

Mahler measure and L-series of K3-hypersurfaces

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Motivation

Generalise **Deninger's guess** (Calgary CMS meeting 1996):

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) = \frac{15}{4\pi^2} L(E, 2) =: L'(E, 0),$$

E elliptic curve, algebraic closure of the zero set of the polynomial, denoted 15a8 (Cremona's notation), of conductor 15, defined by

$$Y^2 + XY + Y = X^3 + X^2$$

with L-series given by the modular form

$$f_{15A}(z) = \eta(z)\eta(3z)\eta(5z)\eta(15z)$$

(Deninger's guess was proved in 2011 by Rogers and Zudilin and again in 2013 by Zudilin.)

Find **explicit** Mahler measures of certain polynomials

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k \quad (\text{Apéry-Fermi pencil})$$

$$Q_k = (x + y + z + 1)(xy + xz + yz + xyz) - (k + 4)xyz$$

in terms of ***L*-series of modular forms** and also in terms of ***L*-series of the variety**.

Definitions and historic background

Introduced by Mahler in 1962,
the logarithmic Mahler measure of a polynomial P is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}$$

and its Mahler measure

$$M(P) = \exp(m(P))$$

where

$$\mathbb{T}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n / |x_1| = \dots = |x_n| = 1\}.$$

By Jensen's formula, $\Omega(P) = M(P)$ if P a one variable polynomial.

Thus the first reference to problems of Mahler measures is **Lehmer's question (1933)** (still unsolved):

Let $P \in \mathbb{Z}[X]$, monic, non cyclotomic and define:

$$\Omega(P) = \prod_{P(\alpha)=0} \max(|\alpha|, 1),$$

does there exist such a P satisfying

$$1 < \Omega(P) < \Omega(P_0) = 1.1762 \dots ?$$

where

$$P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

is the Lehmer polynomial, in fact a Salem polynomial.

For detailed background, see my Copenhagen talk (Aug. 27, 2018) on my web page.

Why this generalisation seemed possible?

- Polynomials P_k and Q_k define elliptic $K3$ -surfaces
- Elliptic curves and elliptic $K3$ -surfaces are Calabi-Yau varieties
- Like elliptic curves, **singular** $K3$ -surfaces defined over \mathbb{Q} are modular

What's an elliptic $K3$ -surface?

- A double covering branched along a plane sextic for example defines a $K3$ -surface X .

In case of the Apéry-Fermi pencil

$$(2z + x + \frac{1}{x} + y + \frac{1}{y} - k)^2 = (x + \frac{1}{x} + y + \frac{1}{y} - k)^2 - 4$$

- An elliptic $K3$ surface X admits a fibration $\pi : X \rightarrow \mathbb{P}^1$ such that the fiber $\pi^{-1}(t)$ is an elliptic curve for all but a finite number of t giving the singular fibers classified by Kodaira.
- Given an elliptic surface as

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

you recognize an elliptic fibration on a $K3$ surface if the degree of polynomials a_i is $\leq 2i$ and is exactly $2i$ for one i .

- How to get an elliptic fibration on the Apéry-Fermi pencil?
First, as did Peters and van Vglut, in cutting by a pencil of planes $X + Y + Z = t$. This gives the elliptic fibration:

$$y^2 - xy(t^2 - kt + 1) = x(x - 1)(x + t^2 - tk)$$

t is called an **elliptic parameter**

But, if we want to obtain Weierstrass equations of all the elliptic fibrations of the pencil, we must use the technique of Elkies's neighbors. (see our recent preprint B. and Lecacheux (2018)).

- There is a unique holomorphic 2-form ω on X up to a scalar.
- $H_2(X, \mathbb{Z})$ is a free group of rank 22.
- With the intersection pairing, $H_2(X, \mathbb{Z})$ is a lattice and

$$H_2(X, \mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

\mathcal{L} is the K3-lattice, U_2 the hyperbolic lattice of rank 2, E_8 the unique even positive-definite unimodular lattice of rank 8.

- $$Pic(X) \subset H_2(X, \mathbb{Z}) \simeq Hom(H^2(X, \mathbb{Z}), \mathbb{Z})$$

where $Pic(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles.

- $$Pic(X) \simeq \mathbb{Z}^{\rho(X)}$$

$\rho(X) :=$ Picard number of X

$$1 \leq \rho(X) \leq 20$$



$$T(X) := (\text{Pic}(X))^\perp$$

is the transcendental lattice of dimension $22 - \rho(X)$

- If $\{\gamma_1, \dots, \gamma_{22}\}$ is a \mathbb{Z} -basis of $H_2(X, \mathbb{Z})$ and ω the holomorphic 2-form,

$$\int_{\gamma_i} \omega$$

is called a period of X and

$$\int_{\gamma} \omega = 0 \text{ for } \gamma \in \text{Pic}(X).$$

- If $\{X_z\}$ is a family of $K3$ surfaces, $z \in \mathbb{P}^1$ with generic Picard number ρ and ω_z the corresponding holomorphic 2-form, then the periods of X_z satisfy a Picard-Fuchs differential equation of order $k = 22 - \rho$. For our family $k = 3$.

- In fact, by Morrison, a \mathcal{M} -polarized K3-surface, with Picard number 19 or 20 has a Shioda-Inose structure, that means

$$\begin{array}{ccc}
 X & & A = E \times E / C_N \\
 & \searrow & \swarrow \\
 & Y = \text{Kum}(A / \pm 1) &
 \end{array}$$

- If the Picard number $\rho = 20$, then the elliptic curve E is CM.

Theorem

(B. 2005) Let $k = -(t + \frac{1}{t}) - 2$ and $t = \frac{\eta(3\tau)^4 \eta(12\tau)^8 \eta(2\tau)^{12}}{\eta(\tau)^4 \eta(4\tau)^8 \eta(6\tau)^{12}}$

$$\begin{aligned}
 m(Q_k) = & \frac{\Im \tau}{8\pi^3} \left\{ \sum_{m, \kappa}' \left(2(2\Re \frac{1}{(m\tau + \kappa)^3 (m\bar{\tau} + \kappa)} + \frac{1}{(m\tau + \kappa)^2 (m\bar{\tau} + \kappa)^2}) \right. \right. \\
 & - 32(2\Re \frac{1}{(2m\tau + \kappa)^3 (2m\bar{\tau} + \kappa)} + \frac{1}{(2m\tau + \kappa)^2 (2m\bar{\tau} + \kappa)^2}) \\
 & - 18(2\Re \frac{1}{(3m\tau + \kappa)^3 (3m\bar{\tau} + \kappa)} + \frac{1}{(3m\tau + \kappa)^2 (3m\bar{\tau} + \kappa)^2}) \\
 & \left. \left. + 288(2\Re \frac{1}{(6m\tau + \kappa)^3 (6m\bar{\tau} + \kappa)} + \frac{1}{(6m\tau + \kappa)^2 (6m\bar{\tau} + \kappa)^2}) \right) \right\}
 \end{aligned}$$

Main ingredients in the proof

- The derivative of $m(Q_k)$ with respect to the parameter k is a period of the $K3$ -surface, hence satisfy the Picard-Fuchs equation.
- Since we know the solutions of such P-F in terms of modular forms we find that $dm(P_k)$ can be expressed as a combination of $E_4(n\tau)$, $n = 1, 2, 3, 6$.
- An integration allows to write the Mahler measure as a modular form.
- Using a Fourier development, it follows $m(Q_k)$ in terms of Eisenstein-Kronecker series.

When $m(Q_k)$ is obtained as the L -series of a modular form?

When the $K3$ -surface is **singular**, that is when its Picard number is 20, that is when τ is imaginary quadratic.

k	-36	-12	-6	-3	0	4	12	60
τ	$\frac{\sqrt{-3}}{3}$	$\frac{\sqrt{-6}}{6}$	$\frac{\sqrt{-3}}{6}$	$\frac{3+\sqrt{-15}}{24}$	$\frac{3+\sqrt{-3}}{12}$	$\frac{-2+\sqrt{-2}}{6}$	$\frac{3+\sqrt{-3}}{6}$	$\frac{3+\sqrt{-15}}{6}$

Denote $f_{12} = \eta(2\tau)^3\eta(6\tau)^3 = q - 3q^3 + 2q^7 + 9q^9 - 22q^{13} + 26q^{19} + \dots$ and more generally f_N is the CM newform of weight 3 with rational Fourier coefficients up to twisting (see Schütt's table in "CM newforms with rational coefficients").

Denote also d_f the Dirichlet series

$$d_f = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2)$$

Table of new results

$m(Q_{-36}) =$	$8 \frac{12\sqrt{12}}{4\pi^3} L(f_{12}, 3) + 2d_4$	B.(2019), Samart (2013)
$m(Q_{-12}) =$	$\frac{3}{2} \frac{24\sqrt{24}}{4\pi^3} L(f_{24}, 3) + \frac{5}{2} d_3$	B.(2019)
$m(Q_{-6}) =$	$\frac{7}{2} \frac{12\sqrt{12}}{4\pi^3} L(f_{12}, 3) + d_4$	B.(2019), Samart (2013)
$m(Q_{-3}) =$	$\frac{8}{5} d_3$	B.(2008)
$m(Q_0) =$	$2 \frac{12\sqrt{12}}{4\pi^3} L(f_{12}, 3)$	B.(2005)
$m(Q_4) =$	$10 \frac{8\sqrt{8}}{4\pi^3} L(f_8, 3)$	B.(2019)
$m(Q_{12}) =$	$8 \frac{12\sqrt{12}}{4\pi^3} L(f_{12}, 3)$	B.(2005)
$m(Q_{60}) =$	$\frac{14}{5} \frac{15\sqrt{15}}{4\pi^3} L(f_{15} \otimes \chi_{-3}) + 4d_3$	B.(2019)

The proof uses the theorem and some tricks to distinguish the modular part from the Dirichlet part.

Relating the Mahler measure to the L -series of the $K3$ -surface

Livné's modularity theorem:

Theorem

Let S be a $K3$ -surface defined over \mathbb{Q} , with Picard number 20 and discriminant N . Its transcendental lattice $T(S)$ is a dimension 2 $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -module thus defines a L series, $L(T(S), s)$.

There exists a weight 3 modular form, f , CM over $\mathbb{Q}(\sqrt{-N})$ satisfying

$$L(T(S), s) \doteq L(f, s).$$

Moreover, if $NS(S)$ is generated by divisors defined over \mathbb{Q} ,

$$L(S, s) \doteq \zeta(s-1)^{20} L(f, s).$$

When, instead of elliptic curves, we consider singular $K3$ -hypersurfaces, the determinant of the Gram matrix of the transcendental lattice plays the role of the conductor of the elliptic curve and for some polynomials defining singular $K3$ -surfaces we expect formulae like

$$m(P_k) \doteq \frac{(\det(T_K))^{3/2}}{4\pi^3} L(Y_k, 3)$$

\doteq meaning up to a rational factor.

Computing the L -series

$$L^*(X, s) := \prod_{p \nmid N}^* Z(X|_{\mathbb{F}_p}, p^{-s}) = \sum_{n \geq 1} \frac{A(n)}{n^s}$$

N is the determinant of the transcendental lattice. Giving a suitable value to the local factors, the L -series of the surface X can be expressed in terms of the Mellin transform of a modular form.

Lemma

Let X be a singular elliptic K3-surface defined over \mathbb{Q} . If a Weierstrass model E of an elliptic fibration of X has rank 1 and possess an infinite section defined over $\mathbb{Q}(\sqrt{d})$, then

$$A_p = - \sum_{x \in \mathbb{P}^1(\mathbb{F}_p), E_x \text{ smooth}} a_p(x) - \sum_{x \in \mathbb{P}^1(\mathbb{F}_p), E_x \text{ singular}} \epsilon_p(x) - \left(\frac{d}{p}\right) p$$

with $a_p(x) = p - 1 - \#E_x(\mathbb{F}_p)$ and $\epsilon_p(x)$ such that

A good Weierstrass model

If interested in the link between the Mahler measure and the L-series of the $K3$ -surface, we need a good Weierstrass model for the family.

By chance, quite recently, [O. Lecacheux](#) found such a model for the family Q_k , namely

$$(F_k) \quad y^2 = x^3 - ((-k^2 + 24)t^2 - 2(k-2)(k+4)^2 t - k(k+4)^3)x^2 - 16t^4(t+k+3)x$$

I explain

k	-36	-12	-4	-6	-3	0	4	12	60
rk(MW_k)	1	1	-1	1	0	0	1	0	1

The model is very good when the rank is 0 thanks to Shioda's formula.

For example

F_0 has singular fibers III^* , I_4^* , I_3 , I_2 , 2-torsion hence

$$|\det T_0| = \frac{2 \times 4 \times 3 \times 2}{2^2} = 12.$$

The results (B. 2019)

When the rank is 1, we need an infinite section for computing the determinant of the transcendental lattice and also applying the lemma. Sometimes it is difficult to find.

$m(Q_0) =$	$2 \frac{12\sqrt{12}}{4\pi^3} L(f_{12}, 3)$	$L(Z_0, 3) = L(f_{12}, 3)$
$m(Q_{12}) =$	$\frac{4 \times 12 \sqrt{4 \times 12}}{4\pi^3} L(f_{12}, 3)$	$L(Z_{12}, 3) = L(f_{12}, 3)$
$m(Q_{-3}) =$	$\frac{8}{5} d_3$	$L(Z_{-3}, 3) = L(f_{15} \otimes \chi_{-3}, 3)$
$m(Q_{-36}) =$	$8 \frac{12\sqrt{12}}{4\pi^3} L(f_{12}, 3) + 2d_4$	$L(Z_{-36}, 3) \stackrel{?}{=} L(f_{12}, 3)$
$m(Q_{-6}) =$	$\frac{7}{16} \frac{48\sqrt{48}}{4\pi^3} L(f_{12}, 3) + d_4$	$L(Z_{-6}, 3) = L(f_{12}, 3)$
$m(Q_4) =$	$\frac{5}{32} \frac{32\sqrt{32}}{4\pi^3} L(f_8, 3)$	$L(Z_4, 3) = L(f_8, 3)$
$m(Q_{60}) =$	$7 \times \frac{3\sqrt{15}}{2\pi^3} L(f_{15} \otimes \chi_{-3}, 3) + 4d_3$	$L(Z_{60}, 3) \stackrel{?}{=} L(f_{15} \otimes \chi_{-3}, 3)$
$m(Q_{-12}) =$	$\frac{3}{16} \frac{4 \times 24 \sqrt{4 \times 24}}{4\pi^3} L(f_{24}, 3) + \frac{5}{2} d_3$	$L(Z_{-12}, 3) = L(f_{24}, 3)$

$K3$ -surfaces with the same L -series

Shioda proved that the transcendental lattice is a birational invariant of $K3$ -surfaces.

The surfaces Z_0 , Z_{12} , Z_{-3} are extremal $K3$ -surfaces of Picard number 20. By [Shimada-Zhang](#) we know

$$T(Z_0) = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}, T(Z_{12}) = \begin{pmatrix} 2 & 0 \\ 0 & 24 \end{pmatrix}, T(Z_{-3}) = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

We can prove that

$$T(Z_{-6}) = \begin{pmatrix} 2 & 0 \\ 0 & 24 \end{pmatrix}$$

Hence, the surfaces Z_{12} and Z_{-6} are birationally equivalent.

To conclude, there are many other relations between these surfaces and also with surfaces of the Apéry-Fermi family suggested by computations of Mahler measures.