NS(Y_k)$, in particular, the formula

$$\rho_k = r_k + 2 + \sum_{\nu}(m_{\nu,k} - 1)$$

where r_k is the rank of $MW(Y_k)$.

Some ingredients in the proof (continued)

- If $k = 2$, since $\rho_2 = 20$ and fibers are of type I_{12}, I_6, I_2, I_2, I_1, I_1, $r_2 = 0$ (easy case).
- So,

$$| \det NS(Y_2) |= \frac{\prod m_{\nu,2}}{\text{Torsion}^2} = 8$$

- If $k = 10$, since $\rho_2 = 20$ and fibers are of type I_{12}, I_3, I_3, I_2, I_2, I_1, I_1, $r_{10} = 1$ (difficult case).
  - So, have to guess an infinite section,
  - have to use Néron's desingularization.

Some ingredients in the proof (continued)

- The value of det $NS(Y_{10})$ gives the CM-field of the elliptic curve in the Shioda-Inose structure.
- Have to count the number of points of the reduction of Y_k modulo q ($q = p^r$).
- In case Y_k modular this allows to determine which modular form gives the equality
$$L(Y_k, s) = L(f, s).$$
- Compare to the expression of the Mahler measure and conclude.

Remarks

- We have

$$\det NS(Y_2) = -8$$

$$\det NS(Y_{10}) = -72$$

so the underlying elliptic curves E_2 and E_{10} are both CM on $\mathbb{Q}(\sqrt{-2})$.

- Since

$$L(Y_2, s) = L(Y_{10}, s) = L(f, s),$$

by Tate's conjecture, Y_2 and Y_{10} are related by an algebraic correspondance.