MAHLER MEASURE OF SOME SINGULAR $K3$-SURFACES

MARIE-JOSÉ BERTIN, AMY FEAVER, JENNY FUSELIER, MATILDE LALÍN, MICHELLE MANES

Abstract. We study the Mahler measure of the three-variable Laurent polynomial $x+1/x+y+1/y+z+1/z-k$ where k is a parameter. The zeros of this polynomial define (after desingularization) a family of $K3$-surfaces. In favorable cases, the $K3$-surface has Picard number 20, and the Mahler measure is related to its L-function. This was first studied by Marie-José Bertin. In this work, we prove several new formulas, extending the earlier work of Bertin.

1. Introduction

Given a nonzero Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) Mahler measure is defined by

$$\begin{aligned} \mathrm{m}(P) &= \int_0^1 \cdots \int_0^1 \log \left| P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n}) \right| d\theta_1 \cdots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}, \end{aligned}$$

where $\mathbb{T}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1| = \cdots = |x_n| = 1\}$ is the unit n-torus.

Jensen's formula relates the Mahler measure of a one-variable polynomial to a very simple formula depending on the roots of the polynomial:

$$\mathrm{m}(P) = \log |a| + \sum_{|r_j|>1} \log |r_j| \quad \text{for} \quad P(x) = a \prod_j (x - r_j).$$

This formula shows, in particular, that the Mahler measure of a polynomial with integral coefficients is the logarithm of an algebraic number.

The situation for several variable polynomials is very different. There are several formulas for specific polynomials yielding special values of L-functions. The first examples were computed by Smyth in the 1970s [Sm71, Bo81] and give special values of the Riemann zeta function and Dirichlet L-series:

$$\begin{aligned} \mathrm{m}(x+y+1) &= \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1), \\ \mathrm{m}(x+y+z+1) &= \frac{7}{2\pi^2}\zeta(3). \end{aligned}$$

Then, in the mid 1990s, Boyd [Bo98] (after a suggestion of Deninger) looked at more complicated polynomials such as the family

$$P_k(x,y) = x + \frac{1}{x} + y + \frac{1}{y} - k, \tag{1.1}$$

where k is an integral parameter. For most k, the zero set $P_k(x,y) = 0$ is a genus-one curve which we will denote by $E_{(k)}$. Boyd found several numerical formulas of the same shape:

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} - k\right) \stackrel{?}{=} s_k L'(E_{(k)}, 0) \quad k \in \mathbb{Z}, \quad |k| \neq 0, 4,$$

2010 *Mathematics Subject Classification.* Primary 11R06; Secondary 11R09, 14J27, 14J28.

Key words and phrases. Mahler measure, polynomial, singular $K3$-surfaces, elliptic surfaces.

This work of ML was partially supported by NSERC Discovery Grant 355412-2008 and FQRNT Subvention établissement de nouveaux chercheurs 144987. The work of MM was partially supported by NSF-DMS 1102858.

where s_k is a rational number and the question mark means that both sides of the equation are equal to at least 25 decimal places. In fact, it suffices to consider k natural since one can easily see that $\mathrm{m}(P_k) = \mathrm{m}(P_{-k})$.

In particular, for $k = 1$,

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} - 1\right) = \frac{15}{4\pi^2} L(E_{15}, 2) = L'(E_{15}, 0), \tag{1.2}$$

which was recently proven by Rogers and Zudilin [RZ11].

The connection with the L-function of the elliptic curve defined by the zeros of the polynomial was explained by Deninger [De97] in a very general context and Rodriguez-Villegas [RV97] for some specific formulas in terms of Beilinson's conjectures. Beilinson's conjectures predict that special values of L-functions (coming from an arithmetic-geometric structure) are given by certain values of the regulator associated to the structure up to a rational number. In favorable cases, Mahler measure can be related to the regulator. In particular, this allowed Rodriguez-Villegas to prove the formulas for the case where E has complex multiplication, since in this case Beilinson's conjectures are known to be true.

More generally, let $P(x, y)$ be a polynomial in two variables with integer coefficients and suppose that P does not vanish on the 2-torus $\mathbb{T}^2$. If P defines an elliptic curve E and the polynomials of the faces P_F of P (defined in terms of the Newton polygon of P) are cyclotomic (in other words, they have measure zero), then the following relation between $\mathrm{m}(P)$ and the L-series of the elliptic curve E is conjectured to hold:

$$\mathrm{m}(P) \stackrel{?}{=} \frac{qN}{4\pi^2} L(E, 2) = qL'(E, 0),$$

where N is the conductor of E and q is a rational number.

A natural extension to this connection involves polynomials whose zeros define Calabi-Yau varieties. One-dimensional Calabi-Yau varieties are elliptic curves, while 2-dimensional Calabi-Yau varieties are elliptic $K3$-surfaces. For example, it is natural to consider the family of polynomials resulting from adding an extra variable to the equation in (1.1). Bertin has been pursuing this program [Be06, Be08a, Be10] with the families

$$P_k(x, y, z) = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k.$$

$$Q_k(x, y, z) = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + xy + \frac{1}{xy} + zy + \frac{1}{zy} + xyz + \frac{1}{xyz} - k.$$

Relating these examples back to the elliptic curve case, one may ask for a natural condition on the faces of the Newton polytope for the polynomials P_k in order to expect relationships between $\mathrm{m}(P_k)$ and the L-series of the associated surface. The situation is more complicated than in the elliptic curve case, since the faces in the above examples have nonzero Mahler measure. This question remains open.

The first step in Bertin's work is to generalize Rodriguez-Villegas's expression of the Mahler measure in terms of Eisenstein–Kronecker series for these two families of polynomials defining $K3$-surfaces. For example, in [Be06] Bertin proves

$$\mathrm{m}(P_k) = \frac{\operatorname{Im}\tau}{8\pi^3} \sum_{j \in \{1,2,3,6\}} \sideset{}{'}\sum_{m,n} (-1)^j 4j^2 \left(2 \operatorname{Re} \frac{1}{(jm\tau + n)^3 (jm\bar{\tau} + n)} + \frac{1}{(jm\tau + n)^2 (jm\bar{\tau} + n)^2} \right).$$

Here $k = w + \frac{1}{w}$ and

$$w = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} \right)^6 = q^{1/2} - 6q^{3/2} + 15q^{5/2} - 20q^{7/2} + \cdots$$

where η denotes the Dedekind eta function.

For exceptional values of k, the corresponding $K3$-surface Y_k is singular (or extremal) and τ is imaginary quadratic. The Eisenstein–Kronecker series can be split into two sums, one with the $\operatorname{Re} \frac{1}{(jm\tau+n)^3 (jm\bar{\tau}+n)}$ terms and the other with the $\frac{1}{(jm\tau+n)^2(jm\bar{\tau}+n)^2}$ terms. The first one is related to the L-series of the surface, while the second one is either zero or may be expressed in terms of a Dirichlet series related to the Mahler measure of the 2-dimensional faces of the Newton polytope of the polynomial P_k.

Bertin obtained

$$\begin{aligned}
\mathrm{m}(P_0) &= d_3 := \frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2),\\
\mathrm{m}(P_2) &= 4\frac{|\det \mathbb{T}(Y_2)|^{3/2}}{4\pi^3}L(\mathbb{T}(Y_2),3) = 4\cdot\frac{8\sqrt{8}}{4\pi^3}L(g_8,3), \text{ and}\\
\mathrm{m}(P_{10}) &= \frac{4}{9}\frac{|\det \mathbb{T}(Y_{10})|^{3/2}}{4\pi^3}L(\mathbb{T}(Y_{10}),3)+2d_3 = \frac{4}{9}\cdot\frac{72\sqrt{72}}{4\pi^3}L(g_8,3)+2d_3,
\end{aligned}$$

where Y_k denotes the $K3$-surface associated to the zero set $P_k(x,y,z) = 0$, $\mathbb{T}$ denotes its transcendental lattice, and $L(g_N,3)$ the L-series at $s=3$ of a modular form of weight 3 and level N.

In this note, we continue the work of Bertin and prove

$$\begin{aligned}
\mathrm{m}(P_3) &= 2\frac{|\det \mathbb{T}(Y_3)|^{3/2}}{4\pi^3}L(\mathbb{T}(Y_3),3) = 2\cdot\frac{15\sqrt{15}}{4\pi^3}L(g_{15},3),\\
\mathrm{m}(P_6) &= 2\frac{|\det \mathbb{T}(Y_6)|^{3/2}}{4\pi^3}L(\mathbb{T}(Y_6),3) = 2\cdot\frac{24\sqrt{24}}{4\pi^3}L(g_{24},3), \text{ and}\\
\mathrm{m}(P_{18}) &= \frac{1}{5}\frac{|\det \mathbb{T}(Y_{18})|^{3/2}}{4\pi^3}L(\mathbb{T}(Y_{18}),3)+\frac{14}{5}d_3 = \frac{1}{5}\cdot\frac{120\sqrt{120}}{4\pi^3}L(g_{120},3)+\frac{14}{5}d_3.
\end{aligned}$$

The case with $k = 18$ is particularly difficult because the corresponding $K3$-surface has an infinite section that is defined over a quadratic field rather than being defined over $\mathbb{Q}$. The method we use to find this infinite section should be useful in other cases.

2. Background on $K3$-surfaces

A $K3$-surface is a complete smooth surface Y that is simply connected and admits a unique (up to scalars) holomorphic 2-form ω. We list here some useful facts about $K3$-surfaces along with notation that will be used throughout. See [Yu04] for general results about Calabi-Yau manifolds including $K3$-surfaces.

- $H_2(Y,\mathbb{Z})$ is a free group of rank 22.
- The Picard group $\mathrm{Pic}(Y) \subset H_2(Y,\mathbb{Z})$ is the group of divisors modulo linear equivalence, parametrized by algebraic cycles:
  $$\mathrm{Pic}(Y) \cong \mathbb{Z}^{\rho(Y)}.$$
  The exponent $\rho(Y)$ is called the Picard number, and over a field of characteristic 0 it satisfies
  $$1 \le \rho(Y) \le 20.$$
  If $\rho(Y) = 20$, we say that the $K3$-surface is *singular*.
- The transcendental lattice is defined by
  $$\mathbb{T}(Y) = (\mathrm{Pic}(Y))^{\perp}.$$
- Let $\{\gamma_1,\dots,\gamma_{22}\}$ be a $\mathbb{Z}$-basis for $H_2(Y,\mathbb{Z})$. Then
  $$\int_\gamma \omega = \begin{cases} 0 & \gamma \in \mathrm{Pic}(Y),\\ \text{period of } Y & \gamma \in \mathbb{T}(Y).\end{cases}$$

2.1. L-functions.

Let Y be a surface. The zeta function is defined by

$$Z(Y,u) = \exp\left(\sum_{n=1}^{\infty} N_n(Y)\frac{u^n}{n}\right), \qquad |u| < \frac{1}{p},$$

where $N_n(Y)$ denotes the number of points on Y in $\mathbb{F}_{p^n}$.

If Y is a $K3$-surface defined over $\mathbb{Q}$, then Y gives a $K3$-surface over $\mathbb{F}_p$ for almost all p and

$$Z(Y,u) = \frac{1}{(1-u)(1-p^2u)P_2(u)},$$

where $\deg P_2(u) = 22$. In fact,

$$P_2(u) = Q_p(u)R_p(u),$$

where the polynomial $R_p(u)$ comes from the algebraic cycles and $Q_p(u)$ comes from the transcendental cycles. Hence, for a singular $K3$-surface, $\deg Q_p = 2$ and $\deg R_p = 20$.

Finally, we will work with the part of the L-function of Y coming from the transcendental lattice, which is given by

$$L(\mathbb{T}(Y), s) = (*) \prod_{p \,\mathrm{good}} \frac{1}{Q_p(p^{-s})} = \sum_{n=1}^{\infty} \frac{A_n}{n^s},$$

where $(*)$ represents finite factors coming from the primes of bad reduction.

2.2. **Elliptic surfaces.** An elliptic surface Y over $\mathbb{P}^1$ is a smooth projective surface Y with an elliptic fibration, i.e., a surjective morphism

$$\Phi : Y \to \mathbb{P}^1$$

such that almost all of the fibers are smooth curves of genus 1 and no fiber contains an exceptional curve of the first kind (with self-intersection -1). Here we list some facts about elliptic surfaces. See [SS10] for a comprehensive reference containing these results.

The group of global sections of the elliptic surface is called the Mordell-Weil group and can be naturally identified with the group of points of the generic fiber. Its rank r can be found from the formula

$$\rho(Y) = r + 2 + \sum_{\nu=1}^{h} (m_\nu - 1) \tag{2.1}$$

due to Shioda [Sh90]. Here m_ν denotes number of irreducible components of the corresponding singular fiber and h is the number of singular fibers.

Global sections can be also thought as part of the Néron-Severi group $\mathrm{NS}(Y)$ given by the divisors modulo algebraic equivalence. It is finitely generated and torsion-free. Intersection of divisors yields a bilinear pairing which gives $\mathrm{NS}(Y)$ the structure of an integral lattice.

The trivial lattice $\mathrm{T}(Y)$ is the subgroup of $\mathrm{NS}(Y)$ generated by the zero section and the fiber components. Its determinant is given by

$$\det \mathrm{T}(Y) = \prod_{\nu=1}^{h} m_\nu^{(1)}, \tag{2.2}$$

where $m_\nu^{(1)}$ indicates the number of single components of the corresponding singular fiber. (See [Sh90, p. 17].) One has that the Mordell-Weil group is isomorphic to $\mathrm{NS}(Y)/\mathrm{T}(Y)$.

The Mordell-Weil group can also be given a lattice structure $\mathrm{MWL}(Y)$. Then

$$\det \mathrm{NS}(Y) = (-1)^r \frac{\det \mathrm{T}(Y) \det \mathrm{MWL}(Y)}{|E_{\mathrm{tors}}|^2}, \tag{2.3}$$

where E is the generic fiber. The bilinear pairing induced by intersection can be used to construct a height that satisfies

$$h(P) = 2\chi(Y) + 2(\overline{P} \cdot \overline{O}) - \sum_{\nu} \mathrm{contr}_\nu(P), \tag{2.4}$$

where $\chi(Y)$ is the arithmetic genus ($\chi(Y) = 2$ for $K3$-surfaces), $\overline{P} \cdot \overline{O} \geq 0$, and the (always nonnegative) correction terms $\mathrm{contr}_\nu(P)$ measure how P intersects the components of the singular fiber over ν. This height is the canonical height that one obtains by thinking about the elliptic surface as an elliptic curve over a function field [Sh90].

2.3. **A particular family of $K3$-surfaces.** In this note, we consider the family of polynomials

$$P_k(x, y, z) = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k.$$

The desingularization of $P_k = 0$ results in a $K3$-hypersurface Y_k. We homogenize the numerator of P_k:

$$x^2yz + xy^2z + xyz^2 + t^2(xy + xz + yz) - kxyzt,$$

and then get an elliptic fibration by setting $t = s(x + y + z)$.

$$Y_k : s^2(x + y)(x + z)(y + z) + (s^2 - ks + 1)xyz = 0. \tag{2.5}$$

To study the components of the singular fibers, one expresses the $K3$-surface Y_k as a double covering of a well-known rational elliptic surface given by Beauville [Bea82]

$$(x+y)(x+z)(y+z)+uxyz=0. \tag{2.6}$$

By analyzing the structure of the singular fibers, we can compute the rank of the group of sections r. In the case of Beauville's surface, the singular fibers are given by

$$\begin{array}{ll} u=\infty & I_6, \\ u=0 & I_3, \\ u=1 & I_2, \text{ and} \\ u=-8 & I_1. \end{array}$$

To conclude this section, we summarize some results from Peters and Stienstra [PS89] on this family of $K3$-surfaces. For generic k, the Picard number is $\rho(Y_k)=19$. We focus on the singular $K3$-surfaces — that is, on k values for which $\rho(Y_k)=20$. The transcendental lattice $\mathbb{T}$ of the general family Y_k has a Gram matrix of the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 12 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.7}$$

Having Picard number $\rho=20$ is equivalent to having a relation between the generic basis $\{\gamma_1,\gamma_2,\gamma_3\}$ of transcendental periods; that is,

$$p\gamma_1+q\gamma_2+r\gamma_3 \tag{2.8}$$

becomes algebraic for some choice of p,q,r.

Now, let $k=w+\frac{1}{w}$. Then w can be represented as a modular function:

$$w=\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^6, \quad \eta(\tau)=e^{\frac{\pi i\tau}{12}}\prod_{n\geq 1}(1-e^{2\pi in\tau}), \quad \tau\in\mathbb{H},$$

where $\mathbb{H}$ denotes the upper half-plane. Furthermore, a period is algebraic precisely when it is orthogonal to $\gamma_1+\tau\gamma_2-6\tau^2\gamma_3$. Combining these facts yields a quadratic equation for τ:

$$-6p\tau^2+12q\tau+r=0. \tag{2.9}$$

Thus to find k-values such that Y_k is a singular $K3$-surface, we look for k values yielding an imaginary quadratic τ. Here are a few such values:

k	0	2	3	6	10	18
τ	$\frac{-3+\sqrt{-3}}{6}$	$\frac{-2+\sqrt{-2}}{6}$	$\frac{-3+\sqrt{-15}}{12}$	$\frac{1}{\sqrt{-6}}$	$\frac{1}{\sqrt{-2}}$	$\sqrt{\frac{-5}{6}}$

Given τ, one may find the parameters p, q, and r, and then find the discriminant of $\mathbb{T}$ up to squares by taking the determinant of the resulting Gram matrix. See Section 4 for details in the cases where $k=3$, $k=6$, and $k=18$.

3. Main results and the general strategy for the proof

Theorem 3.1. *We have the following formulas:*

$$\begin{aligned} \mathrm{m}(P_3) &= \frac{15\sqrt{15}}{2\pi^3}L(g_{15},3)=2\frac{|\det\mathbb{T}(Y_3)|^{3/2}}{4\pi^3}L(\mathbb{T}(Y_3),3), \\ \mathrm{m}(P_6) &= \frac{24\sqrt{24}}{2\pi^3}L(g_{24},3)=2\frac{|\det\mathbb{T}(Y_6)|^{3/2}}{4\pi^3}L(\mathbb{T}(Y_6),3), \text{ and} \\ \mathrm{m}(P_{18}) &= \frac{120\sqrt{120}}{20\pi^3}L(g_{120},3)+\frac{14}{5}d_3=\frac{1}{5}\frac{|\det\mathbb{T}(Y_{18})|^{3/2}}{4\pi^3}L(\mathbb{T}(Y_{18}),3)+\frac{14}{5}d_3, \end{aligned}$$

where Y_k is the $K3$-hypersurface defined by the zeros of $P_k(x,y,z)$, $\mathbb{T}(Y_k)$ is its transcendental lattice, and g_N is a CM modular form of level N.

The strategy for proving these formulas is as follows:

- Understand the transcendental lattice and the group of sections.

- Relate the Mahler measure $\mathrm{m}(P_k)$ to the L-function of a modular form.
- Relate the L-function of the surface Y_k to the L-function of that same modular form.

4. The Transcendental Lattice and the Rank

We will prove the following:

- For $k = 6$, $|\det \mathbb{T}(Y_6)| = 24$, rank $= 0$.
- For $k = 3$, $|\det \mathbb{T}(Y_3)| = 15$, rank $= 1$.
- For $k = 18$, $|\det \mathbb{T}(Y_{18})| = 120$, rank $= 1$.

4.1. **The transcendental lattice and the rank for Y_6.** When $k = 6$, we see from the table on page 5 that $\tau = \frac{1}{\sqrt{-6}}$. Thus, it satisfies the equation $-6\tau^2 - 1 = 0$, so in equation (2.9) we take $p = 1$, $q = 0$, and $r = -1$. By equation (2.8), the vector $\gamma_1 - \gamma_3$ becomes algebraic over Y_6. That is, $v = \gamma_1 - \gamma_3 \in \mathrm{Pic}(Y_6)$.

To find the transcendental lattice, we use the Gram matrix (2.7) to find vectors orthogonal to v. A simple computation yields: $\{\gamma_2, \gamma_1 + \gamma_3\}$; hence these span a sublattice of $\mathbb{T}$. We again use (2.7), this time to find the Gram matrix for the space spanned by these two vectors:

$$\begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix}.$$

Thus the discriminant of $\mathbb{T}$, up to a square, is equal to 24. It remains to decide if it is 6 or 24.

Equation (2.5) expresses Y_6 as a double-covering of the Beauville surface (2.6), with $u = (s^2 - 6s + 1)/s^2$.

$$Y_6 : s^2(x + y)(x + z)(y + z) + (s^2 - 6s + 1)xyz = 0.$$

Since we know the singular fibers of the Beauville surface, we easily find the singular fibers of Y_6:

$$\begin{array}{lll}
s = 0 & I_{12} & \text{double over } u = \infty, \\
s = \alpha & I_3 & \text{over } u = 0, \\
s = \beta & I_3 & \text{over } u = 0, \\
s = \frac{1}{6} & I_2 & \text{over } u = 1, \\
s = \infty & I_2 & \text{over } u = 1, \text{ and} \\
s = \frac{1}{3} & I_2 & \text{double over } u = -8.
\end{array}$$

(Here α and β are the two distinct roots of $s^2 - 6s + 1 = 0$.)

Applying Shioda's formula (2.1), we have

$$20 = r + 2 + (12 - 1) + (3 - 1) + (3 - 1) + (2 - 1) + (2 - 1) + (2 - 1) = r + 20,$$

so the rank of the group of sections is 0. A Weierstrass form is given by

$$y^2 + (s^2 - 6s + 1)xy = x(x - s^4)(x + s^2 - 6s^3).$$

We can compute the torsion group directly. A point of order 6 is given by

$$\big(s^2(6s - 1), 0\big)$$

and the only point of order 2 is $(0, 0)$.

Applying formula (2.3), we have

$$|\det \mathbb{T}(Y_6)| = |\det \mathrm{NS}(Y_6)| = \frac{12 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 2}{|E_{\mathrm{tors}}|^2} = \frac{2^5 \cdot 3^3}{|E_{\mathrm{tors}}|^2}.$$

This means that either $|E_{\mathrm{tors}}| = 6$ and $|\det \mathbb{T}_{Y_6}| = 24$, or $|E_{\mathrm{tors}}| = 12$ and $|\det \mathbb{T}_{Y_6}| = 6$. By the work of Miranda and Persson [MP89], $|E_{\mathrm{tors}}| = 12$ implies that the torsion is given by $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ which is not possible since there is only one point of order 2. Therefore, $|E_{\mathrm{tors}}| = 6$ and

$$|\det \mathbb{T}(Y_6)| = 24.$$

4.2. **The transcendental lattice and the rank for Y_3.** In this case we have $\tau = \frac{-3+\sqrt{-15}}{12}$ (see the table on page 5), which satisfies the quadratic equation $-6 \cdot 4\tau^2 - 12\tau - 4 = 0$. So in equation (2.9) we take $p = 4$, $q = -1$, and $r = -4$. By equation (2.8), $v = 4\gamma_1 - \gamma_2 - 4\gamma_3 \in \mathrm{Pic}(Y_3)$. Using the Gram matrix (2.7), we find that $\{\gamma_1 + \gamma_3, \gamma_2 + 3\gamma_3\}$ generate a sublattice of $\mathbb{T}$, and their Gram matrix is:

$$\begin{pmatrix} 2 & 3 \\ 3 & 12 \end{pmatrix}.$$

Since the determinant of this matrix is square-free, we conclude that $|\det \mathbb{T}(Y_3)| = 15$.

The equation

$$s^2(x+y)(x+z)(y+z) + (s^2 - 3s + 1)xyz = 0,$$

expresses Y_3 as a double-covering of the Beauville surface (2.6) with $u = (s^2 - 3s + 1)/s^2$. In this case, the singular fibers are:

$$\begin{array}{lll} s = 0 & I_{12} & \text{double over } u = \infty, \\ s = \alpha_1 & I_3 & \text{over } u = 0, \\ s = \beta_1 & I_3 & \text{over } u = 0, \\ s = \frac{1}{3} & I_2 & \text{over } u = 1, \\ s = \infty & I_2 & \text{over } u = 1, \\ s = \alpha_2 & I_1 & \text{over } u = -8, \text{ and} \\ s = \beta_2 & I_1 & \text{over } u = -8. \end{array}$$

Here, α_1, β_1 are the two distinct roots of $s^2 - 3s + 1 = 0$, and α_2, β_2 are the roots of $9s^2 - 3s + 1 = 0$.

By Shioda's formula (2.1), the rank is 1. A Weierstrass model around infinity is given by:

$$y^2 + (\sigma^2 - 3\sigma + 1)xy = x(x-1)(x + \sigma^2 - 3\sigma) = x^3 + (\sigma^2 - 3\sigma - 1)x^2 + (-\sigma^2 + 3\sigma)x.$$

With the aid of Pari/gp or Sage [PARI, St11] we find a point ρ_6 of order 6. Indeed,

$$\begin{aligned} \rho_6 &= \left(-\sigma(\sigma-3), \sigma(\sigma-3)(\sigma^2 - 3\sigma + 1)\right), \\ 2\rho_6 &= \left(1, -\sigma^2 + 3\sigma - 1\right), \\ 3\rho_6 &= (0,0), \\ 4\rho_6 &= (1,0), \text{ and} \\ 5\rho_6 &= \left(-\sigma^2 + 3\sigma, 0\right). \end{aligned}$$

By the work of Miranda and Persson [MP89], since the rank is 1 and $\chi = 2$, the torsion must have order 6, and therefore it must be generated by ρ_6.

With the aid of Pari/gp or Sage we also find the following point in each fiber:

$$\left(-(\sigma-3)(\sigma-1)^2, (\sigma-3)(\sigma-2)(\sigma-1)(\sigma^2 - 3\sigma + 1)\right).$$

Since this point is not generically among the torsion points of each fiber, it must give an infinite section, which is in particular defined over $\mathbb{Q}$. In fact, this point is a generator of the infinite section, but we do not need this fact for our computation.

4.3. **The transcendental lattice and the rank for Y_{18}.** When $k = 18$, the table shows $\tau = \sqrt{\frac{-5}{6}}$, which satisfies $-6\tau^2 - 5 = 0$. Take $p = 1$, $q = 0$, and $r = -5$ in equation (2.9), so $v = \gamma_1 - 5\gamma_3 \in \mathrm{Pic}(Y_{18})$. The vectors $\{\gamma_2, \gamma_1 + 5\gamma_3\}$ are orthogonal to v, and the corresponding Gram matrix is

$$\begin{pmatrix} 12 & 0 \\ 0 & 10 \end{pmatrix}. \tag{4.1}$$

The determinant of this matrix is 120, so the discriminant of the transcendental lattice is either 30 or 120.

The double-cover of the Beauville surface is given by:

$$Y_{18} : s^2(x+y)(x+z)(y+z) + (s^2 - 18s + 1)xyz = 0,$$

where $u = (s^2 - 18s + 1)/s^2$. The singular fibers are

$$\begin{array}{lll} s = 0 & I_{12} & \text{double over } u = \infty, \\ s = \alpha_1 & I_3 & \text{over } u = 0, \\ s = \beta_1 & I_3 & \text{over } u = 0, \\ s = \frac{1}{18} & I_2 & \text{over } u = 1, \\ s = \infty & I_2 & \text{over } u = 1, \\ s = \alpha_2 & I_1 & \text{over } u = -8, \text{ and} \\ s = \beta_2 & I_1 & \text{over } u = -8. \end{array}$$

Here α_1, β_1 are the two distinct roots of $s^2 - 18s + 1 = 0$, and α_2, β_2 are the roots of $9s^2 - 18s + 1 = 0$.

From Shioda's formula (2.1), we see that the rank is 1. A Weierstrass model around infinity is given by

$$y^2 + (\sigma^2 - 18\sigma + 1)xy = x(x-1)(x + \sigma^2 - 18\sigma) = x^3 + (\sigma^2 - 18\sigma - 1)x^2 + (-\sigma^2 + 18\sigma)x. \tag{4.2}$$

With the aid of Pari/gp or Sage [PARI, St11], we find a point ρ_6 of order 6. Indeed,

$$\begin{aligned} \rho_6 &= \left(-\sigma(\sigma - 18), \sigma(\sigma - 18)\left(\sigma^2 - 18\sigma + 1\right)\right), \\ 2\rho_6 &= \left(1, -\sigma^2 + 18\sigma - 1\right), \\ 3\rho_6 &= (0,0), \\ 4\rho_6 &= (1,0), \text{ and} \\ 5\rho_6 &= \left(-\sigma^2 + 18\sigma, 0\right). \end{aligned}$$

Again by the work of Miranda and Persson [MP89], $r = 1$ and $\chi = 2$ implies that the torsion must have order 6, and hence must be generated by ρ_6.

If P is a generator of the infinite part of the group of sections, then $\det \mathrm{MWL}(Y_{18}) = h(P)$. Applying formulas (2.2) and (2.3), we have

$$|\det \mathbb{T}(Y_{18})| = |\det \mathrm{NS}(Y_{18})| = \frac{12 \cdot 3^2 \cdot 2^2 h(P)}{6^2} = 12h(P). \tag{4.3}$$

By the remark following (4.1), $|\det \mathbb{T}(Y_{18})| = 30$ or 120. Hence either $|\det \mathbb{T}(Y_{18})| = 30$ and $h(P) = 5/2$ or $|\det \mathbb{T}(Y_{18})| = 120$ and $h(P) = 10$.

Finding the infinite section for Y_{18} is more difficult than for Y_3 because the infinite section is not defined over $\mathbb{Q}$. Details of the method used to find the infinite section, prove that we have a generator, and compute its height are in Section 7. The outcome of the computations is a generator p_σ defined over $\mathbb{Q}(\sqrt{-3})$ satisfying $h(p_\sigma) = 10$; hence

$$|\det \mathbb{T}(Y_{18})| = 120.$$

5. Relating the Mahler Measure to a newform

The main ingredient we use to relate Mahler mesure to newforms is the following result from [Be06].

Theorem 5.1 (Bertin). *Let $k = w + \frac{1}{w}$ with*

$$w = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^6, \quad \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n \geq 1} (1 - e^{2\pi i n \tau}).$$

Then

$$\mathrm{m}(P_k) = \frac{\operatorname{Im}\tau}{8\pi^3}\left[\sum_{m,n}\left(-4\left(2\operatorname{Re}\frac{1}{(m\tau+n)^3(m\bar\tau+n)}+\frac{1}{(m\tau+n)^2(m\bar\tau+n)^2}\right)\right.\right.$$
$$+16\left(2\operatorname{Re}\frac{1}{(2m\tau+n)^3(2m\bar\tau+n)}+\frac{1}{(2m\tau+n)^2(2m\bar\tau+n)^2}\right)$$
$$-36\left(2\operatorname{Re}\frac{1}{(3m\tau+n)^3(3m\bar\tau+n)}+\frac{1}{(3m\tau+n)^2(3m\bar\tau+n)^2}\right)$$
$$\left.\left.+144\left(2\operatorname{Re}\frac{1}{(6m\tau+n)^3(6m\bar\tau+n)}+\frac{1}{(6m\tau+n)^2(6m\bar\tau+n)^2}\right)\right)\right].$$

The evaluation of the Eisenstein–Kronecker series often leads to Hecke L-functions. Let K be an imaginary quadratic number field and $\mathfrak{m}$ be an ideal of $\mathcal{O}_K$. A Hecke character of K modulo $\mathfrak{m}$ with ∞-type ℓ is a homomorphism ϕ on the group of fractional ideals of K which are prime to $\mathfrak{m}$ such that for all $\alpha \in K^*$ with $\alpha \equiv 1 \mod \mathfrak{m}$,

$$\phi((\alpha)) = \alpha^\ell.$$

The ideal $\mathfrak{m}$ is called the conductor of ϕ if it is minimal in the following sense: if ϕ is defined modulo $\mathfrak{m}'$, then $\mathfrak{m}|\mathfrak{m}'$.

Let

$$L(\phi, s) = \sum_{\mathfrak{a}\text{ integral}} \frac{\phi(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{cl(\mathfrak{a})} \frac{\phi(\mathfrak{a})}{N(\mathfrak{a})^{2-s}}\frac{1}{2}\sum_{\lambda\in\mathfrak{a}}{}' \frac{\bar\lambda^2}{(\lambda\bar\lambda)^s}.$$

The Mellin transform gives a Hecke eigenform:

$$f_\phi = \sum_{n\in\mathbb{N}} a_n q^n = \sum_{\mathfrak{a}\text{ integral}} \phi(\mathfrak{a}) q^{N(\mathfrak{a})}.$$

A theorem of Hecke and Shimura implies that f_ϕ has weight $\ell+1$ and level $\Delta_K N(\mathfrak{m})$. If ℓ is even,

$$f_\phi \in S_{\ell+1}(\Gamma_0(\Delta_K N(\mathfrak{m})), \chi_K)$$

where $-\Delta_K$ is the discriminant of the field, and χ_K is its quadratic character.

A newform $f = \sum a_n q^n \in S_k(\Gamma_1(N))$ is said to have complex multiplication (CM) by a Dirichlet character ϕ if $f = f \otimes \phi$, where

$$f \otimes \phi = \sum_{n\in\mathbb{N}} \phi(n) a_n q^n.$$

By a result of Ribet, a newform has CM by a quadratic field K iff it comes from a Hecke character of K. In particular, K is imaginary and unique. Schütt [Sc08] proves that there are only finitely many CM newforms with rational coefficients for certain fixed weights (including 3) up to twisting, and he gives a comprehensive table for these.

5.1. **The relation with a newform for P_6.** From Theorem 5.1,

$$\mathrm{m}(P_6) = \frac{24\sqrt{6}}{\pi^3}\left(\frac{1}{2}\sum_{m,k}{}'\left(\frac{m^2-6k^2}{(m^2+6k^2)^3}+\frac{3k^2-2m^2}{(3k^2+2m^2)^3}\right)\right).$$

This summation can be viewed (see [Be06]) as a Hecke L-series on the field $\mathbb{Q}(\sqrt{-6})$. This field has discriminant -24 and class number 2, with the nontrivial class represented by $(2, \sqrt{-6})$. That is, we have

$$\mathrm{m}(P_6) = \frac{24\sqrt{6}}{\pi^3} L_{\mathbb{Q}(\sqrt{-6})}(\phi, 3), \text{ where } \phi(2, \sqrt{-6}) = -2.$$

By the results of Hecke and Shimura, we look for a correspondence to a (quadratic) twist of a newform of weight 3 and level 24. According to Schütt's table [Sc08], there is only one newform (up to twisting) of weight 3 and level 24. The twist must be of the form $\left(\frac{d}{p}\right)$ for d dividing 24, and we can compute the twist exactly by comparing the first few coefficients, as shown in the following table.

a_p	2	3	5	7	11	13	17	19	23	29	31
newform of level 24	2	-3	-2	-10	10	0	0	0	0	-50	38
coef. of $L_{\mathbb{Q}(\sqrt{-6})}(\phi, s)$	-2	3	2	-10	-10	0	0	0	0	50	38

We find that the twist is given by $\left(\frac{-3}{p}\right)$. Therefore,

$$\mathrm{m}(P_6) = \frac{24\sqrt{6}}{\pi^3} L\left(f_{24} \otimes \left(\frac{-3}{\cdot}\right), 3\right). \tag{5.1}$$

5.2. **The relation with a newform for P_3.** This case was also considered in [Be06] as a Hecke L-series on the field $\mathbb{Q}(\sqrt{-15})$. This field has discriminant -15 and class number 2, with the nontrivial class represented by $\left(2, \frac{1+\sqrt{-15}}{2}\right)$.

$$\begin{aligned}
\mathrm{m}(P_3) &= \frac{15\sqrt{15}}{2\pi^3}\left(\frac{1}{4}\sum_{m,k}{}' \left(\frac{2m^2+2mk-7k^2}{(m^2+mk+4k^2)^3} - \frac{m^2+8mk+k^3}{(2m^2+mk+2k^2)^3}\right)\right)\\
&= \frac{15\sqrt{15}}{2\pi^3} L_{\mathbb{Q}(\sqrt{-15})}(\phi, 3),
\end{aligned}$$

where $\phi\left(2, \frac{1+\sqrt{-15}}{2}\right) = -2$.

There is only one newform of level 15 and weight 3 in Schütt's table. We compare the first few coefficients.

a_p	2	3	5	7	11	13	17	19	23	29	31
newform of level 15	-1	3	-5	0	0	0	14	-22	-34	0	2
coef. of $L_{\mathbb{Q}(\sqrt{-15})}(\phi, s)$	-1	3	-5	0	0	0	14	-22	-34	0	2

Therefore,

$$\mathrm{m}(P_3) = \frac{15\sqrt{15}}{2\pi^3} L\left(f_{15}, 3\right). \tag{5.2}$$

5.3. **The relation with a newform for P_{18}.** After some algebraic manipulation, one can find a Hecke series in $\mathbb{Q}(\sqrt{-30})$. This field has discriminant -120 and class number 4, with the class group generated by $(2, \sqrt{-30})$ and $(3, \sqrt{-30})$. We have

$$\begin{aligned}
\mathrm{m}(P_{18}) = {}& \frac{6\sqrt{120}}{\pi^3}\left(\frac{1}{2}\sum_{m,k}{}'\left(\frac{5m^2-6k^2}{(5m^2+6k^2)^3} - \frac{10m^2-3k^2}{(10m^2+3k^2)^3} + \frac{15m^2-2k^2}{(15m^2+2k^2)^3} - \frac{30m^2-k^2}{(30m^2+k^2)^3}\right)\right)\\
&+ \frac{3\sqrt{30}}{\pi^3}\sum_{m,k}{}'\left(-\frac{1}{(5m^2+6k^2)^2} + \frac{1}{(10m^2+3k^2)^2} - \frac{1}{(15m^2+2k^2)^2} + \frac{1}{(30m^2+k^2)^2}\right)\\
={}& \frac{6\sqrt{120}}{\pi^3} L_{\mathbb{Q}(\sqrt{-30})}(\phi, 3) + \frac{14}{5} d_3,
\end{aligned}$$

where $\phi(2, \sqrt{-30}) = -2$ and $\phi(3, \sqrt{-30}) = 3$. The equality for the term $\frac{14}{5} d_3$ was proved by Bertin [Be11] by examining identities of certain Epstein zeta functions.

There is only one newform of weight 3 and level 120 in Schütt's table.

a_p	2	3	5	7	11	13	17	19	23	29	31
newform of level 120	2	3	-5	0	2	-14	-26	0	-14	38	-58
coef. of $L_{\mathbb{Q}(\sqrt{-30})}(\phi, s)$	-2	3	5	0	-2	-14	26	0	14	-38	-58

The final results yields

$$L_{\mathbb{Q}(\sqrt{-30})}(\phi,3) = L\left(f_{120}\otimes\left(\frac{-3}{\cdot}\right),3\right),$$

$$\mathrm{m}(P_{18}) = \frac{6\sqrt{120}}{\pi^3}L\left(f_{120}\otimes\left(\frac{-3}{\cdot}\right),3\right)+\frac{14}{5}d_3. \tag{5.3}$$

6. Relating $L(\mathbb{T}(Y),s)$ to a newform

The main tool for this section is the following result from [Sc08].

Theorem 6.1 (Schütt)**.** *The following classification of singular $K3$-surfaces over $\mathbb{Q}$ are equivalent.*

- *By the discriminant d of the transcendental lattice of the surface up to square.*
- *By the discriminant $-d$ of the Néron-Severi lattice of the surface up to square.*
- *By the associated newform up to twisting.*
- *By the level of the associated newform up to square.*
- *By the CM field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.*

This theorem depends on Livné's modularity theorem for singular $K3$-surfaces that predicts that $L(\mathbb{T}(Y),s)$ is modular and that the corresponding modular form has weight 3.

The first step in finding the corresponding modular form is to compute the first few coefficients A_p from $L(\mathbb{T}(Y),s)$; then the coefficients are compared to the tables that can be found in [Sc08] in order to identify the corresponding CM newform. Tackling the first step requires the following result from [Be10].

Theorem 6.2 (Bertin)**.** *Let Y be an elliptic $K3$-surface defined over $\mathbb{Q}$ and rank $r(Y)=0$. Then*

$$A_p = -\sum_{s\in\mathbb{P}^1(\mathbb{F}_p)} a_p(s), \tag{6.1}$$

where

$$a_p(s) = p+1-\#Y_s(\mathbb{F}_p).$$

Now suppose that $r(Y)=1$ and that there is an infinite section defined over $\mathbb{Q}(\sqrt{d})$. Then

$$A_p = -\sum_{s\in\mathbb{P}^1(\mathbb{F}_p)} a_p(s) - \left(\frac{d}{p}\right)p. \tag{6.2}$$

Notice that the result stated in [Be10] requires a *generator* of $\mathrm{MWL}(Y)$ to be defined over $\mathbb{Q}(\sqrt{d})$. But it is not hard to see that it suffices to find any element of infinite order to be defined over $\mathbb{Q}(\sqrt{d})$.

6.1. Relating $L(\mathbb{T}(Y_6),s)$ to a newform.

We know from Section 4.1 that $r(Y_6)=0$ and that $|\det\mathbb{T}(Y_6)| = 24$, so we use equation (6.1). With the help of Pari/gp or Sage we compute several coefficients A_p and compare them to the coefficients of the newform of level 24 from Schütt's table in [Sc08].

a_p	5	7	11	13	17	19	23	29	31
newform of level 24	-2	-10	10	0	0	0	0	-50	38
A_p	2	-10	-10	0	0	0	0	50	38

We see that

$$L(\mathbb{T}(Y_6),3) = L\left(f_{24}\otimes\left(\frac{-3}{\cdot}\right),3\right),$$

and combining this with equation (5.1) gives the final result

$$\mathrm{m}(P_6) = \frac{24\sqrt{6}}{\pi^3}L(\mathbb{T}(Y_6),3).$$

6.2. **Relating $L(\mathbb{T}(Y_3), s)$ to a newform.** In this case, $r(Y_3) = 1$ and the infinite section is defined over $\mathbb{Q}$. We apply equation (6.2) to compute the A_p values and compare with the table from [Sc08] in order to obtain

$$L(\mathbb{T}(Y_3), 3) = L\left(f_{15}, 3\right).$$

Combining this with equation (5.2) gives the final result

$$\mathrm{m}(P_3) = \frac{15\sqrt{15}}{2\pi^3} L(\mathbb{T}(Y_3), 3).$$

6.3. **Relating $L(\mathbb{T}(Y_{18}), s)$ to a newform.** In this case, $r(Y_{18}) = 1$ and the infinite section is defined over $\mathbb{Q}(\sqrt{-3})$. We again apply equation (6.2) to compute the A_p values and compare with the table from [Sc08] in order to obtain

$$L(\mathbb{T}(Y_{18}), 3) = L\left(f_{120} \otimes \left(\frac{-3}{\cdot}\right), 3\right).$$

Combining this with equation (5.3) gives the final result

$$\mathrm{m}(P_{18}) = \frac{120\sqrt{120}}{20\pi^3} L(\mathbb{T}(Y_{18}), 3) + \frac{14}{5} d_3.$$

As a final note, we remark that one could have started the computations from this subsection without knowing that the infinite section is defined over $\mathbb{Q}(\sqrt{-3})$. Computing several values of A_p with equation (6.1) and comparing with the table from [Sc08] will reveal the necessary correction factor. This allows one to *predict* that the infinite section is defined over $\mathbb{Q}(\sqrt{-3})$, and armed with this knowledge the infinite section is more easily computed (see Section 7.1).

7. Infinite section for Y_{18}

We now describe the computations used to find an infinite section p_σ for the elliptic surface given in equation (4.2), show that our p_σ is a generator for the infinite part of the group of sections, and prove that $h(p_\sigma) = 10$.

7.1. **Finding the infinite section.** As noted above, we can predict that the infinite section is defined over $\mathbb{Q}(\sqrt{-3})$. Therefore, we twist equation (4.2) by -3 in order to get an elliptic surface with the infinite section defined over $\mathbb{Q}$. We denote this twist Y_{-3} (we drop the Y_{18} notation in this case because there is no ambiguity). Applying the general formula for a quadratic twist [Co99, Chapter 4], we have

$$Y_{-3} : y^2 + (\sigma^2 - 18\sigma + 1)xy = x^3 + (-\sigma^4 + 36\sigma^3 - 329\sigma^2 + 90\sigma + 2)x^2 + 9\sigma(-\sigma + 18)x.$$

For each σ, the fiber Y_σ is a curve in Y_{18} and the fiber $Y_{\sigma,-3}$ is a curve in Y_{-3}. These curves satisfy the following exact sequence (see [IR90], Proposition 20.5.4):

$$0 \to Y_{\sigma,-3}(\mathbb{Q}) \to Y_\sigma\left(\mathbb{Q}(\sqrt{-3})\right) \xrightarrow{\mathrm{Tr}_{\mathbb{Q}(\sqrt{-3})/\mathbb{Q}}} Y_\sigma(\mathbb{Q}) \to Y_\sigma(\mathbb{Q})/2Y_\sigma(\mathbb{Q}) \to 0.$$

More specifically, we have

$$0 \to Y_{\sigma,-3}(\mathbb{Q}) \to Y_\sigma\left(\mathbb{Q}(\sqrt{-3})\right) \to \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

A computation verifies that a section for Y_{-3} is given by $p_{-3} = (x_{-3}(\sigma), y_{-3}(\sigma))$ where

$$x_{-3}(\sigma) = -\frac{2^4 3^6 \sigma(\sigma-18)(\sigma-21)^2(\sigma+3)^2}{(\sigma-9)^2(\sigma^2-21\sigma+72)^2(\sigma^2-15\sigma+18)^2}, \text{ and}$$

$$\begin{aligned} y_{-3}(\sigma) = &-\frac{2^2 3^4 \sigma(\sigma-18)(\sigma-21)(\sigma+3)}{(\sigma-9)^3(\sigma^2-21\sigma+72)^3(\sigma^2-15\sigma+18)^3} \\ &\cdot (\sigma^{10} - 108\sigma^9 + 4455\sigma^8 - 87822\sigma^7 + 771363\sigma^6 - 294840\sigma^5 - 44001711\sigma^4 \\ &\quad + 281168010\sigma^3 - 545848956\sigma^2 + 132322248\sigma + 128490624). \end{aligned}$$

The curve $Y_{\sigma,-3}$ has good reduction modulo 5 when $\sigma \equiv 1, 2 \pmod 5$. In those cases, one finds that $Y_{\sigma,-3}(\mathbb{F}_5)$ has 6 elements and is generated by the point $(3, 1)$. Hence the torsion of $Y_{\sigma,-3}(\mathbb{Q})$ injects into $\mathbb{Z}/6\mathbb{Z}$. With the help of Pari/gp or Sage [PARI, St11], it is easy to compute $[6]p_{-3}$ and see that the result is different from $O_{\sigma,-3}$. Therefore this point is not torsion.

Reversing the change of coordinates, one finds an infinite section $p_\sigma = (x(\sigma), y(\sigma))$ for the surface Y_{18}:

$$
\begin{aligned}
(7.1)\qquad x(\sigma) &= \frac{2^4 3^5 \sigma(\sigma-18)(\sigma-21)^2(\sigma+3)^2}{(\sigma-9)^2(\sigma^2-21\sigma+72)^2(\sigma^2-15\sigma+18)^2}, \text{ and}\\
y(\sigma) &= \frac{2^2 3^2 \sqrt{-3}\sigma(\sigma-21)(\sigma-18)(\sigma+3)}{(\sigma-9)^3(\sigma^2-21\sigma+72)^3(\sigma^2-15\sigma+18)^3}\left(\sigma^2+3(-6+\sqrt{-3})\sigma+9(5-3\sqrt{-3})\right)\\
&\quad\cdot\left(\sigma^3+3(-9+\sqrt{-3})\sigma^2+9(19-6\sqrt{-3})\sigma+9(-9+11\sqrt{-3})\right)\\
&\quad\cdot\left(\sigma^5+3(-15+4\sqrt{-3})\sigma^4+27(19-16\sqrt{-3})\sigma^3\right.\\
&\qquad\left.+81(9+52\sqrt{-3})\sigma^2+162(-139-36\sqrt{-3})\sigma+5832(1-\sqrt{-3})\right).
\end{aligned}
$$

It is clear from these formulas that p_σ and the zero section $[0:1:0]$ have simple intersections over $\sigma = 9$, and over the distinct roots of $(\sigma^2 - 21\sigma + 72)$ and $(\sigma^2 - 15\sigma + 18)$. Therefore $\overline{p_\sigma} \cdot \overline{O} = 5$. Applying equation (2.4), we see that

$$h(p_\sigma) \quad = \quad 2\chi(Y_{18}) + 2\left(\overline{p_\sigma}\cdot\overline{O}\right) - \sum_v \mathrm{contr}_\nu(P) = 2\cdot 2 + 2\cdot 5 - \sum_v \mathrm{contr}_\nu(P).$$

From this, we have

$$
\begin{aligned}
14 \geq \quad h(p_\sigma) \quad &\geq 14 - \frac{6\cdot 6}{12} - \frac{1\cdot 1}{2} - \frac{1\cdot 1}{2} - \frac{1\cdot 2}{3} - \frac{1\cdot 2}{3}\\
14 \geq \quad h(p_\sigma) \quad &\geq \frac{26}{3}.
\end{aligned}
$$

From the remarks following equation (4.3), we know that the height of a generator must be either $5/2$ or 10. This means that $h(p_\sigma) = 10$, since it must be a square multiple of the height of a generator. In Section 7.3, we show this fact directly by analyzing the intersection with the singular fibers.

7.2. **Proof that p_σ is a generator.** Let $K = \mathbb{Q}(\sqrt{-3})(\sigma)$. To prove that p_σ is indeed a generator of the infinite section, we need to see that we cannot write $p_\sigma + k\rho_6 = [2]P$ for any $P \in E(K)$ and $k = 0, \ldots, 5$. In fact, it suffices to prove that $p_\sigma + k\rho_6 = [2]P$ has no solution $P \in E(K)$ for $k = 0, 3$. We will use the following theorem.

Theorem 7.1 ([Co99], Proposition 1.7.5(b))**.** *Let*

$$E : y^2 = x(x^2 + ax + b)$$

be an elliptic curve defined over a field K with char $K \neq 2$, *and suppose $a^2 - 4b \notin K^{*2}$. Let $Q = (x, y) \in E(K)$ with $x \neq 0$. Then there exists $P \in E(K)$ such that $Q = [2]P$ iff* (i) *$x \in K^{*2}$, say $x = r^2$; and* (ii) *one of $q_\pm = 2x + a \pm 2y/r \in K^{*2}$.*

In order to apply this result, we need to eliminate the term xy from the Weierstrass equation (4.2), which we do by making the change $Y = y + \frac{(\sigma^2-18\sigma+1)x}{2}$. This gives

$$Y^2 = x\left(x^2 + \frac{\sigma^4 - 36\sigma^3 + 330\sigma^2 - 108\sigma - 3}{4}x + (-\sigma^2 + 18\sigma)\right).$$

From equation (7.1), we see that $x(\sigma)$ is not a square in K, hence there is no $P \in E(K)$ such that $p_\sigma = [2]P$.

Now write $p_\sigma + 3\rho_6 = (x'(\sigma), Y'(\sigma))$. A computation yields

$$x'(\sigma) = -\frac{(\sigma-9)^2(\sigma^2-21\sigma+72)^2(\sigma^2-15\sigma+18)^2}{2^4\cdot 3^5(\sigma-21)^2(\sigma+3)^2}$$

which is a square in K, so take

$$r = \frac{(\sigma-9)(\sigma^2-21\sigma+72)(\sigma^2-15\sigma+18)}{2^2\cdot 3^2\sqrt{-3}(\sigma-21)(\sigma+3)}.$$

To compute $q_\pm$ as in Theorem 7.1, we first find

$$Y'(\sigma) = \frac{\sqrt{-3}(\sigma-9)(\sigma^2-15\sigma+18)(\sigma^2-21\sigma+72)(\sigma^3-12\sigma^2-171\sigma+1350)}{2^6 3^8(\sigma+3)^3(\sigma-21)^3} \cdot (\sigma^3-42\sigma^2+369\sigma-216)(\sigma^4-36\sigma^3+351\sigma^2-486\sigma-486).$$

It is then a simple matter to compute

$$\begin{aligned} q_+ &= -\frac{1}{2^2\cdot 3^5}(\sigma-21)^2(\sigma+3)^2(\sigma^2-18\sigma+9) \\ q_- &= -\frac{3^5(\sigma^2-18\sigma+1)^3}{(\sigma-21)^2(\sigma+3)^2}, \end{aligned}$$

and neither of these are squares in K.

7.3. **Height computation.** In order to compute $h(p_\sigma)$, we need to study the intersection of p_σ with the singular fibers, since the correction terms in formula (2.4) are given by

$$\mathrm{contr}_\nu(P) = \frac{j(m-j)}{m},$$

when P intersects the component $\Theta_{s,j}$ of the singular fiber over s of type I_m. We need the following theorem from [Ne64]:

Theorem 7.2 (Néron)**.** *Let E_s be an elliptic curve defined over $\mathbb{C}[s]$ given by a Weierstrass model, and denote by v the s-adic valuation. Suppose that E_0 has a double point with distinct tangents and $v(j(E_s)) = -m < 0$ (this happens if and only if E_0 is singular of type I_m in Kodaira's classification). Then, for every integer $l > m/2$, there exists a Weierstrass model $\mathcal{E}_s$ deduced from E_s by a transformation of the form*

$$\begin{aligned} X &= x+qz, \\ Y &= y+ux+rz, \\ Z &= z, \end{aligned}$$

with $q, r, u \in \mathbb{C}[s]$. A Weierstrass model $\mathcal{E}_s$ is given by

$$Y^2Z + \lambda XYZ + \mu YZ^2 = X^3 + \alpha X^2 Z + \beta XZ^2 + \gamma Z^3 \tag{7.2}$$

with coeffcients satisfying

$$v(\lambda^2+4\alpha) = 0, \quad v(\mu)\geq l, \quad v(\beta) \geq l, \quad v(\gamma) = m, \text{ and } \quad v(j(\mathcal{E}_s)) = -m. \tag{7.3}$$

We now follow the argument in [Be08b], and refer the interested reader there for details. A singular fiber of type I_m over $s = 0$ is composed of the nonsingular rational curves $\Theta_{0,0}, \Theta_{0,1}, \ldots, \Theta_{0,m-1}$. If $m = 2h$, the configuration of the these curves can be found in $\left(\mathbb{P}^2\right)^h$, with a point $[X : Y : Z] \in Y_{18}$ over $s = 0$ corresponding to the point

$$[X:Y:Z^{(1)}]\times[X:Y:Z^{(2)}]\times\cdots\times[X:Y:Z^{(h)}] \in (\mathbb{P}^2)^h, \quad \text{where } [X:Y:Z^{(i+1)}] = [X:Y:sZ^{(i)}]. \tag{7.4}$$

So in particular,

$$[X:Y:Z^{(1)}] = [X:Y:sZ] \text{ and inductively } [X:Y:Z^{(h)}] = [X:Y:s^hZ].$$

If $[X : Y : Z]$ satisfies equation (7.2), then $[X : Y : Z^{(h)}]$ must satisfy the equation

$$Y^2Z^{(h)} + \lambda XYZ^{(h)} + (\mu/s^h)Y(Z^{(h)})^2 = s^hX^3 + \alpha X^2Z^{(h)} + (\beta/s^h)X(Z^{(h)})^2 + (\gamma/s^{2h})(Z^{(h)})^3.$$

Now, given the valuations in (7.3) and the fact that $2h = m$, at $s = 0$ this simplifies to

$$Y^2Z^{(h)} + \lambda_0 XYZ^{(h)} = \alpha_0 X^2 Z^{(h)} + \gamma_m^0 (Z^{(h)})^3, \tag{7.5}$$

where the subscript 0 indicates evaluation at $s = 0$, and $\gamma_m^0 = (\gamma/s^m)|_{s=0}$.

In fact, we can describe the components $\Theta_{0,i}$ exactly. We give here only the fibers relevant in the sequel:

$$\begin{aligned} &\Theta_{0,0} = [X:Y:0]\times\cdots\times[X:Y:0] \in (\mathbb{P}^2)^h, \text{ and} \\ &\Theta_{0,h} = [0:0:1]\times\cdots\times[0:0:1]\times[X_0:Y_0:Z_0] \in (\mathbb{P}^2)^h, \end{aligned} \tag{7.6}$$

where $Z_0 \neq 0$ and $[X_0 : Y_0 : Z_0]$ is on the conic (7.5).

7.3.1. *The fiber over $s = 0$.* This is a singularity of type I_{12}. Let $(x'(\sigma), y'(\sigma))$ represent the infinite section in equation (7.1). The change of variables

$$x(s) = s^4 x'(1/s), \qquad y(s) = s^6 y'(1/s), \qquad \sigma = 1/s$$

yields an infinite section for the Weierstrass model around 0 given by the equation

$$y^2 + (s^2 - 18s + 1)xy = x^3 + s^2(-s^2 - 18s + 1)x^2 + (-s^6 + 18s^7)x.$$

A second change of variables

$$x = X + 2s^6, \qquad\qquad y = Y - sX - 2s^7 - s^6$$

gives the $\mathcal{E}_s$ model

$$\begin{aligned} Y^2 + (s^2 - 20s + 1)XY + (2s^8 - 40s^7)Y &= X^3 + (6s^6 - s^4 - 17s^3 - 18s^2 + s)X^2 \\ &+ (12s^{12} - 4s^{10} - 68s^9 - 71s^8 + 2s^7)X + (8s^{18} - 4s^{16} - 68s^{15} - 70s^{14} - s^{12}). \end{aligned}$$

The same change of variables applied to the infinite section $(x(s), y(s))$ yields

$$(X(s), Y(s)) = \left(s^6 f_1(s), s^6 g_1(s)\right)$$

where $f_1(0) = -2$ and $g_1(0) = 1$. So by equation (7.4) this corresponds to the point

$$[0:0:1]\times[0:0:1]\times[0:0:1]\times[0:0:1]\times[0:0:1]\times[-2:1:1]$$

in the $\mathcal{E}_s$ model. From (7.6) we see that this point is on $\Theta_{0,6}$ because $[-2 : 1 : 1]$ is on the conic

$$Y^2 + XY + Z^2 = 0.$$

7.3.2. *The fiber over $s = \infty$.* This is a singularity of type I_2, and the infinite section given in equation (7.1) is for the model around infinity given by the Weierstrass equation

$$y^2 + (\sigma^2 - 18\sigma + 1)xy = x(x-1)(x + \sigma^2 - 18\sigma) = x^3 + (\sigma^2 - 18\sigma - 1)x^2 + (-\sigma^2 + 18\sigma)x.$$

So we work with the singular fibers over $\sigma = 0$ just as we did above with $s = 0$. The change of variables

$$x = \frac{X}{9} + 12\sigma, \qquad\qquad y = \frac{Y}{27} + \frac{X}{9} - 6\sigma$$

gives the $\mathcal{E}_\sigma$ model

$$\begin{aligned} Y^2 + (3\sigma^2 - 54\sigma + 9)XY + (324\sigma^3 - 5832\sigma^2)Y &= X^3 + (324\sigma - 27)X^2 \\ &+ (1458\sigma^3 + 8667\sigma^2)X + (157464\sigma^4 - 1583388\sigma^3 + 78732\sigma^2). \end{aligned}$$

The same change of variables applied to equation (7.1) yields the infinite section

$$(X(\sigma), Y(\sigma)) = (\sigma f_2(\sigma), \sigma g_2(\sigma)), \qquad \text{where } f_2(0) = -\frac{1011}{8} \text{ and } g_2(0) = \frac{9099 - 1575\sqrt{-3}}{16}.$$

From (7.6), the corresponding point on the $\mathcal{E}_\sigma$ model is

$$[0:0:1]\times\left[-\frac{1011}{8} : \frac{9099 - 1575\sqrt{-3}}{16} : 1\right],$$

which is on the component $\Theta_{\infty,1}$ since the second point is on the conic

$$Y^2 + 9XY + 27X^2 - 78732Z^2 = 0.$$

7.3.3. *The fiber over* $s = \frac{1}{18}$. This is also a singularity of type I_2. We consider the change of variables

$$X = -y - (\sigma^2 - 18\sigma + 1)x, \qquad Y = y, \qquad Z = x + (\sigma^2 - 18\sigma)z, \tag{7.7}$$

which takes the Weierstrass equation at infinity to

$$(X + Y)(X + Z)(Y + Z) + (\sigma^2 - 18\sigma + 1)XYZ = 0. \tag{7.8}$$

When $s = \frac{1}{18}$, we have $\sigma = 18$, and the equation is a product of two rational curves

$$(X + Y + Z)(XY + XZ + YZ) = 0,$$

so this is our Néron model. The component $\Theta_{\frac{1}{18},0}$ is the one meeting the zero section, which is given by $[x : y : z] = [0 : 1 : 0]$. From the change of coordinates in (7.7), this corresponds to $[X : Y : Z] = [-1 : 1 : 0]$. So we have

$$\Theta_{\frac{1}{18},0} : X + Y + Z = 0 \quad \text{and} \quad \Theta_{\frac{1}{18},1} : XY + XZ + YZ = 0.$$

Applying the change of coordinates in (7.7) to the infinite section in (7.1), one calculates

$$XY + XZ + YZ = -\frac{2^4 3^5 (\sigma - 18)\sigma(\sigma - 21)^2(\sigma + 3)^2}{(\sigma - 9)^2(\sigma^2 - 21\sigma + 72)^2(\sigma^2 - 15\sigma + 18)^2},$$

which means that it cuts $\Theta_{\frac{1}{18},1}$.

7.3.4. *The fibers over* $s = \alpha_1$, β_1, α_2, β_2. Recall that α_1 and β_1 are the two distinct roots of $s^2 - 18s + 1 = 0$, and since $\sigma = 1/s$ they are also roots of $\sigma^2 - 18\sigma + 1$. These fibers are of type I_3. We again use the change of coordinates in (7.7). From (7.8), both fibers become a product of three rational curves

$$(X + Y)(X + Z)(Y + Z) = 0.$$

Again, the zero section is $[X : Y : Z] = [-1 : 1 : 0]$, which satisfies $X + Y = 0$. So we identify

$$\Theta_{\alpha_1,0} : X + Y = 0 \quad \text{and} \quad \Theta_{\beta_1,0} : X + Y = 0.$$

After the change of coordinates in (7.7), the infinite section satisfies

$$X + Y = (\sigma^2 - 18\sigma + 1)f_3(\sigma)$$

with $f_3(\sigma)$ a rational function not divisible by $(\sigma^2 - 18\sigma + 1)$. Hence the infinite section cuts $\Theta_{\alpha_1,0}$ and $\Theta_{\beta_1,0}$.

Finally, note that the fibers over α_2 and β_2 are of type I_1, so we know that the infinite section cuts $\Theta_{\alpha_2,0}$ and $\Theta_{\beta_2,0}$ because that is the only choice.

Recall from the discussion in section 7 that $\overline{p_\sigma} \cdot \overline{O} = 5$. With these considerations, equation (2.4) tells us that

$$h(p_\sigma) = 2 \cdot 2 + 2 \cdot 5 - \frac{6 \cdot 6}{12} - \frac{1 \cdot 1}{2} - \frac{1 \cdot 1}{2} = 10,$$

which completes the proof.

Acknowledgements. The authors would like to thank the Banff International Research Station for sponsoring the second Women in Numbers workshop and for providing a productive and enjoyable environment for our initial work on this project. We also thank Kiran Kedlaya, Joseph Silverman, and Bianca Viray for some helpful discussions.

References

[Bea82] Beauville, A. Les familles stables de courbes elliptiques sur $\mathbf{P}^1$ admettant quatre fibres singulières. *C. R. Acad. Sci. Paris Sér. I Math.* **294** (1982), no. 19, 657–660.

[Be06] Bertin, M. J. Mahler's measure and L-series of K3 hypersurfaces. *Mirror symmetry.* V, 3–18, AMS/IP Stud. Adv. Math., 38, Amer. Math. Soc., Providence, RI, 2006.

[Be08a] Bertin, M. J. Mesure de Mahler d'hypersurfaces K3. *J. Number Theory* **128** (2008), no. 11, 2890–2913.

[Be08b] Bertin, M. J. The Mahler measure and the L-series of a singular $K3$-surface. arXiv:0803.0413

[Be10] Bertin, M. J. Measure de Mahler et série L d'une surface $K3$ singulière. *Actes de la Conférence "Fonctions L et Arithmétique",* pp. 5–28 Publ. Math. Besançon Algèbre Théorie Nr., Lab. Math. Besançon, Besançon, 2010.

[Be11] Bertin, M. J. Fonction zêta d'Epstein et dilogarithme de Bloch–Wigner. *J. Théor. Nombres Bordeaux* **23** (2011), no. 1, 21–34.

[Bo81] Boyd, D. W. Speculations concerning the range of Mahler's measure. *Canad. Math. Bull.* **24** (1981), no. 4, 453 – 469.

[Bo98] Boyd, D. W. Mahler's measure and special values of L-functions, *Experiment. Math.* **7** (1998), 37 – 82.

[Co99] Connell, I. *Elliptic Curve Handbook,* http://www.math.mcgill.ca/connell/public/ECH1/.

[De97] Deninger, C. Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbb{Z}^n$-actions, *J. Amer. Math. Soc.* **10** no. 2 (1997), 259 – 281.

[IR90] Ireland, K., Rosen, M. *A classical introduction to modern number theory.* Second edition. Graduate Texts in Mathematics, 84. Springer-Verlag, New York, (1990).

[Kn92] Knapp, A. W., *Elliptic curves.* Mathematical Notes, 40. Princeton University Press, Princeton, NJ, 1992. xvi+427 pp.

[MP89] Miranda, R., Persson, U. Torsion groups of elliptic surfaces. *Compositio Math.* **72** (1989), no. 3, 249–267.

[Ne64] Néron, A., Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. *Inst. Hautes Études Sci. Publ.Math.* **21** (1964) 128.

[PARI] The PARI Group. *PARI/GP, version* 2.5.0 (2011) http://pari.math.u-bordeaux.fr/.

[PS89] Peters, C., Stienstra, J. A pencil of $K3$-surfaces related to Apéry's recurrence for $\zeta(3)$ and Fermi surfaces for potential zero. *Arithmetic of complex manifolds (Erlangen, 1988)*, 110–127, Lecture Notes in Math., 1399, *Springer, Berlin,* 1989.

[RV97] Rodriguez-Villegas, F. Modular Mahler measures I, *Topics in number theory* (University Park, PA 1997), 17–48, Math. Appl., 467, Kluwer Acad. Publ. Dordrecht (1999).

[RZ11] Rogers, M. , Zudilin, W. On the Mahler measures of $1 + X + 1/X + Y + 1/Y$. Preprint, March 2011.

[Sc08] Schütt, M. CM newforms with rational coefficients. *Ramanujan J.* **19** (2009), no. 2, 187–205.

[SS10] Schütt, M., Shioda, T. Elliptic surfaces. *Algebraic geometry in East Asia-Seoul 2008,* 51–160, Adv. Stud. Pure Math., 60, *Math. Soc. Japan, Tokyo,* 2010.

[Sh90] Shioda, T. On the Mordell-Weil lattices. *Comment. Math. Univ. St. Paul.* **39** (1990), no. 2, 211–240.

[Sm71] Smyth, C. J. On the product of the conjugates outside the unit circle of an algebraic integer, Bull. Lond. Math. Soc. 3 (1971), 169–175.

[St11] Stein, W. et al. *Sage Mathematics Software (Version 4.7.2)*, http://www.sagemath.org.

[Yu04] Yui, N. Arithmetic of Calabi-Yau varieties. *Mathematisches Institut, Georg-August-Universität Göttingen: Seminars Summer Term 2004, 9–29, Universitätsdrucke Göttingen, Göttingen,* 2004.

Marie-José Bertin: Université Pierre et Marie Curie (Paris 6), Institut de Mathématiques, 175 rue du Chevaleret, 75013 Paris, France

E-mail address: bertin@math.jussieu.fr

Amy Feaver: Department of Mathematics, University of Colorado at Boulder, Campus Box 395, Boulder, CO 80309, USA

E-mail address: amy.feaver@colorado.edu

Jenny Fuselier: Department of Mathematics & Computer Science, Drawer 31, High Point University, 833 Montlieu Ave., High Point, NC 27262, USA

E-mail address: jfuselie@highpoint.edu

Matilde Lalín: Département de mathématiques et de statistique, Université de Montréal. CP 6128, succ. Centre-ville. Montreal, QC H3C 3J7, Canada

E-mail address: mlalin@dms.umontreal.ca

Michelle Manes: Department of Mathematics, University of Hawaii, 2565 McCarthy Mall, Honolulu, HI 96822, USA

E-mail address: mmanes@math.hawaii.edu